FACTORS FROM TREES

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ABSTRACT. We construct factors of type $\mathrm{III}_{1/n}$ for $n \in \mathbb{N}, n \geq 2$ from group actions on homogeneous trees and their boundaries. Our result is a discrete analogue of a result of R.J Spatzier, namely [S, Proposition 1], where the hyperfinite factor of type III_1 is constructed from a group action on the boundary of the universal cover of a manifold.

1. Introduction

Let Γ be a group acting simply transitively on the vertices of a homogeneous tree $\mathcal{T}$ of degree $n+1 < \infty$. Then, by [FTN, Ch. I, Theorem 6.3],

$$\Gamma \cong \mathbb{Z}_2 * \cdots * \mathbb{Z}_2 * \mathbb{Z} * \cdots * \mathbb{Z}$$

where there are s factors of $\mathbb{Z}_2$, t factors of $\mathbb{Z}$, and $s + 2t = n + 1$. Thus Γ has a presentation

$$\Gamma = \left\langle a_1, \ldots, a_{s+t} : a_i^2 = 1 \text{ for } i \in \{1, \ldots, s\} \right\rangle,$$

we can identify the Cayley graph of Γ constructed via right multiplication with $\mathcal{T}$ and the action of Γ on $\mathcal{T}$ is equivalent to the natural action of Γ on its Cayley graph via left multiplication.

We can associate a natural boundary to $\mathcal{T}$, namely the set Ω of semi-infinite reduced words in the generators of Γ. The action of Γ on $\mathcal{T}$ induces an action of Γ on Ω.

For each $x \in \Gamma$, let

$$\Omega^x = \{\omega \in \Omega : \omega = x \cdots\}$$

be the set of semi-infinite reduced words beginning with x. The set $\{\Omega^x\}_{x \in \Gamma}$ is a set of basic open sets for a compact Hasudorff topology on Ω. Denote by $|x|$ the length of a reduced expression for x. Let $V^m = \{x \in \Gamma : |x| = m\}$ and define $N_m = |V^m|$. Then Ω is the disjoint union of the N_m sets Ω^x for $x \in V^m$.

We can also endow Ω with the structure of a measure space. Ω has a unique distinguished Borel probability measure ν such that

$$\nu(\Omega^x) = \frac{1}{n+1} \left(\frac{1}{n}\right)^{|x|-1}$$

for every nontrivial $x \in \Gamma$. The sets Ω^x, $x \in \Gamma$ generate the Borel σ-algebra.

This measure ν on Ω is quasi-invariant under the action of Γ, so that Γ acts on the measure space (Ω, ν) and enabling us to extend the action of Γ to an action on $L^\infty(\Omega, \nu)$ via

$$g \cdot f(\omega) = f(g^{-1} \cdot \omega)$$

for all $g \in \Gamma$, $f \in L^\infty(\Omega, \nu)$, and $\omega \in \Omega$. We may therefore consider the von Neumann algebra $L^\infty(\Omega, \nu) \rtimes \Gamma$ which we shall write as $L^\infty(\Omega) \rtimes \Gamma$ for brevity.

2. The Factors

We note that the action of Γ on Ω is free since if $g\omega = \omega$ for some $g \in \Gamma$ and $\omega \in \Omega$ then we must have either $\omega = ggg\cdots$ or $\omega = g^{-1}g^{-1}g^{-1}\cdots$ and

$$\nu\left(ggg\cdots, g^{-1}g^{-1}g^{-1}\cdots\right) = 0.$$

The action of Γ on Ω is also ergodic by the proof of [PS, Proposition 3.9], so that $L^\infty(\Omega) \rtimes \Gamma$ is a factor. Establishing the type of the factor is not quite as straightforward. We begin by recalling some classical definitions.

