Higher rank Cuntz-Krieger algebras.

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Part I

Cuntz-Krieger algebras

Cuntz-Krieger algebras

The Cuntz-Krieger algebra $\mathcal{O}_A$ associated with a nondegenerate $n \times n$ matrix A with entries in $\{0, 1\}$ is the universal C^*-algebra generated by partial isometries $s_1, \cdots, s_n$ satisfying

$$s_1 s_1^* + \cdots + s_n s_n^* = 1 \tag{1a}$$

$$s_i^* s_i = \sum_{j=1}^{n} A(i,j) s_j s_j^* \tag{1b}$$

$\mathcal{O}_A$ is simple if and only if the matrix A is irreducible and not a permutation matrix.

K-theory

Murray–von Neumann equivalence of projections.
If p, q are projections in a C^*-algebra $\mathfrak{A}$, say that $p \sim q$ if and only if $p = s^*s$ and $q = ss^*$ for some $s \in \mathfrak{A}$.
Suppose that $\mathfrak{A}$ is a *purely infinite* C^*-algebra with unit. Then

General Fact: Cuntz 1981

- $K_0(\mathfrak{A}) = \{[p] : p \text{ is a nonzero projection in } \mathcal{A}\}$
  addition : $[p] + [q] = [p + q]$, if $pq = 0$,
  zero element : $[p - p']$, where $p \sim p' < p$.
- $K_1(\mathfrak{A}) = \mathcal{U}(\mathfrak{A})/\mathcal{U}_0(\mathfrak{A})$
  where $\mathcal{U}_0(\mathfrak{A})$ is the connected component of $\mathbf{1}$.

Computation of $K_*(\mathcal{O}_A)$

The matrix A defines a homomorphism $\mathbb{Z}^n \to \mathbb{Z}^n$.

Theorem

$K_0(\mathcal{O}_A) \cong \mathbb{Z}^n/(I - A^t)\mathbb{Z}^n; \qquad K_1(\mathcal{O}_A) \cong ker(I - A^t).$

- Simple C-K algebras are classified up to stable isomorphism by their K_0-group.
- The K-theory of a C-K algebra $\mathcal{O}_A$ can be characterized as follows:

$$K_0(\mathcal{O}_A) \cong (\text{finite abelian group}) \oplus \mathbb{Z}^k;\ K_1(\mathcal{O}_A) \cong \mathbb{Z}^k.$$

- The simple algebras $\mathcal{O}_A$ are classified up to isomorphism by the group $K_0(\mathcal{O}_A)$ together with the class $[\mathbf{1}] \in K_0(\mathcal{O}_A)$.

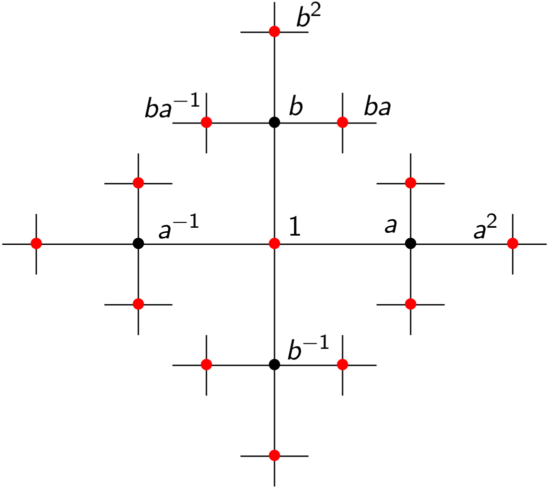
Example: Boundary action of free group of rank 2

$\Gamma = \langle a, b \rangle$ has a Cayley graph T, with vertex set Γ and edges

$$x \xrightarrow{\ s\ } xs$$

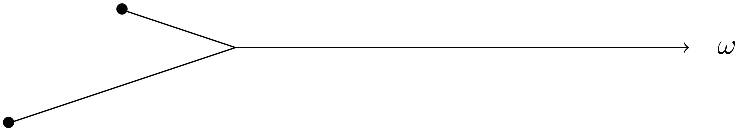
where $s \in S = \{a, a^{-1}, b, b^{-1}\}$

