Singular masas of von Neumann algebras: examples from the geometry of spaces of nonpositive curvature

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Abstract.

If Γ is a group, then the von Neumann algebra $VN(\Gamma)$ is the convolution algebra

$$\mathrm{VN}(\Gamma) = \{f \in \ell^2(\Gamma) : f \star \ell^2(\Gamma) \subseteq \ell^2(\Gamma)\}\,.$$

Let T^r be a totally geodesic flat torus in a compact locally symmetric space $M = \Gamma\backslash G/K$ of rank r. Let $\Gamma_0 = \pi(T^r) \cong \mathbb{Z}^r$, embedded naturally in $\Gamma = \pi(M)$. Then $\mathrm{VN}(\Gamma_0)$ is a maximal abelian $\star$-subalgebra (masa)of VN(Γ). If in addition diam(T^r) is less than the length of a shortest closed geodesic in M then $\mathrm{VN}(\Gamma_0)$ is a *singular masa* : its unitary normalizer is as small as possible. This result (Theorem B below) is joint work with A. M. Sinclair and R. R. Smith.

Let Γ be a countable group. Suppose Γ is ICC :

$$\#\{y^{-1}xy : y \in \Gamma\} = \infty \quad \text{if} \quad x \in \Gamma - \{1\}.$$

Consider the convolution algebra

$$\mathrm{VN}(\Gamma) = \{f \in \ell^2(\Gamma) : f \star \ell^2(\Gamma) \subseteq \ell^2(\Gamma)\}\,.$$

This von Neumann algebra is a II_1 **factor** :

(a) $\mathrm{VN}(\Gamma)$ is a strongly closed $\star$-subalgebra of $B(\ell^2(\Gamma))$, with trivial centre;
(b) there is a faithful trace on $\mathrm{VN}(\Gamma)$ defined by $\mathrm{tr}(f) := f(1)$.

Γ embeds as a subgroup of the unitary group of $\mathrm{VN}(\Gamma)$ via $x \mapsto \delta_x$.

Theorem. (A. Connes, 1976) If Γ_1,Γ_2 are amenable ICC groups then $\mathrm{VN}(\Gamma_1) \cong \mathrm{VN}(\Gamma_2)$. [*The hyperfinite* II_1 *factor.*]

At the opposite extreme from amenable groups there is the

Rigidity Conjecture (A. Connes) If ICC groups Γ_1,Γ_2 have Property (T), then

$$\mathrm{VN}(\Gamma_1) \cong \mathrm{VN}(\Gamma_2) \Rightarrow \Gamma_1 \cong \Gamma_2.$$

Compare this with the

Rigidity Theorem (Mostow-Margulis-Prasad) For $i = 1, 2$, let Γ_i be a lattice in G_i, a connected non-compact simple Lie group with trivial centre, $G_1 \neq \mathrm{PSL}_2(\mathbb{R})$. Then

$$\Gamma_1 \cong \Gamma_2 \Rightarrow G_1 \cong G_2.$$

In Mostow's proof of rigidity (the cocompact, higher rank case), maximal flats of the associated symmetric spaces play an important role.

Let $\mathcal{A}$ be a maximal abelian $\star$-subalgebra (**masa**) of VN(Γ).

Say that $\mathcal{A}$ is a **singular masa** if :
$u \in$ VN(Γ), u unitary, $u\mathcal{A}u^* = \mathcal{A} \Rightarrow u \in \mathcal{A}$.

Singular masas always exist (S. Popa, 1983), but are hard to construct explicitly.

Aside: If the unitary normalizer of $\mathcal{A}$ generates VN(Γ) then $\mathcal{A}$ is a *Cartan* masa. VN(Γ) may not contain a Cartan masa: e.g. $\Gamma = \mathbb{F}_2$.