Definition 2.1. Given a group Γ acting on a measure space Ω, we define the **full group**, $[\Gamma]$, of Γ by

$$[\Gamma] = \{T \in \mathrm{Aut}(\Omega) : T\omega \in \Gamma\omega \text{ for almost every } \omega \in \Omega\}.$$

The set $[\Gamma]_0$ of measure preserving maps in $[\Gamma]$ is then given by

$$[\Gamma]_0 = \{T \in [\Gamma] : T\circ\nu = \nu\}$$

Definition 2.2. Let G be a countable group of automorphisms of the measure space (Ω, ν). Following W. Krieger, define the **ratio set** $r(G)$ to be the subset of $[0, \infty)$ such that if $\lambda \geq 0$ then $\lambda \in r(G)$ if and only if for every $\epsilon > 0$ and Borel set $\mathcal{E}$ with $\nu(\mathcal{E}) > 0$, there exists a $g \in G$ and a Borel set $\mathcal{F}$ such that $\nu(\mathcal{F}) > 0$, $\mathcal{F} \cup g\mathcal{F} \subseteq \mathcal{E}$ and

$$\left| \frac{d\nu\circ g}{d\nu}(\omega) - \lambda \right| < \epsilon$$

for all $\omega \in \mathcal{F}$.

Remark 2.3. The ratio set $r(G)$ depends only on the quasi-equivalence class of the measure ν, see [HO, §I-3, Lemma 14]. It also depends only on the full group in the sense that

$$[H] = [G] \Rightarrow r(H) = r(G).$$

The following result will be applied in the special case where $G = \Gamma$. However, since the simple transitivity of the action doesn't play a role in the proof, we can state it in greater generality.

Proposition 2.4. *Let G be a countable subgroup of $\mathrm{Aut}(\mathcal{T}) \leq \mathrm{Aut}(\Omega)$. Suppose there exist an element $g \in G$ such that $d(ge, e) = 1$ and a subgroup K of $[G]_0$ whose action on Ω is ergodic. Then*

$$r(G) = \left\{n^k : k \in \mathbb{Z}\right\} \cup \left\{0\right\}.$$

Proof. By Remark 2.3, it is sufficient to prove the statement for some group H such that $[H] = [G]$. In particular, since $[G] = [\langle G, K\rangle]$ for any subgroup K of $[G]_0$, we may assume without loss of generality that $K \leq G$.

By [FTN, Chapter 2, part 1)], for each $g \in G$ and $\omega \in \Omega$ we have

$$\frac{d\nu\circ g}{d\nu}(\omega) \in \left\{n^k : k \in \mathbb{Z}\right\} \cup \left\{0\right\}.$$

Since G acts ergodically on Ω, $r(G) \setminus \{0\}$ is a group. It is therefore enough to show that $n \in r(G)$. Write $x = ge$ and note that $\nu_x = \nu\circ g^{-1}$. By [FTN, Chapter 2, part1)] we have

$$\frac{d\nu_x}{d\nu}(\omega) = n, \text{ for all } \omega \in \Omega_e^x. \tag{1}$$

Let $\mathcal{E} \subseteq \Omega$ be a Borel set with $\nu(\mathcal{E}) > 0$. By the ergodicity of K, there exist $k_1, k_2 \in K$ such that the set

$$\mathcal{F} = \{\omega \in \mathcal{E} : k_1\omega \in \Omega_e^x \text{ and } k_2 g^{-1} k_1\omega \in \mathcal{E}\}$$

has positive measure.

Finally, let $t = k_2 g^{-1} k_1 \in G$. By construction, $\mathcal{F} \cup t\mathcal{F} \subseteq \mathcal{E}$. Moreover, since K is measure-preserving,

$$\frac{d\nu\circ t}{d\nu}(\omega) = \frac{d\nu\circ g^{-1}}{d\nu}(k_1\omega) = \frac{d\nu_x}{d\nu}(k_1\omega) = n \text{ for all } \omega \in \mathcal{F}$$

by (1), since $k_1 \in \Omega_e^x$. This proves $n \in r(G)$, as required. □

Corollary 2.5. *If, in addition to the hypotheses for Proposition 2.4, the action of G is free, then $L^\infty(\Omega) \rtimes G$ is a factor of type $\mathrm{III}_{1/n}$.*

Proof. Having determined the ratio set, this is immediate from [C1, Corollaire 3.3.4]. □

Thus, if we can find a countable subgroup $K \leq [\Gamma]_0$ whose action on Ω is ergodic we will have shown that $L^\infty(\Omega) \rtimes \Gamma$ is a factor of type $\mathrm{III}_{1/n}$. To this end, we prove the following sufficiency condition for ergodicity.