- T is a **tree**
- Γ acts on T (by left multiplication)


b^2
ba^{-1}
b
ba
a^{-1}
1
a
a^2
b^{-1}

A **boundary point** $\omega \in \partial T$ is :

1. An equivalence class of semi-geodesics



2. A semi-geodesic with origin $\mathbf{1}$

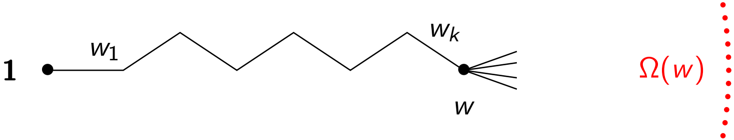
$$\mathbf{1} \bullet\!\!\longrightarrow \omega$$

3. An infinite reduced word

$$\omega = w_1 w_2 w_3 \ldots$$

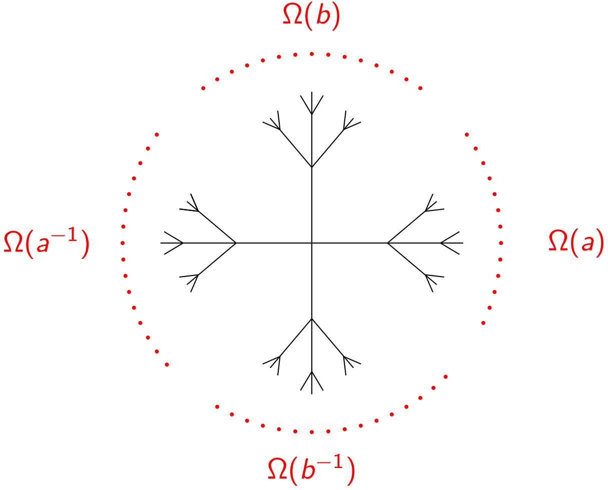
where $w_i \in S$

For $w = w_1 \dots w_k \in \Gamma - \{\mathbf{1}\}$, let
$t(w) = w_k \in S$
$\Omega(w)$: all infinite words beginning with w



$\Omega(w)$ is a basic open set for a totally disconnected compact topology on ∂T.
Denote by $p_w = \mathbf{1}_{\Omega(w)} \in C(\Omega)$ the characteristic function of $\Omega(w)$.

The boundary is partitioned into four parts according to the four possible initial letters of $\omega \in \Omega$.



Γ acts on $C(\partial T)$:

$$\gamma(f)(\omega) = f(\gamma^{-1}\omega)$$

Study the action $(\Gamma, \partial T)$ by forming the **crossed product** C^*-**algebra** :

$$\mathcal{A}_\Gamma = C(\partial T) \rtimes \Gamma = C^*\langle C(\partial T) \cup \Gamma;\ \gamma(f) = \gamma f \gamma^{-1}\rangle$$

Here

$$\begin{array}{ll} C(\partial T) \subset \mathcal{A}_\Gamma & \textit{an abelian subalgebra} \\ \Gamma \subset \mathcal{A}_\Gamma & \textit{a group of unitaries} \end{array}$$

If $u, v \in \Gamma$ and $t(u) = t(v)$, define

$$s_{u,v} = \gamma p_v \in C(\Omega) \rtimes \Gamma$$

where $\gamma = uv^{-1}$.

Covariance implies that $\gamma p_v = p_u \gamma$, so that $s_{u,v}$ is a partial isometry with initial projection p_v and final projection p_u. Let $\mathcal{A} = C^*\{s_{u,v}\,;\ u, v \in \Gamma,\ t(u) = t(v)\} \subseteq C(\Omega) \rtimes \Gamma$. Then

$\mathcal{A} = C(\Omega) \rtimes \Gamma.$

Reason: $\mathcal{A}$ contains $C(\Omega)$, since it contains $\{p_w\ ;\ w \in \Gamma\}$, which generates $C(\Omega)$. Also each element $u \in \Gamma$ lies in $\mathcal{A}$, since

$$u = \sum_{|x|=|u|+1} u p_x = \sum_{|x|=|u|+1} s_{ux,x}.$$

$\mathcal{A}$ is a C-K algebra.