S. Popa has recently used Cartan masas to construct isomorphism invariants for certain II_1 factors.

Subalgebras from subgroups

If $\Gamma_0 < \Gamma$, then VN(Γ_0) $\subseteq$ VN(Γ) via $f \mapsto \overline{f}$, where

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

Masas from abelian subgroups

Lemma. Let $\Gamma_1 < \Gamma_0 < \Gamma$, with Γ_0 abelian. Suppose that, for all $x \notin \Gamma_0$,

$$A_x = \{x_1^{-1} x x_1 : x_1 \in \Gamma_1\} \text{ is infinite.}$$

Then $\mathsf{VN}(\Gamma_1)' = \mathsf{VN}(\Gamma_0)$.

$'$ means commutant.

In particular, $\mathsf{VN}(\Gamma_0)$ is a masa of $\mathsf{VN}(\Gamma)$.

Proof. Let $f \in \mathsf{VN}(\Gamma_1)'$ and $x \notin \Gamma_0$.
Then $\delta_{x_1^{-1}} * f * \delta_{x_1} = f \ \ (\forall x_1 \in \Gamma_1)$
$\Rightarrow f$ is constant on A_x
$\Rightarrow f = 0$ on A_x $\quad$ (*since $f \in \ell^2(\Gamma)$ and $\#A_x = \infty$*)
$\Rightarrow f(x) = 0$ $\quad$ (for all $x \notin \Gamma_0$)
$\Rightarrow f \in \mathsf{VN}(\Gamma_0)$. $\square$

There is a **conditional expectation**

$$\mathbb{E}_{\mathcal{A}} : \mathsf{VN}(\Gamma) \to \mathcal{A}$$

onto any masa $\mathcal{A}$ which extends to an orthogonal projection on $\ell^2(\Gamma)$. If $\mathcal{A} = \mathsf{VN}(\Gamma_0)$, where $\Gamma_0 < \Gamma$, then

$$\mathbb{E}_{\mathcal{A}} f(x) = \begin{cases} f(x) & \text{if } x \in \Gamma_0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition. A masa $\mathcal{A}$ is **strongly singular** if

$$\|\mathbb{E}_{u\mathcal{A}u^*} - \mathbb{E}_{\mathcal{A}}\|_{\infty,2} \geq \|u - \mathbb{E}_{\mathcal{A}}(u)\|_2$$

for all unitaries $u \in \mathrm{VN}(\Gamma)$.
$\| \cdot \|_{\infty,2}$ means : operator norm on domain, ℓ^2 norm on range.

Theorem. [RSS] *The following condition implies that* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$*:*
(SS) *If* $x_1, \ldots, x_m \in \Gamma$ *and*

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 x_j \,,$$

then $x_i \in \Gamma_0$ *for some* i*.*

This condition can be verified by geometric methods for certain groups acting on spaces of nonpositive curvature.

Let G be a semisimple Lie group of rank r, no centre and no compact factors;
Γ a torsion free cocompact lattice in G;
$X = G/K$, with $K < G$ maximal compact.
Then :
$M = \Gamma\backslash X$ is compact, locally symmetric.
Universal covering : $p : X \to M$.
Fundamental group: $\pi(M) = \Gamma$.

Let $T^r \subset M$, a totally geodesic flat r-torus, and $\xi \in T^r$.
The inclusion map i induces a monomorphism

$$i_* : \pi(T^r, \xi) \to \pi(M, \xi).$$

[No geodesic loop in M can be null-homotopic.]

Let $\Gamma_0 = i_*\pi(T^r, \xi) \cong \mathbb{Z}^r < \Gamma$.

Theorem A. $\mathrm{VN}(\Gamma_0)$ *is a masa of* $\mathrm{VN}(\Gamma)$*.*

Theorem B. *Let σ be the length of a shortest closed geodesic in M. If* $\mathrm{diam}(T^r) < \sigma$ *then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$*.*

Theorem A$'$. Let $x_1 \in \Gamma_0$ be the class of a *regular* closed geodesic c in T^r, and

$$\Gamma_1 = \langle x_1 \rangle \cong \mathbb{Z} < \Gamma_0.$$

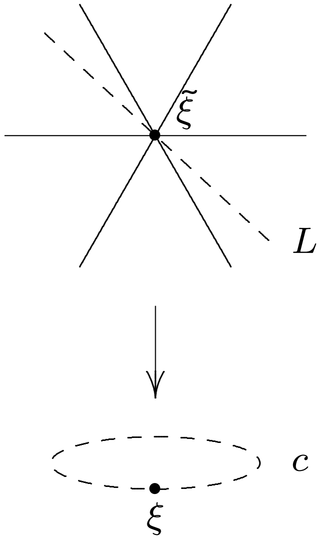
Then $\mathrm{VN}(\Gamma_1)' = \mathrm{VN}(\Gamma_0)$.