Lemma 2.6. *Let K be group which acts on Ω. If K acts transitively on the collection of sets $\{\Omega^x : x \in \Gamma, |x| = m\}$ for each natural number m, then K acts ergodically on Ω.*

Proof. Suppose that $X_0 \subseteq \Omega$ is a Borel set which is invariant under K and such that $\nu(X_0) > 0$. We show that this necessarily implies $\nu(\Omega \setminus X_0) = 0$, thus establishing the ergodicity of the action.

Define a new measure μ on Ω by $\mu(X) = \nu(X \cap X_0)$ for each Borel set $X \subseteq \Omega$. Now, for each $g \in K$,

$$\begin{aligned}
\mu(gX) &= \nu(gX \cap X_0) = \nu(X \cap g^{-1}X_0) \\
&\leq \nu(X \cap X_0) + \nu(X \cap (g^{-1}X_0 \setminus X_0)) \\
&= \nu(X \cap X_0) \\
&= \mu(X),
\end{aligned}$$

and therefore μ is K-invariant. Since K acts transitively on the basic open sets Ω^x associated to words x of length m this implies that

$$\mu(\Omega^x) = \mu(\Omega^y)$$

whenever $|x| = |y|$. Since Ω is the union of N_m disjoint sets Ω^x, $x \in V^m$, each of which has equal measure with respect to μ, we deduce that

$$\mu(\Omega^x) = \frac{c}{N_m}$$

for each $x \in V^m$, where $c = \mu(X_0) = \nu(X_0) > 0$. Thus $\mu(\Omega^x) = c\nu(\Omega^x)$ for every $x \in \Gamma$.

Since the sets Ω^x, $x \in \Gamma$ generate the Borel σ-algebra, we deduce that $\mu(X) = c\nu(X)$ for each Borel set X. Therefore

$$\begin{aligned}
\nu(\Omega \setminus X_0) &= c^{-1}\mu(\Omega \setminus X_0) \\
&= c^{-1}\nu((\Omega \setminus X_0) \cap X_0) = 0,
\end{aligned}$$

thus proving ergodicity. □

In the last of our technical results, we give a constructive proof of the existence of a countable ergodic subgroup of $[\Gamma]_0$.

Lemma 2.7. *There is a countable ergodic group $K \leq \mathrm{Aut}(\Omega)$ such that $K \leq [\Gamma]_0$.*

Proof. Let $x, y \in V^m$. We construct a measure preserving automorphism $k_{x,y}$ of Ω such that

(1) $k_{x,y}$ is almost everywhere a bijection from Ω^x onto Ω^y,
(2) $k_{x,y}$ is the identity on $\Omega \setminus (\Omega^x \cup \Omega^y)$.

It then follows from Lemma 2.6 that the group

$$K = \langle k_{x,y} : \{x, y\} \subseteq V^m, m \in \mathbb{N}\rangle$$

acts ergodically on Ω and the construction will show explicitly that $K \leq [\Gamma]_0$.

Fix $x, y \in V^m$ and suppose that we have reduced expressions $x = x_1 \dots x_m$, and $y = y_1 \dots y_m$.

Define $k_{x,y}$ to be left multiplication by yx^{-1} on each of the sets Ω^{xz} where $|z| = 1$ and $z \notin \{x_m^{-1}, y_m^{-1}\}$. Then $k_{x,y}$ is a measure preserving bijection from each such set onto Ω^{yz}. If $y_m = x_m$ then $k_{x,y}$ is now well defined everywhere on Ω^x.