Proof. For each $x \in S$ let

$$r_x = \sum_{y \in S;\, |xy|=2} s_{xy,y} = \sum_{y \in S;\, |xy|=2} x p_y .$$

Then

$$r_x r_x^* = \sum_{y \in S;\, |xy|=2} p_{xy} = p_x ,$$

$$r_x^* r_x = \sum_{y \in S;\, |xy|=2} p_y = \sum_{y \in S;\, |xy|=2} r_y r_y^* .$$

Also

$$\sum_{x \in S} r_x r_x^* = \sum_{x \in S} p_x = \mathbf{1}.$$

Therefore $\{r_x;\, x \in S\}$ satisfies the C-K relations (1).

For $u, v \in \Gamma$ with $t(u) = t(v)$, write

$$r_u = r_{u(0)} r_{u(1)} \dots r_{t(u)} = \sum_{y \in S;\, |uy| = |u|+1} s_{uy,y}.$$

Then $r_u r_v^* = s_{u,v}$. Hence $\mathcal{A}$ is generated by $\{r_x; x \in S\}$.

It follows that $C(\Omega) \rtimes \Gamma = \mathcal{A} = \mathcal{O}_A$, where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Generalization

Fact

A group which acts regularly on the vertices of a tree T has the form $\Gamma = \mathbb{Z}^{*m} * (\mathbb{Z}_2)^{*n}$, where T has degree $q = 2m + n$.

Theorem

$C(\partial T) \rtimes \Gamma$ is a C-K algebra.

Exercise

If $\Gamma = \mathbb{Z} * \mathbb{Z}_2$ then $\mathcal{A}_\Gamma = C(\partial T) \rtimes \Gamma \cong \mathcal{O}_A$ where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

and $K_0(\mathcal{A}_\Gamma) = K_1(\mathcal{A}_\Gamma) = \mathbb{Z}$, with $[\mathbf{1}] = 0$.

Part II

Higher rank Cuntz-Krieger algebras

Higher rank Cuntz-Krieger algebras

Notation

- $\mathbb{Z}_+$ is the set of nonnegative integers.
- $[m, n] = \{m, m+1, \ldots, n\}$, where $m \leq n$ are integers.
- If $m, n \in \mathbb{Z}^r$, $m \leq n$ if $m_j \leq n_j$ for $1 \leq j \leq r$.
- If $m \leq n$, let $[m, n] = [m_1, n_1] \times \cdots \times [m_r, n_r]$.
- Fix a finite set A (an *alphabet*).
- A $\{0, 1\}$-*matrix* is a matrix with entries in $\{0, 1\}$.

r-DIMENSIONAL WORDS

Choose nonzero $\{0,1\}$-matrices $M_1, M_2, \ldots, M_r$ with elements $M_j(b,a) \in \{0,1\}$, $a, b \in A$.

If $m, n \in \mathbb{Z}^r$ with $m \leq n$, let

$W_{[m,n]} = \{w : [m,n] \to A\,;\ M_j(w(l{+}e_j), w(l)) = 1, \quad l, l{+}e_j \in [m,n]\}.$

(e_j is the standard unit basis vector.)

Let $W_m = W_{[0,m]}$ if $m \geq 0 = (0, 0, \ldots, 0)$.

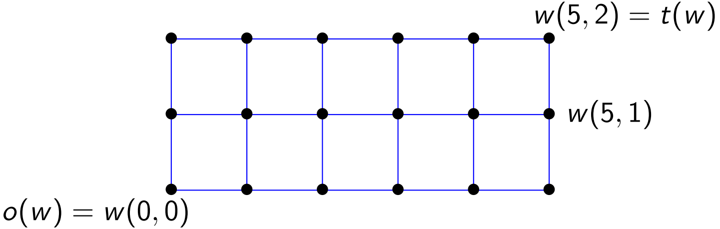
Identify A with W_0 and define the initial and final maps

$$o : W_m \to A \text{ and } t : W_m \to A$$

by

$$o(w) = w(0) \text{ and } t(w) = w(m).$$

Representation of a two dimensional word of shape $m = (5, 2)$.



Say that $w \in W_m$ has shape m, and write $\sigma(w) = m$.

Decorated and periodic words

Fix a finite or countable set of *decorations* $D \neq \emptyset$.

Fix a *decorating* map $\delta : D \to A$.

