HARMONIC COCHAINS AND K-THEORY FOR $\tilde{A}_2$ GROUPS

GUYAN ROBERTSON

Abstract. If $\Gamma$ is a torsion free $\tilde{A}_2$ group acting on an $\tilde{A}_2$ building $\Delta$ and $\mathfrak{A}_\Gamma$ is the associated boundary $C^*$-algebra, it is proved that $K_0(\mathfrak{A}_\Gamma) \otimes \mathbb{R} \cong \mathbb{R}^{2\beta_2}$, where $\beta_2 = \dim_{\mathbb{R}} H^2(\Gamma, \mathbb{R})$.

1. Introduction

Let $\Gamma$ be an $\tilde{A}_2$ group acting on an $\tilde{A}_2$ building $\Delta$ of order $q$. The Furstenberg boundary $\Omega$ of $\Delta$ is the set of chambers of the spherical building at infinity, endowed with a natural compact totally disconnected topology. The topological action of $\Gamma$ on $\Omega$ is encoded in the full crossed product $C^*$-algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$, which is studied in [10, 11, 12]. This full crossed product is isomorphic to the reduced crossed product, since the action of $\Gamma$ on $\Omega$ is amenable [10, Section 4.2]. As the notation suggests, $\mathfrak{A}_\Gamma$ depends only on $\Gamma$ [12]. Motivated by rigidity theorems of Mostow, Margulis and others, whose proofs rely on the study of boundary actions, it is of interest to determine the extent to which the boundary $C^*$-algebra $\mathfrak{A}_\Gamma$ determines the group $\Gamma$.

In [12], T. Steger and the author computed the K-theory of $\mathfrak{A}_\Gamma$ for many $\tilde{A}_2$ groups with $q \leq 13$. The computations were done for all the $\tilde{A}_2$ groups in the cases $q = 2, 3$ and for several representative groups for each of the other values of $q \leq 13$. If $q = 2$ there are precisely eight $\tilde{A}_2$ groups $\Gamma$, all of which embed as lattices in $\text{PGL}(3, K)$, where $K = F_2((X))$ or $K = \mathbb{Q}_2$. If $q = 3$ there are 89 possible $\tilde{A}_2$ groups, of which 65 are “exotic” in the sense that they do not embed naturally in linear groups. Exotic $\tilde{A}_n$ groups only exist if $n = 2$, since all locally finite Euclidean buildings of dimension $\geq 3$ are associated to linear algebraic groups. This justifies, to some extent, the focus on $\tilde{A}_2$ groups.

For each $\tilde{A}_2$ group $\Gamma$, the $C^*$-algebra $\mathfrak{A}_\Gamma$ has the structure of a rank 2 Cuntz-Krieger algebra [11, Theorem 7.7]. These algebras are classified up to isomorphism by their $K$-groups [11, Remark 6.5] and it was proved in [12, Theorem 2.1] that

$$K_0(\mathfrak{A}_\Gamma) = K_1(\mathfrak{A}_\Gamma) = \mathbb{Z}^{2r} \oplus T,$$

where $r \geq 0$ and $T$ is a finite group. The computations in [12] led to some striking observations. For example, the three torsion-free $\tilde{A}_2$ subgroups of $\text{PGL}_3(\mathbb{Q}_2)$ are distinguished from each other by $K_0(\mathfrak{A}_\Gamma)$. There was also evidence for the conjecture that, for any torsion free $\tilde{A}_2$ group $\Gamma$, the integer $r$ in the equation (1) is equal to the second Betti number of $\Gamma$. The purpose of this article is to prove that this is indeed the case.

