Rapid Decay for Free Groups and Higher Rank Analogues

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The Free Group

Γ : free group of finite rank.

For $f \in \ell^2(\Gamma), g \in \ell^1(\Gamma)$, let

$$\rho(g)f = f * g.$$

Then

$$\|\rho(g)\| \leq \|g\|_1.$$

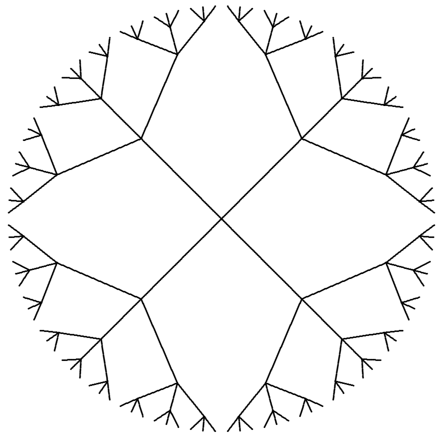
U. Haagerup (1979) proved the inequality (H):

if $\operatorname{supp} g \subseteq \{x \in \Gamma : |x| = n\}$ then

$$\|\rho(g)\| \leq (n+1)\|g\|_2.$$

Geometry

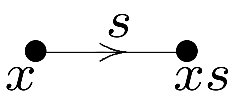
Γ acts on a tree :



The figure illustrates $\Gamma = \langle a, b \rangle$, a free group of rank 2.

Vertices : elements $x \in \Gamma$

Edges :

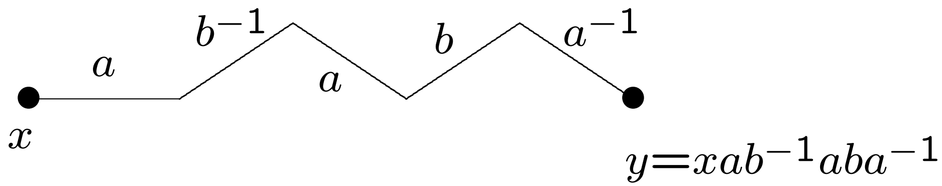


$s \in \left\{a, a^{-1}, b, b^{-1}\right\}$.

Γ acts on the left (preserving the tree structure).

If $x, y \in \Gamma$ then:

- $x^{-1}y$ labels the edges between x and y;
- $d(x, y) = |x^{-1}y|$.

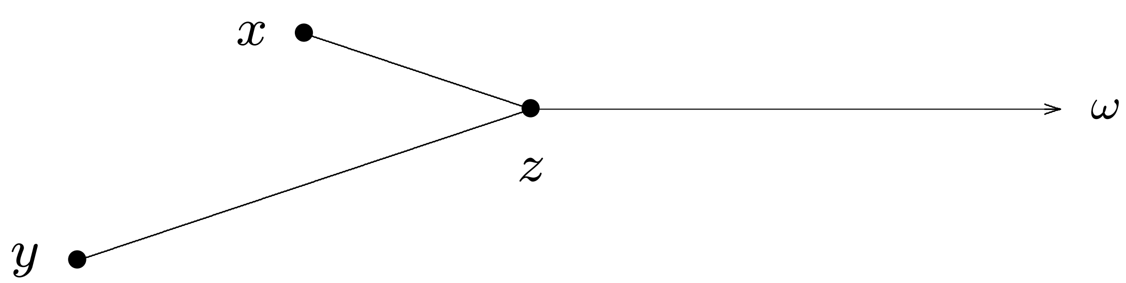


Proof of inequality (H) − geometric version.