- The set of *decorated words* of shape m is $\overline{W}_m = \{(d, w) \in D \times W_m;\ o(w) = \delta(d)\}$.
- Identify D with $\overline{W}_0$ via $d \mapsto (d, \delta(d))$.
- $W := \bigcup_m W_m$ and $\overline{W} := \bigcup_m \overline{W}_m$.
- $o(d, w) := d,\ t(d, w) = t(w)$ and $\sigma((d, w)) := \sigma(w)$.

- If $j \le k \le l \le m$ and $w : [j, m] \to A$, define $w|_{[k,l]} \in W_{l-k}$ by

$$w|_{[k,l]} = w'$$

where $w'(i) = w(i + k)$ for $0 \le i \le l - k$.

- If $\overline{w} = (d, w) \in \overline{W}_m$, define

$$\begin{aligned} & \overline{w}|_{[k,l]} = w|_{[k,l]} \in W_{l-k} \text{ if } k \ne 0, \\ \text{and} \qquad & \overline{w}|_{[0,l]} = (d, w|_{[0,l]}) \in \overline{W}_l. \end{aligned}$$

- If $w \in W_l$ and $k \in \mathbb{Z}^r$, define $\tau_k w : [k, k + l] \to A$ by

$$(\tau_k w)(k + j) = w(j).$$

- If $w \in W_l$ where $l \ge 0$ and if $p \ne 0$, say that w is *p-periodic* if $\tau_p w$ satisfies

$$\tau_p w|_{[0,l]\cap[p,p+l]} = w|_{[0,l]\cap[p,p+l]}.$$

Fundamental hypotheses

Assume that the following conditions hold.

(H0) Each M_i is a nonzero $\{0,1\}$-matrix.

(H1) Let $u \in W_m$ and $v \in W_n$. If $t(u) = o(v)$ then there exists a unique $w \in W_{m+n}$ such that

$$w\big|_{[0,m]} = u \qquad \text{and} \qquad w\big|_{[m,m+n]} = v.$$

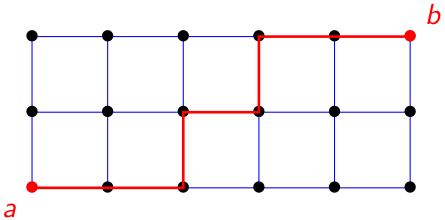
Write $w = uv$ and say that the product uv exists.

(H2) Consider the directed graph with vertex set A and a directed edge from a to b for each i such that $M_i(b,a) = 1$. This graph is irreducible.

(H3) Let $p \in \mathbb{Z}^r$, $p \neq 0$. There exists some $w \in W$ which is not p-periodic.

Condition (H2)

For each $a, b \in A$ there exists a directed path



Condition (H1) holds if the matrices M_i, $1 \leq i \leq r$ satisfy the following three conditions.

(H1A) $M_i M_j = M_j M_i$.

(H1B) For $i < j$, $M_i M_j$ is a $\{0, 1\}$-matrix.

(H1C) For $i < j < k$, $M_i M_j M_k$ is a $\{0, 1\}$-matrix.

The higher rank C-K algebra $\mathcal{A}$

Definition

The C^*-algebra $\mathcal{A}$ (or $\mathcal{A}_D$) is the universal C^*-algebra generated by a family of partial isometries $\{s_{u,v};\ u, v \in \overline{W} \text{ and } t(u) = t(v)\}$ satisfying the relations

$$s_{u,v}{}^* = s_{v,u} \tag{2a}$$

$$s_{u,v}s_{v,w} = s_{u,w} \tag{2b}$$

$$s_{u,v} = \sum_{\substack{w \in W; \sigma(w) = e_j, \\ o(w) = t(u) = t(v)}} s_{uw,vw}, \text{ for } 1 \leq j \leq r \tag{2c}$$

$$s_{u,u}s_{v,v} = 0, \text{ for } u, v \in \overline{W}_0, u \neq v. \tag{2d}$$

Example

If $r = 1$, $M = M_1$, $D = A$ and $\delta = id$, then $\mathcal{A} \cong \mathcal{O}_{M^t}$.

Proof.