Note: Such elements x_1 exist.

Consequence: $\mathrm{VN}(\Gamma_0)$ is the unique masa containing $\mathrm{VN}(\Gamma_1)$.

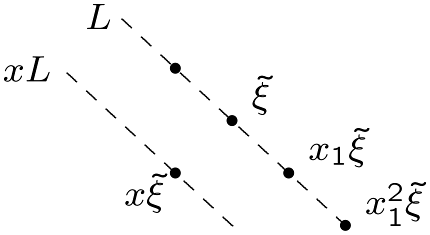
The meaning of *regularity*.

Lift c to a geodesic L in X through $\tilde{\xi}$, where $p(\tilde{\xi}) = \xi$.



L lies in a *unique* maximal flat F_0 and $p(F_0) = T^r$.

Proof of Theorem A′. (Using the Lemma.)
x_1 acts on L by translation,



Suppose $A_x = \{x_1^{-n}xx_1^n : n \in \mathbb{Z}\}$ is finite.
Let

$$\delta = \sup\{d(\eta, x_1^{-n}xx_1^n\eta) : \eta \in [\tilde{\xi}, x_1\tilde{\xi}], n \in \mathbb{Z}\}.$$

Then

$$d(x_1^n\eta, xx_1^n\eta) \le \delta \qquad (\eta \in [\tilde{\xi}, x_1\tilde{\xi}],\ n \in \mathbb{Z}).$$

Therefore

$$d(\zeta, x\zeta) \le \delta \text{ for all } \zeta \in L.$$

i.e. L is a parallel translate of xL
$\Rightarrow$ L and xL lie in a common maximal flat
$\Rightarrow xL \subseteq F_0$
$\Rightarrow x\tilde{\xi} \in F_0$
$\Rightarrow p[\tilde{\xi}, x\tilde{\xi}]$ is a closed geodesic in T^r
$\Rightarrow x \in \Gamma_0$. □

Corollary to Theorem B. *Let* $\Gamma = \pi(M_g)$, M_g *a compact Riemann surface of genus* $g \geq 2$. *Let* c *be a closed geodesic of minimal length* σ *in* M_g. *Let* $x_0 = [c] \in \Gamma$, *and* $\Gamma_0 = \langle x_0 \rangle \cong \mathbb{Z}$. *Then* $\mathrm{VN}(\Gamma_0)$ *is a strongly singular masa of* $\mathrm{VN}(\Gamma)$.

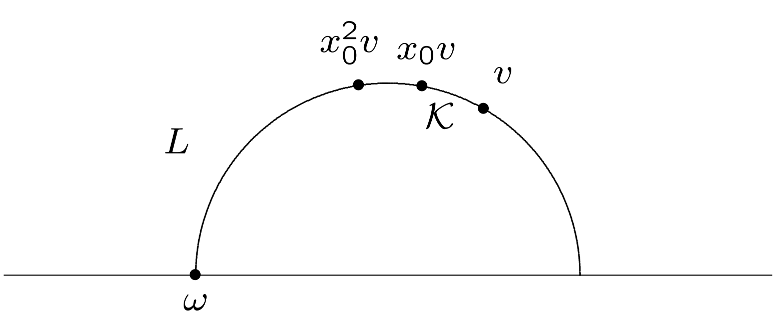
Proof. (Using condition (SS).) The universal covering of M_g is

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}.$$

Γ acts isometrically on $\mathfrak{H}$.