2000 Mathematics Subject Classification. Primary 46L80; secondary 58B34, 51E24, 20G25.
Key words and phrases. Euclidean building, boundary, operator algebra.
Theorem 1.1. If $\Gamma$ is a torsion free $\tilde{A}_2$ group acting on an $\tilde{A}_2$ building $\Delta$ of order $q$, then

$$K_0(\mathfrak{A}_\Gamma) \otimes \mathbb{R} \cong \mathbb{R}^{2\beta_2},$$

where $\beta_2 = \dim_{\mathbb{R}} H^2(\Gamma, \mathbb{R}) = \frac{1}{3}(q - 2)(q^2 + q + 1)$.

The article [12] identified the integer $r$ in (1) with the rank of a certain finitely generated abelian group $C(\Gamma)$. Two new ideas lead to the proof of Theorem 1.1. The local structure of the building $\Delta$, together with the fact that $\Gamma$ has Kazhdan’s property (T), is used to show that $C(\Gamma) \otimes \mathbb{R}$ is isomorphic to the space of $\Gamma$-invariant $\mathbb{R}$-valued harmonic 2-cochains on $\Delta$, in the sense of [1, 4]. Then, according to an isomorphism of Garland [5], this space is isomorphic to $H^2(\Gamma, \mathbb{R})$.

Remark 1.2. An $\tilde{A}_2$ group is a natural analogue of a free group, which acts freely and transitively on the vertex set of a tree (which is a building of type $\tilde{A}_1$). If the tree is homogeneous of degree $q + 1$, with $q \geq 2$, then $\Gamma$ is a free group on $\frac{1}{2}(q + 1)$ generators and one can again form the full crossed product $C^*$-algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$, where $\Omega$ is the space of ends of the tree. The analogue of Theorem 1.1 states that $K_0(\mathfrak{A}_\Gamma) \otimes \mathbb{R} \cong \mathbb{R}^{\beta_1}$, where $\beta_1 = \dim_{\mathbb{R}} H^1(\Gamma, \mathbb{R}) = \frac{1}{2}(q + 1)$ [8, Theorem 1].

Remark 1.3. Another simple $C^*$-algebra associated with the $\tilde{A}_2$ group $\Gamma$ is the reduced group $C^*$-algebra $C^*_r(\Gamma)$. It is shown in [9, Theorem 6.1] that $K_0(C^*_r(\Gamma)) = \mathbb{Z}^{\chi(\Gamma)}$. This is a consequence of the fact that $\tilde{A}_2$ groups belong to the class of groups for which the Baum-Connes conjecture is known to be true.

Remark 1.4. This paper is a sequel to the articles [11], [12]. The key results used are [11, Theorem 7.7], which shows that $\mathfrak{A}_\Gamma$ is isomorphic to a rank 2 Cuntz-Krieger algebra, and [12, Theorem 2.1] which shows that the $K$-theory of this algebra is given by equation (1).

What happens in the case of a torsion free $\tilde{A}_n$-group $\Gamma$ ($n \geq 3$)? There seems to be no fundamental obstruction to generalising [11, Theorem 7.7], to identify the boundary crossed product algebra with a higher rank Cuntz-Krieger algebra, in the sense of [11]. The arguments of the present paper should also generalise, but additional conditions which are vacuous in the rank 2 case would need to be verified [1, Theorem 2.3 (C),(D)]. However it would be more difficult to generalise [12, Theorem 2.1]. This is because the proof of that result uses a Kasparov spectral sequence [12, Proposition 4.1] whose limit is clear only in the rank 2 case.

2. $\tilde{A}_2$ Groups

Consider a locally finite building $\Delta$ of type $\tilde{A}_2$. Each vertex $v$ of $\Delta$ has a type $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber (maximal simplex) of $\Delta$ has exactly one vertex of each type. Each edge $e$ is directed, with initial vertex of type $i$ and final vertex of type $i + 1$. An automorphism $\alpha$ of $\Delta$ is type rotating if there exists $i \in \mathbb{Z}/3\mathbb{Z}$ such that $\tau(\alpha(v)) = \tau(v) + i$ for all vertices $v$ of $\Delta$.