Fix n and note that

$$\langle \rho(g)\delta_x, \delta_y \rangle = g(x^{-1}y).$$

(1) Fix a **boundary point** ω:



Define $T^{(m)} : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by

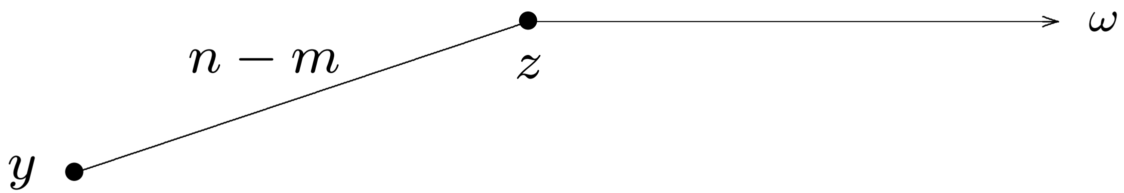
$$\langle T^{(m)}\delta_x, \delta_y\rangle = \begin{cases} g(x^{-1}y) & d(x,z) = m,\ d(y,z) = n-m \\ 0 & \text{otherwise}. \end{cases}$$

Then $\rho(g) = T^{(0)} + T^{(1)} + \cdots + T^{(n)}$.

Must show $\|T^{(m)}\| \le \|g\|_2$.

(2) If $z \in \Gamma$, let $H_z \subset \ell^2(\Gamma)$ be the span of δ_x such that

and $K_z \subset \ell^2(\Gamma))$ the span of δ_y such that



Then $\ell^2(\Gamma) = \bigoplus_z H_z = \bigoplus_z K_z$ and

$$T^{(m)} = \bigoplus_z T_z^{(m)}$$

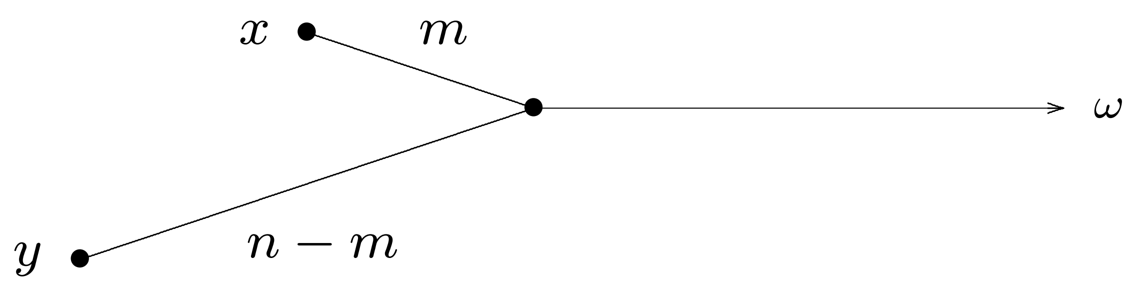
where $T_z^{(m)} : H_z \to K_z$.

Must show $\|T_z^{(m)}\| \leq \|g\|_2$.

(3) Fix m and z. Then $T_z^{(m)}$ has matrix coefficients

$$\alpha_{y,x} = \langle T_z^{(m)} \delta_x, \delta_y \rangle = g(x^{-1}y) \qquad \delta_x \in H_z, \delta_y \in K_z.$$

In the diagram



$x^{-1}y$ determines x and y. Therefore

$$\begin{aligned}\|T_z^{(m)}\| &\leq \|T_z^{(m)}\|_{\mathsf{HS}} \\ &= (\sum |\alpha_{y,x}|^2)^{\frac{1}{2}} \\ &\leq \|g\|_2.\end{aligned}$$

Corollary. (**Rapid Decay**) If $g : \Gamma \to \mathbb{C}$ has finite support then

$$\|\rho(g)\| \leq 2\left(\sum_{x\in\Gamma}|g(x)|^2(1+|x|)^4\right)^{\frac{1}{2}}.$$

U.Haagerup used Rapid Decay to prove that $C_r^*(\Gamma)$ has the **Metric Approximation Property**, despite being non-nuclear.

How to generalise this?

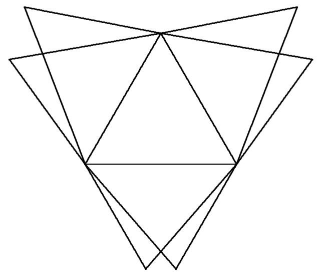
A free group is a **lattice** in $\mathrm{SL}_2(\mathbb{Q}_p)$ and acts on a **tree**.

A lattice $\Gamma < \mathrm{SL}_3(\mathbb{Q}_p)$ acts on a **building**.