Lift c to a geodesic L in $\mathcal{H}$.
Fix $v \in L$, and let $\mathcal{K} = [v, x_0 v]$. Then

$$L = \bigcup_{n \in \mathbb{Z}} x_0^n \mathcal{K} = \Gamma_0 \mathcal{K}\,. \qquad (1)$$



Suppose $x_1, \dots, x_m \in \Gamma$ and

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 x_j \,. \tag{2}$$

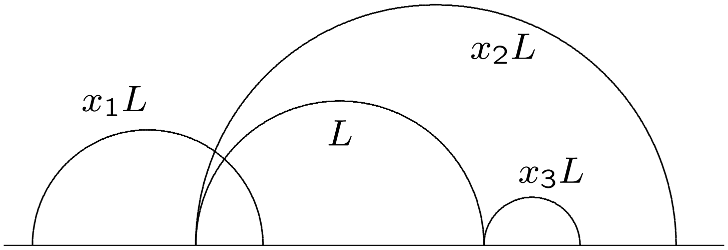
Let $\delta = \max\{d(x_j\kappa, \kappa); 1 \leq j \leq m, \kappa \in \mathcal{K}\}$. Then

$$\begin{aligned} x_j\mathcal{K} \underset{\delta}{\subset} \mathcal{K} \;&\Rightarrow\; \Gamma_0 x_j \mathcal{K} \underset{\delta}{\subset} \Gamma_0\mathcal{K} = L \\ &\Rightarrow\; x_i\Gamma_0 x_j \mathcal{K} \underset{\delta}{\subset} x_i L \end{aligned}$$

$P \underset{\delta}{\subset} Q$ means that $d(p, Q) \leq \delta$, for all $p \in P$.

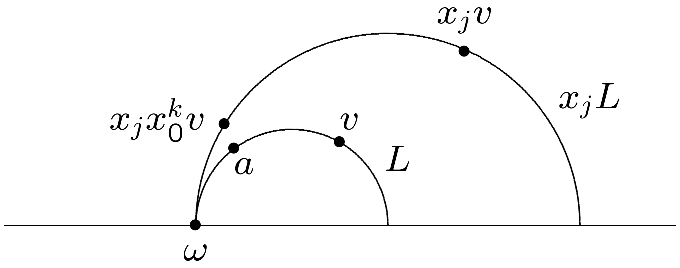
$$L = \Gamma_0 \mathcal{K} \underset{\delta}{\subset} x_1 L \cup x_2 L \cup \cdots \cup x_m L \,.$$

Therefore each boundary point of L is a boundary point of some $x_j L$.

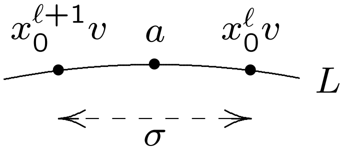


In particular $\omega = x_0^\infty v$ is a boundary point of some $x_j L$.
Claim: $x_j \in \Gamma_0$.
Choose $k \in \mathbb{Z}$, $a \in L$ such that

$$d(x_j x_0^k v, a) < \frac{\sigma}{2}.$$



Choose $\ell \in \mathbb{Z}$ such that $d(a, x_0^\ell v) \leq \frac{\sigma}{2}$:



Then

$$\begin{aligned} d(x_0^{-\ell} x_j x_0^k v, v) &= d(x_j x_0^k v, x_0^\ell v) \\ &\leq d(x_j x_0^k v, a) + d(a, x_0^\ell v) \\ &< \sigma \end{aligned}$$

$[v, x_0^{-\ell} x_j x_0^k v]$ projects to a closed geodesic in M_g of length $< \sigma$.