Suppose that $\Gamma$ is a group of type rotating automorphisms of $\Delta$, which acts freely and transitively on the vertex set of $\Delta$. Such a group is called an $\tilde{A}_2$ group. The theory of $\tilde{A}_2$ groups has been developed in [3] and some, but not all, $\tilde{A}_2$ groups embed as lattice subgroups of $\text{PGL}_3(\mathbb{R})$. Any $\tilde{A}_2$ group can be constructed as follows [3, I, Section 3]. Let $(P, L)$ be a finite projective plane of order $q$. There are
Let $\lambda : P \to L$ be a bijection and write $\lambda(\xi) = \xi^\lambda$. A triangle presentation compatible with $\lambda$ is a set $T$ of ordered triples $(\xi_i, \xi_j, \xi_k)$ where $\xi_i, \xi_j, \xi_k \in P$, with the following properties.

(i) Given $\xi_i, \xi_j \in P$, then $(\xi_i, \xi_j, \xi_k) \in T$ for some $\xi_k \in P$ if and only if $\xi_j$ and $\xi_i$ are incident, i.e. $\xi_j \in \xi_i$.

(ii) $(\xi_i, \xi_j, \xi_k) \in T \Rightarrow (\xi_j, \xi_k, \xi_i) \in T$.

(iii) Given $\xi_i, \xi_j \in P$, then $(\xi_i, \xi_j, \xi_k) \in T$ for at most one $\xi_k \in P$.

In [3] there is exhibited a complete list of triangle presentations for $q = 2$ and $q = 3$. Given a triangle presentation $T$, one can form the group

$$
\Gamma = \Gamma_T = \langle P \mid \xi_i \xi_j \xi_k = 1 \text{ for } (\xi_i, \xi_j, \xi_k) \in T \rangle.
$$

The Cayley graph of $\Gamma$ with respect to the generating set $(\xi_i, \xi_j, \xi_k)$ with $\gamma \in \Gamma$ is labeled by the generator $\xi \in P$.

The link of a vertex $\gamma$ of $\Delta$ is the incidence graph of the projective plane $(P, L)$, where the lines in $L$ correspond to the inverses in $\Gamma$ of the generators in $P$. In other words, $\xi = \xi^{-1}$ for $\xi \in P$.

![Figure 1. A chamber based at a vertex $\gamma$](image)

For the rest of this article, $\Gamma$ is assumed to be torsion free. Therefore $\Gamma$ acts freely on $\Delta$ and $X = \Gamma \setminus \Delta$ is a 2-dimensional cell complex with universal covering $\Delta$. Let $X^k$ denote the set of oriented $k$-cells of $X$ for $k = 0, 1, 2$. Thus $X^1$ may be identified with $P$ and $X^2$ may be identified with the set of orbits of elements of $T$ under cyclic permutations.

Let $\hat{\Delta}^2$ be the directed version of $\Delta^2$ in which each 2-simplex has a specified base vertex, so that $\mathbb{Z}/3\mathbb{Z}$ acts naturally on $\hat{\Delta}^2$. Let $\hat{X} := \hat{\Delta}^2/\Gamma$, the set of directed 2-cells of $X$. Then $\hat{X}^2$ may be identified with $T$. From now on $a = \langle a_0, a_1, a_2 \rangle$ will denote an element of $T$, regarded as a directed 2-cell. Figure 2 illustrates the three directed 2-cells associated with an oriented 2-cell of $X$. In the diagram, the 2-cells are thought of as being directed upwards and the symbol $\bullet$ is placed opposite the “top” edge to indicate that direction.

3. K-theory

Transition matrices $M = (m_{ab})_{a, b \in \hat{X}^2}$ and $N = (n_{ab})_{a, b \in \hat{X}^2}$ are defined as follows. If $a, b \in \hat{X}^2$ then $m_{ab} = 1$ if and only if there are labeled triangles representing $a, b$ in the building $\Delta$ which lie as shown on the right of Figure 3. If no such diagram is possible then $m_{ab} = 0$.