The building of $\mathrm{SL}_3(\mathbb{Q}_p)$

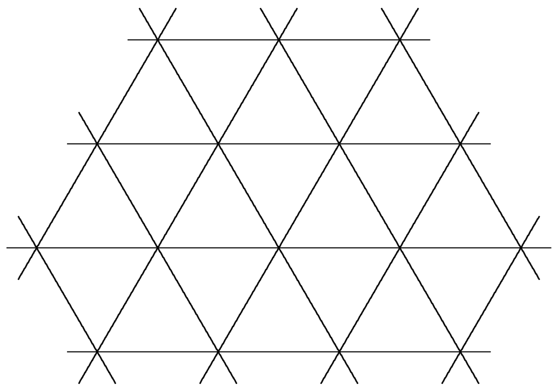
A simply connected simplicial complex Δ, $\dim \Delta = 2$.

Each edge lies on $p+1$ triangles.



$p = 2$

Δ is a union of apartments: flat subcomplexes isomorphic to a tessellation of $\mathbb{R}^2$ by equilateral triangles.



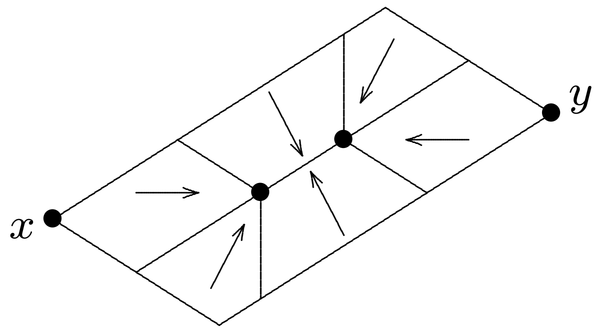
Theorem. A lattice $\Gamma < \mathrm{SL}_3(\mathbb{Q}_p)$ has Rapid Decay.

(J Ramagge, G Robertson, T Steger – 1998)

The **proof** involves reducing to the case where Γ is torsion free and proving a version of inequality (H), by decomposing

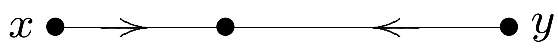
$$\rho(g) = \sum_D T_D$$

according to possible diagrams



where arrows point towards a fixed boundary point of the building.

Tree diagram:



V. Lafforgue used this geometric method to prove Rapid Decay for a cocompact lattice $\Gamma < \mathrm{SL}_3(\mathbb{R}), \mathrm{SL}_3(\mathbb{C})$. Consequently :

Theorem. (Lafforgue) The Baum-Connes conjecture is true for any cocompact lattice $\Gamma < \mathrm{SL}_3(\mathbb{F})$, where $\mathbb{F}$ is a local field.

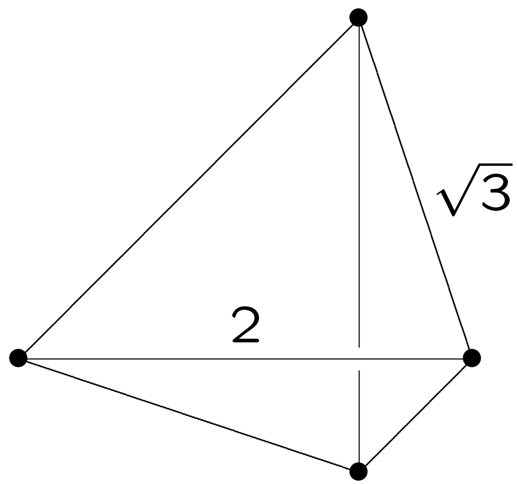
Remark. Haagerup's proof of MAP fails, because $x \mapsto |x|$ is not negative definite.

What about SL_4?

Underlying **obstruction** to generalising the method:

An apartment in the building of $\mathrm{SL}_4(\mathbb{Q}_p)$ is an $\tilde{A}_3$ Coxeter complex:

Fundamental simplex:



Dihedral angles: $\frac{\pi}{3}$, $\boxed{\frac{\pi}{2}}$.