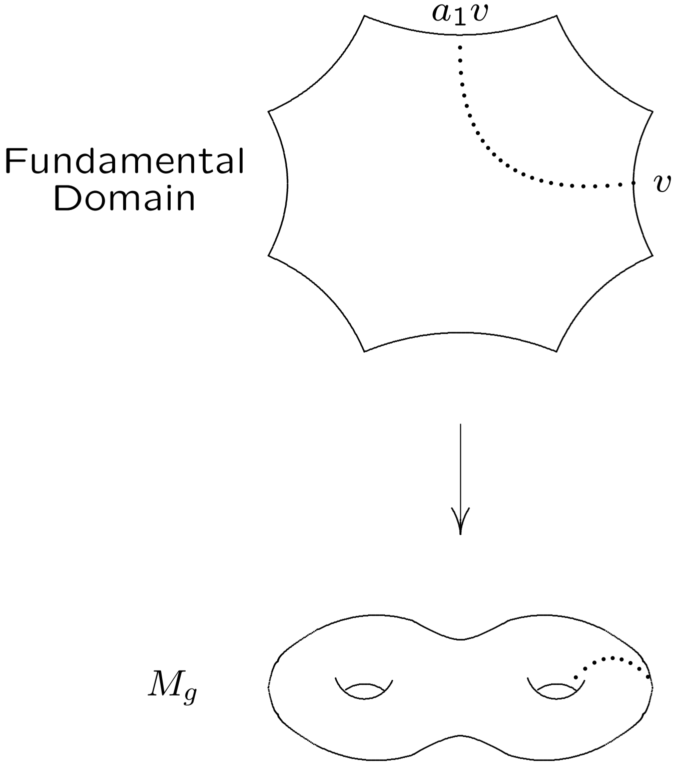
This implies $x_0^{-\ell} x_j x_0^k = 1$.

Therefore $x_j = x_0^{\ell - k} \in \Gamma_0$. $\square$

In the usual presentation of $\pi(M_g)$,

$$\Gamma = \left\langle a_1, \dots, a_g, b_1, \dots, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

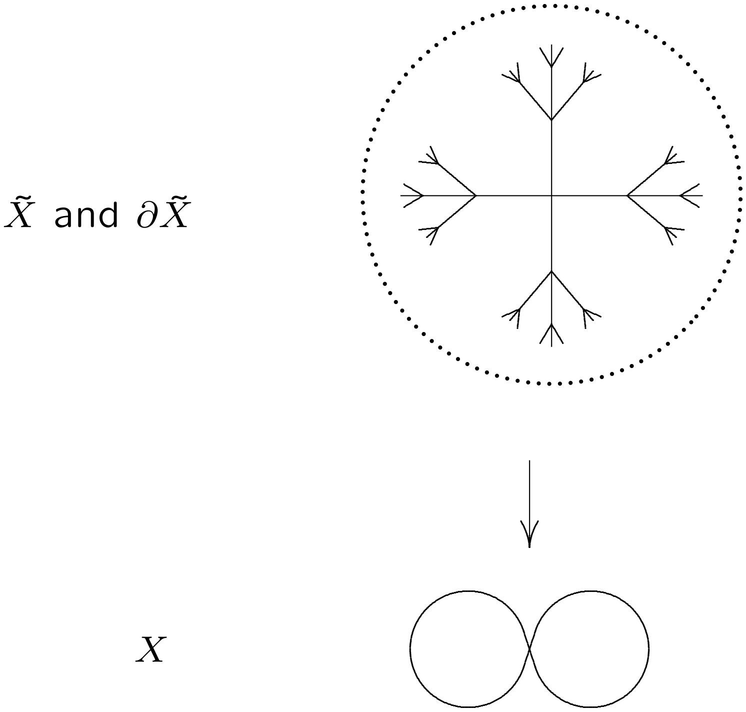
we can take $x_0 \in \{a_i^{\pm 1}, b_j^{\pm 1}\}$.



Free Group Analogue

$\Gamma = \mathbb{Z} \star \mathbb{Z} = \langle a, b \rangle$, acts freely on a tree $\tilde{X}$ with boundary $\partial \tilde{X}$.
$\Gamma = \pi(X)$, where $X = \Gamma \backslash \tilde{X}$.

If $\Gamma_0 = \langle a \rangle \cong \mathbb{Z}$ then $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$. (Same proof.)



Euclidean Buildings

Suppose Γ acts freely and transitively on the vertex set of a euclidean building Δ and Γ_0 is an abelian subgroup which acts transitively on the vertex set of an apartment (flat). Then $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$. (Same proof.)