In terms of the projective plane $(P, L)$, the matrix $M$ is defined by

$$
m_{ab} = 1 \iff b_2 \notin \pi_2, \quad b_1 = a_0 \lor b_2.
$$
It follows that each row or column of $M$ has precisely $q^2$ nonzero entries.

Similarly, the matrix $N$ is defined by

$$n_{ac} = 1 \iff a_1 \notin c_1, \ c_2 = \overline{a_0} \wedge \overline{c_1}.$$ as illustrated on the left of Figure 3.

Let $r$ be the rank, and $T$ the torsion part, of the abelian group $C(\Gamma)$ with generating set $\hat{X}^2$ and relations

$$(4) \quad a = \sum_{b \in \hat{X}^2} m_{ab}b = \sum_{b \in \hat{X}^2} n_{ab}b, \ a \in \hat{X}^2.$$ Thus $C(\Gamma) \cong \mathbb{Z}^r \oplus T$. The following result was proved in [12, Theorem 2.1].

**Theorem 3.1.** Let $\Gamma$ be an $\tilde{A}_2$ group, and let $r$ be the rank, and $T$ the torsion part of $C(\Gamma)$. Then

$$(5) \quad K_0(\mathfrak{A}_{\Gamma}) = K_1(\mathfrak{A}_{\Gamma}) = \mathbb{Z}^{2r} \oplus T.$$ Given $\xi \in P$, let

$$(6) \quad \langle \xi \rangle = \sum_{\begin{subarray}{c} a \in \hat{X}^2 \\ a_2 = \xi \end{subarray}} a \in C(\Gamma).$$ It is sometimes convenient to write such sums pictorially as

$$(7) \quad \langle \xi \rangle = \sum_{\begin{subarray}{c} a \in \hat{X}^2 \\ a \notin \xi \end{subarray}} a.$$
Note that \( a \in \hat{X}^2 \) with \( a_2 = \xi \) if and only if \( a = \langle a_0, a_1, a_2 \rangle \) where \( a_2 = \xi \) and \( a_0 \in \xi \) (and \( a_1 \) is then uniquely determined). There are \( q + 1 \) such choices of \( a_0 \) and so there are \( q + 1 \) terms in the sum (6). Similar remarks apply to the element \( \langle \xi \rangle \in C(\Gamma) \) defined by

\[
\langle \xi \rangle = \sum_{a \in \hat{X}^2} a = \sum_{a_1 = \xi} \left( \sum_{a_0} \right) a
\]

In what follows the element \( \varepsilon = \sum_{a \in \hat{X}^2} a \) plays a special role. An important observation, which is needed subsequently, is that \( \varepsilon \) has finite order in \( C(\Gamma) \). The statement and its proof are very like [12, Proposition 8.2].

**Lemma 3.2.** In the group \( C(\Gamma) \), \((q^2 - 1)\varepsilon = 0\).

*Proof.* Using relations (4) and the fact that each column of the matrix \( M \) has precisely \( q^2 \) nonzero entries,

\[
\varepsilon = \sum_{a \in \hat{X}^2} a = \sum_{a \in \hat{X}^2} \sum_{b \in \hat{X}^2} m_{ab}b = \sum_{b \in \hat{X}^2} \left( \sum_{a \in \hat{X}^2} m_{ab} \right) b = \sum_{b \in \hat{X}^2} q^2 b = q^2 \varepsilon.
\]

\[ \square \]

**Lemma 3.3.** If \( \langle a_0, a_1, a_2 \rangle \in \hat{X}^2 \) then, in the group \( C(\Gamma) \),

\[
\begin{align*}
\langle a_1 \rangle - \langle a_2, a_0, a_1 \rangle &= \langle a_2 \rangle - \langle a_1, a_2, a_0 \rangle; \\
\langle a_0 \rangle + \langle a_1 \rangle + \langle a_2 \rangle &= \varepsilon.
\end{align*}
\]