Suppose now that $y_m \neq x_m$. Then $k_{x,y}$ is defined on the set $\Omega^x \setminus \Omega^{xy_m^{-1}}$, which it maps bijectively onto $\Omega^y \setminus \Omega^{yx_m^{-1}}$. Now define $k_{x,y}$ to be left multiplication by $yx_m^{-1}y_mx^{-1}$ on each of the sets $\Omega^{xy_m^{-1}z}$ where $|z| = 1$ and $z \notin \{x_m, y_m\}$. Then $k_{x,y}$ is a measure preserving bijection of each such $\Omega^{xy_m^{-1}z}$ onto $\Omega^{yx_m^{-1}z}$.

Thus we have extended the domain of $k_{x,y}$ so that it is now defined on the set $\Omega^x \setminus \Omega^{xy_m^{-1}x_m}$, which it maps bijectively onto $\Omega^y \setminus \Omega^{yx_m^{-1}y_m}$.

Next define $k_{x,y}$ to be left muliplication by $yx_m^{-1}y_mx_m^{-1}y_mx^{-1}$ on the sets $\Omega^{xy_m^{-1}x_mz}$ where $|z| = 1$ and $z \notin \{x_m^{-1}, y_m^{-1}\}$.

Continue in this way. At the jth step $k_{x,y}$ is a measure preserving bijection from $\Omega^x \setminus X_j$ onto $\Omega^y \setminus Y_j$ where $\nu(X_j) \to 0$ as $j \to \infty$ so that eventually $k_{x,y}$ is defined almost everywhere on Ω. Finally, define

$$k_{x,y}(xy_m^{-1}x_my_m^{-1}x_my_m^{-1}x_m \dots) = yx_m^{-1}y_mx_m^{-1}y_mx_m^{-1}y_m \dots$$

thus defining $k_{x,y}$ everywhere on Ω in such a way that its action is pointwise approximable by Γ almost everywhere. Hence

$$K = \langle k_{x,y} : \{x, y\} \subseteq V^m, m \in \mathbb{N}\rangle$$

is a countable group with an ergodic measure-preserving action on Ω and $K \leq [\Gamma]_0$. $\square$

We are now in a position to prove our main result.

Theorem 2.8. *The von Neumann algebra $L^\infty(\Omega) \rtimes \Gamma$ is the hyperfinite factor of type* $\mathrm{III}_{1/n}$.

Proof. By applying Corollary 2.5 with $G = \Gamma$, $g \in \Gamma$ any generator of Γ, and K as in Lemma 2.7 we conclude that $L^\infty(\Omega) \rtimes \Gamma$ is a factor of type $\mathrm{III}_{1/n}$.

To see that the factor is hyperfinite simply note that the action of Γ is amenable as a result of [A, Theorem 5.1]. We refer to [C2, Theorem 4.4.1] for the uniqueness of the hyperfinite factor of type $\mathrm{III}_{1/n}$. □

Remark 2.9. Taking different measures on Ω should yield hyperfinite factors of type III_λ for any $0 < \lambda < 1$. We have concentrated on the geometrically interesting case.

Remark 2.10. In [Sp1], Spielberg constructs III_λ factor states on the algebra $\mathcal{O}_2$. The reduced C^*-algebra $C(\Omega) \rtimes_r \Gamma$ is a Cuntz-Krieger algebra $\mathcal{O}_A$ by [Sp2]. What we have done is construct a type $\mathrm{III}_{1/n}$ factor state on some of these algebras $\mathcal{O}_A$.

Remark 2.11. From [C2, p. 476], we know that if $\Gamma = \mathbb{Q} \rtimes \mathbb{Q}^*$ acts naturally on $\mathbb{Q}_p$, then the crossed product $L^\infty(\mathbb{Q}_p) \rtimes \Gamma$ is the hyperfinite factor of type $\mathrm{III}_{1/p}$. This may be proved geometrically as above by regarding the the boundary of the homogeneous tree of degree $p+1$ as the one point compactification of $\mathbb{Q}_p$ as in [CKW].

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