$\exists$ many examples where $\Gamma < \mathrm{PGL}_3(\mathbb{K})$, $\mathbb{K}$ a nonarchimedean local field.
(Cartwright, Mantero, Steger, Zappa)

Example: $\mathbb{K} = \mathbf{F}_4((X))$, Γ is the torsion free group with generators $x_i, 0 \le i \le 20$, and relations (written modulo 21):

$$\begin{cases} x_j x_{j+7} x_{j+14} = x_j x_{j+14} x_{j+7} = 1 & 0 \le j \le 6, \\ x_j x_{j+3} x_{j-6} = 1 & 0 \le j \le 20. \end{cases}$$

For each j, $0 \le j \le 6$,

$$\Gamma_0 = \langle x_j, x_{j+7}, x_{j+14} \rangle \cong \mathbb{Z}^2$$

satisfies the hypotheses.

A Borel subgroup

Proposition. Let Γ be the upper triangular subgroup of $\mathrm{PSL}_n(\mathbb{Q})$, $n \geq 2$, and let Γ_0 be the diagonal subgroup of Γ. Then $\mathrm{VN}(\Gamma_0)$ is a strongly singular masa of $\mathrm{VN}(\Gamma)$.

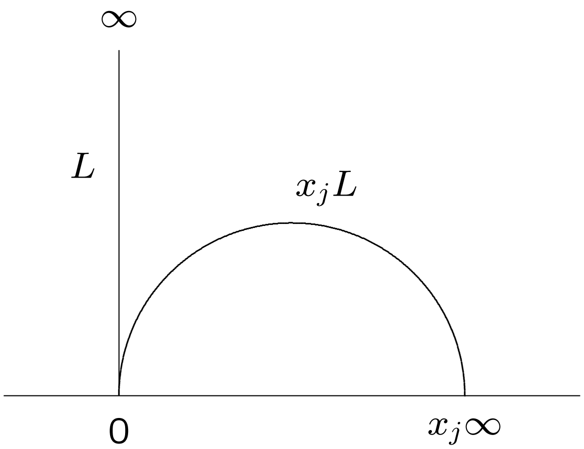
$n = 2$: [J. Dixmier, 1954 (for singularity)]
$\Gamma = \{g \in \mathrm{PSL}_2(\mathbb{Q}) : g\infty = \infty\} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$
acts on $\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}$.
$\Gamma_0 = \{g \in \mathrm{PSL}_2(\mathbb{Q}) : g0 = 0\} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$.

Proof. Suppose $x_1, \ldots, x_m \in \Gamma$ and

$$\Gamma_0 \subseteq \bigcup_{i,j} x_i \Gamma_0 x_j \,,$$

Note that Γ_0 stabilizes the geodesic

$$\begin{aligned} L &= \mathbb{R}^+ i \\ &= \Gamma_0 K, \qquad \text{where } K = [i, 2i]. \end{aligned}$$



Arguing as before, $L \underset{\delta}{\subset} x_1L \cup x_2L \cup \cdots \cup x_mL$, for some $\delta > 0$. Therefore 0 is a boundary point of some x_jL. Now

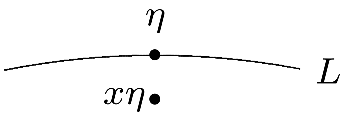
$$\begin{aligned} x_j \in B &\Rightarrow x_j\infty = \infty \\ &\Rightarrow x_jL = L \\ &\Rightarrow x_j \in \Gamma_0 . \end{aligned}$$

The ICC property.

Let X be a symmetric space or euclidean building.
A cocompact group Γ of isometries of X is ICC.

Proof. (The case $X = \mathcal{H}$.)
$\Gamma\mathcal{K} = \mathcal{H}$ where $\mathcal{K} \subset \mathcal{H}$ is compact.
Let $x \in \Gamma - \{1\}$.
Suppose that $C = \{y^{-1}xy : y \in \Gamma\}$ is finite.
Let $\delta = \max\{d(\kappa, y^{-1}xy\kappa) : \kappa \in \mathcal{K}, y \in \Gamma\}$.
Then
$d(y\kappa, xy\kappa) = d(\kappa, y^{-1}xy\kappa) \leq \delta$, $y \in \Gamma, \kappa \in \mathcal{K}$.
Therefore, for all $\xi \in \mathcal{H}$, $d(\xi, x\xi) \leq \delta$.