*Proof.* Fix the base vertex \( 1 \in \Delta \). Any generator \( a \) for \( C(\Gamma) \) has a unique representative directed chamber \( \sigma_a \) based at \( 1 \). The chamber \( \sigma_a \) has vertices \( 1, a_1^{-1}, a_2 \). By [6, Section 15.4], each chamber based at \( 1 \), other than \( \sigma_a \) lies in a common apartment with \( \sigma_a \), in exactly one of the five positions \( \tau_2, \tau_3, \tau_4, \tau_5, \tau_6 \) in Figure 4. As before, directed chambers will be pointed.

![Figure 4](image-url)
The left side of (9a) is equal to the sum of all the elements $b \in \hat{X}^2$ represented by directed chambers $\sigma_b$ in position $\tau_6$, as illustrated in Figure 5(a). Each such element $b$ satisfies $b_2 = a_1$, and the relations (4) imply that

$$b = \sum_{c \in \hat{X}^2} m_{bc}c.$$ 

That is, $b$ is the sum of all the elements $c \in \hat{X}^2$ with representative directed chambers $\sigma_c$ lying in position $\tau_4$, as illustrated in Figure 5(a). Moreover, if $\sigma_c$ is any directed chamber with base vertex 1, lying in position $\tau_4$, then it arises in this way from a unique chamber $\sigma_b$ in position $\tau_6$. To see this, it is enough to take the convex hull of any such chamber $\sigma_c$, which completely determines the whole hexagon in Figure 5(a). Therefore the left side of (9a) is equal to the sum of all the elements $c \in \hat{X}^2$ represented by directed chambers $\sigma_c$ based at 1 which lie in position $\tau_4$ of Figure 4.

Similarly, the right side of (9a) is equal to the sum of all the elements $b \in \hat{X}^2$ represented by directed chambers $\sigma_b$ in position $\tau_2$ as illustrated in Figure 5(b). The relations (4) imply that, for each such chamber $b$,

$$b = \sum_{b \in \hat{X}^2} n_{ab}c.$$ 

It follows that the right side of (9a) is also equal to the sum of all the elements $c \in \hat{X}^2$ represented by directed chambers based at 1 which lie in position $\tau_4$ of Figure 4. This proves that the left and right sides of (9a) are equal.

The next task is to prove (9b). Recall that $\varepsilon$ is the sum of all the elements of $\hat{X}^2$, and representative directed chambers for elements of this sum are $\sigma_a$ together with all chambers lying in any of the five positions $\tau_2, \tau_3, \tau_4, \tau_5, \tau_6$ in Figure 4.

The set of chambers based at the vertex 1 representing the elements of the sum $\langle a_2 \rangle$ consists of $\sigma_a$ together with all directed chambers lying in the position $\tau_2$, as in Figure 6(a). Here it may also be convenient to refer back to equation (7).

Using the relations (4), the sum $\langle a_1 \rangle$ is equal to the sum of elements represented by chambers lying in the position $\tau_3$ or $\tau_4$, as in Figure 6(b).

Finally, the sum $\langle a_0 \rangle$ is equal to the sum of elements represented by directed chambers lying in the position $\tau_5$ or $\tau_6$, as in Figure 6(c). For, cyclically permuting the indices in the equation (9a) gives

$$\langle a_0 \rangle - \langle a_1, a_2, a_0 \rangle = \langle a_1 \rangle - \langle a_0, a_1, a_2 \rangle.$$
Lemma 3.5. The relations (11) imply the relations (4).

\[(11b)\]

Proof. By Lemma 3.2, \(\xi\) for \(\theta\) in \(\langle a_0, a_1, a_2 \rangle\). Therefore, by (9b),

\[\langle \xi \rangle \otimes 1 = 0 = \langle \xi \rangle \otimes 1.\]

This completes the proof that \(\langle a_0 \rangle + \langle a_1 \rangle + \langle a_2 \rangle = \varepsilon.\)

The next lemma is a major step in the proof of the main theorem. It depends on the fact that \(\Gamma\) has Kazhdan’s property (T), which in turn depends only on the local structure of the building \(\Delta\). See, for example, the proof of [2, Theorem 5.7.7].