Choose $\eta \in \mathcal{H}$ such that $x\eta \neq \eta$.
Choose a geodesic L in $\mathcal{H}$ with $\eta \in L$, $x\eta \notin L$.



Now $L \underset{\delta}{\subset} xL \Rightarrow L = xL \Rightarrow x\eta \in L$,
a contradiction. $\square$

Appendix: Symmetric Spaces

Let G be a semisimple Lie group with no centre and no compact factors.

The *symmetric space* is $X = G/K$ where K is a maximal compact subgroup.

The *rank* r of X is the dimension of a maximal *flat* in X. That is, the maximal dimension of an isometrically embedded euclidean space in X.

A geodesic L in X is *regular* if it lies in only one maximal flat; it is called *singular* if it is not regular.

Let F be a maximal flat in X and let $\xi \in F$. Let S_ξ denote the union of all the singular geodesics in F through ξ. A connected component of $F - S_\xi$ is called a *Weyl chamber* with origin ξ.

Example $[r = 2]$

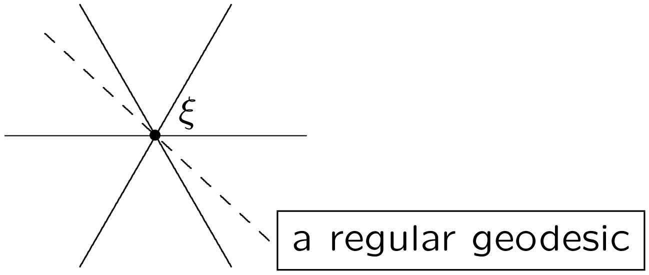
$$\begin{aligned} G &= \mathrm{SL}_3(\mathbb{R}) \\ X &= \{x \in \mathrm{SL}_3(\mathbb{R}) : x \text{ is positive definite } \} \end{aligned}$$

metric : $\dot{s}^2 = Tr(x^{-1}\dot{x})^2$

G acts transitively on X by $x \mapsto gxg^t$ and the stabilizer of I is $\mathrm{SO}_3(\mathbb{R})$. Therefore

$$X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_3(\mathbb{R})$$

A maximal flat F is 2-dimensional. There are six Weyl chambers in F with a given origin $\xi \in F$.



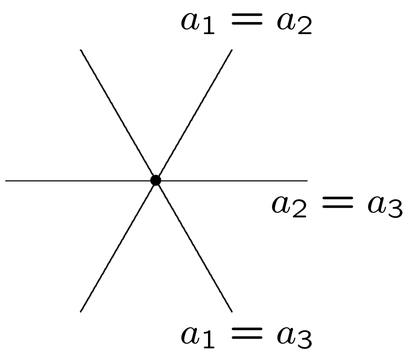
A flat through I has the form $\exp \mathfrak{a}$, where $\mathfrak{a}$ is a linear subspace of

$$S_n(\mathbb{R}) = \{x \in \mathrm{M}_n(\mathbb{R}) : x = x^t, \mathrm{trace}(x) = 0\}$$

such that $xy = yx$ for all $x, y \in \mathfrak{a}$.

The geodesic $t \mapsto \exp tx$ through I in X is regular if and only if the eigenvalues of $x \in S_n(\mathbb{R})$ are all distinct.

For example, let
$\mathfrak{a}_0 = \{\text{diag}(a_1, a_2, a_3) : a_1 + a_2 + a_3 = 0\}$.



If a_1, a_2, a_3 are all distinct then a matrix which commutes with

$$a = \text{diag}(a_1, a_2, a_3)$$

is diagonal and so lies in $\mathfrak{a}_0$.

Thus the geodesic $t \mapsto \exp ta$ lies in a unique maximal flat $\exp \mathfrak{a}_0$

Locally Symmetric Spaces

Let Γ be a torsion free cocompact lattice in G.

$M = \Gamma\backslash X$ is a compact *locally symmetric space* of nonpositive curvature.

$X = G/K$ is the universal covering space of M and the fundamental group of M is $\pi(M) = \Gamma$.