Lemma 3.4. In the group \(C(\Gamma) \otimes \mathbb{R}\), for all \(\langle a_0, a_1, a_2 \rangle \in \hat{X}^2\) and \(\xi \in P\),

\[(10a)\]

\[\langle a_0, a_1, a_2 \rangle \otimes 1 = \langle a_1, a_2, a_0 \rangle \otimes 1 = \langle a_2, a_0, a_1 \rangle \otimes 1;\]

\[(10b)\]

\[\langle \xi \rangle \otimes 1 = 0 = \langle \xi \rangle \otimes 1.\]

Proof. By Lemma 3.2, \(\varepsilon\) has finite order in \(C(\Gamma)\) and hence \(\varepsilon \otimes 1\) is zero in \(C(\Gamma) \otimes \mathbb{R}\). Therefore, by (9b),

\[\langle a_0 \rangle \otimes 1 + \langle a_1 \rangle \otimes 1 + \langle a_2 \rangle \otimes 1 = 0,\]

for \(\langle a_0, a_1, a_2 \rangle \in \hat{X}^2\). It follows from the presentation of \(\Gamma\) that the map \(\xi \mapsto \langle \xi \rangle \otimes 1, \xi \in P\), induces a homomorphism \(\theta\) from \(\Gamma\) into the abelian group \(C(\Gamma) \otimes \mathbb{R}\).

The \(\tilde{A}_2\) group \(\Gamma\) has Kazhdan’s property (T), by [2, Theorem 5.7.7]. It follows that the range of \(\theta\) is finite [2, Corollary 1.3.6] and hence zero, since \(C(\Gamma) \otimes \mathbb{R}\) is torsion free. Therefore \(\langle \xi \rangle \otimes 1 = 0, \xi \in P\). Similarly, \(\langle \xi \rangle \otimes 1 = 0, \xi \in P\). This proves (10b). The relation (9a) then implies that \(\langle a_1, a_2, a_0 \rangle \otimes 1 = \langle a_2, a_0, a_1 \rangle \otimes 1\) and the rest of (10a) follows by symmetry.

Let \(C_0(\Gamma)\) be the abelian group with generating set \(\hat{X}^2\) and the following relations:

\[(11a)\]

\[\langle a_0, a_1, a_2 \rangle = \langle a_1, a_2, a_0 \rangle = \langle a_2, a_0, a_1 \rangle, \quad \langle a_0, a_1, a_2 \rangle \in \hat{X}^2;\]

\[(11b)\]

\[\langle \xi \rangle = 0 = \langle \xi \rangle, \quad \xi \in P.\]

Lemma 3.5. The relations (11) imply the relations (4).
Proof. Let \( a = (a_0, a_1, a_2) \in \hat{X}^2 \). Then, using the relations (11), and referring to Figure 7,

\[
a = \langle a_0, a_1, a_2 \rangle = \langle a_1, a_2, a_0 \rangle \quad \text{[using (11a)]} \\
= - \sum_{(c_2, b_1, a_0) \in \hat{X}^2} \langle c_2, b_1, a_0 \rangle \quad \text{[using (11b), with } \xi = a_0] \\
= - \sum_{(c_2, b_1, a_0) \in \hat{X}^2} \left( - \sum_{b_0 \neq c_2} \langle b_0, b_1, b_2 \rangle \right) \quad \text{[using (11b) again]} \\
= \sum_{b \in X^2} m_{ab} b.
\]

The proof of the relations \( a = \sum_{b \in \hat{X}^2} n_{ab} b \) in (4) is similar. \( \square \)

**Proposition 3.6.** If \( \Gamma \) is a torsion free \( \hat{A}_2 \) group, then \( C(\Gamma) \otimes \mathbb{R} = C_0(\Gamma) \otimes \mathbb{R} \).

**Proof.** The groups have the same set of generators. By Lemmas 3.4 and 3.5, the relations in each group imply the relations in the other. The groups are therefore equal. \( \square \)

4. **Harmonic cochains and proof of the main result**

A harmonic 2-cochain [4] is a function \( c : \hat{X}^2 \to \mathbb{R} \) satisfying the following conditions for all \( a \in \hat{X}^2 \) and for all \( \xi \in P \).

\[
\begin{align*}
(12a) \quad c((a_0, a_1, a_2)) &= c((a_1, a_2, a_0)) = c((a_2, a_0, a_1)); \\
(12b) \quad c(\xi) &= c(\bar{\xi}) = 0.
\end{align*}
\]

Denote the set of harmonic 2-cochains by \( C^2_{\text{har}}(\hat{X}^2) \). Since the group \( \Gamma \) acts freely on \( \Delta \), \( C^2_{\text{har}}(\hat{X}^2) \) may be identified with the space of \( \Gamma \)-invariant harmonic cochains \( c : \hat{\Delta}^2 \to \mathbb{R} \), in the sense of [1]. Now \( C^2_{\text{har}}(\hat{X}^2) \) is the algebraic dual of \( C_0(\Gamma) \otimes \mathbb{R} \). The next result is therefore an immediate consequence of Proposition 3.6.

**Proposition 4.1.** \( C^2_{\text{har}}(\hat{X}^2) \) is isomorphic to \( C(\Gamma) \otimes \mathbb{R} \).

The proof of Theorem 1.1 can now be completed. By Theorem 3.1, it is sufficient to show that the rank \( r \) of \( C(\Gamma) \) is equal to \( \beta_2 = \dim H^2(\Gamma, \mathbb{R}) \). Garland’s isomorphism [1, Section 3.1] states that \( H^2(\Gamma, \mathbb{R}) \cong C^2_{\text{har}}(\hat{X}^2) \). Note that the account of
Garland’s Theorem in [1] relates to the case where \( \Gamma \) is a lattice in \( \text{PGL}(3, \mathbb{K}) \), but the proof applies without change to all torsion free \( \tilde{A}_2 \) groups.

It follows from Proposition 4.1 that \( C(\Gamma) \otimes \mathbb{R} \cong H^2(\Gamma, \mathbb{R}) \). Theorem 3.1 now implies that \( K_0(\mathbb{R}_\Gamma) \otimes \mathbb{R} \cong \mathbb{R}^{2\beta_2} \).

It remains to identify \( \beta_2 \) explicitly. The Euler characteristic of \( \Gamma \) is \( \chi(\Gamma) = \frac{1}{2} (q-1)(q^2-1) \) [7, Section 4]. Now \( \chi(\Gamma) = \beta_0 - \beta_1 + \beta_2 \) where \( \beta_i = \dim_{\mathbb{R}} H_i(\Gamma, \mathbb{R}) \). Since \( \Gamma \) has Kazhdan’s property (T), the abelianisation \( \Gamma/[\Gamma, \Gamma] \) is finite [2, Corollary 1.3.6], and so \( \beta_1 = 0 \). Also \( \beta_0 = 1 \). Therefore \( \beta_2 = \chi(\Gamma) - 1 = \frac{1}{2} (q-2)(q^2 + q + 1) \).

This completes the proof.

\[ \square \]

References


School of Mathematics and Statistics, University of Newcastle, NE1 7RU, U.K.
E-mail address: a.g.robertson@ncl.ac.uk