$\mathcal{O}_{M^t}$ is generated by $\{S_a; a \in A\}$ satisfying

$$S_a^* S_a = \sum_b M(b,a) S_b S_b^*.$$

If $u \in W$, let $S_u = S_{u(0)} S_{u(1)} \ldots S_{t(u)}$. If $v \in W$ with $t(u) = t(v)$, define $S_{u,v} = S_u S_v^*$. The map

$$s_{u,v} \mapsto S_{u,v}$$

establishes an isomorphism of $\mathcal{A}$ with $\mathcal{O}_{M^t}$. The inverse is given by

$$S_a \mapsto \sum_{M(b,a)=1} s_{ab,b}$$

□

Example

If $\mathcal{A}_1$, $\mathcal{A}_2$ are rank one C-K algebras, then $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the rank two C-K algebra.

In obvious notation, the alphabet is $A_1 \times A_2$ and the of matrices are $M_1 \otimes I, I \otimes M_2$.

Theorem

*The C^*-algebra $\mathcal{A}$ is purely infinite, simple and nuclear. Any nontrivial C^*-algebra with generators $S_{u,v}$ satisfying relations (2) is isomorphic to $\mathcal{A}$ via a unique *-isomorphism ϕ such that $\phi(s_{u,v}) = S_{u,v}$.*

Suppose that D is finite. Then $\sum_{u \in \overline{W}_0} s_{u,u}$ is an identity for $\mathcal{A}$. Therefore $\mathcal{A}$ is a p.i.s.u.n. C^*-algebra. It also satisfies the Universal Coefficient Theorem. By the Classification Theorem of Kirchberg-Phillips, $\mathcal{A}$ is classified by its K-groups and the class of the identity in K_0.

Stabilization

Lemma

Given a decoration $\delta : D \to A$*, define another decoration* $\delta' : D \times \mathbb{N} :\to A$ *by* $\delta'((d,i)) = \delta(d)$*. Then*

$$\mathcal{A}_{D\times\mathbb{N}} \cong \mathcal{A}_D \otimes \mathcal{K}.$$

Proof.

If $u, v \in W$, the isomorphism is given by

$$s_{((d,i),u),((d',j),v)} \mapsto s_{(d,u),(d',v)} \otimes e_{ij},$$

where the e_{ij} are matrix units for $\mathcal{K}$. $\square$

Remark

Similarly, $\mathcal{A}_{D\times\{1,2,\dots,r\}}$ is isomorphic to $\mathcal{A}_D \otimes M_r$.

Corners

Lemma

Let $D' \subset D$, $\delta' = \delta|_{D'}$ and $e = \sum_{a \in D'} s_{a,a}$. Then

$$\mathcal{A}_{D'} \cong e\mathcal{A}_D e$$

Proof

The generators of $\mathcal{A}_D$ satisfy

$$es_{u,v}e = \begin{cases} s_{u,v} & \text{if } o(u), o(v) \in D', \\ 0 & \text{otherwise.} \end{cases}$$

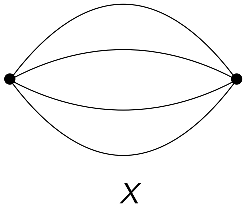
The claim follows by uniqueness of $\mathcal{A}_{D'}$.

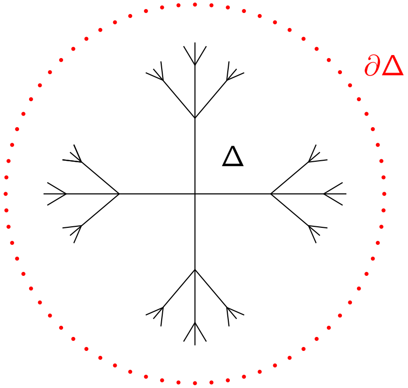
Consequences - after considerable work

- For a fixed alphabet A and fixed transition matrices M_j, the isomorphism class of $\mathcal{A}_D \otimes \mathcal{K}$ is independent of D.
- $\mathcal{A}_D \otimes \mathcal{K} \cong \mathcal{F} \rtimes \mathbb{Z}^r$, where $\mathcal{F}$ is an AF algebra.

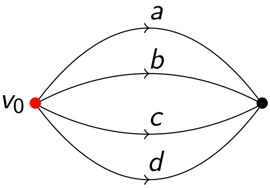
Decorated rank one examples

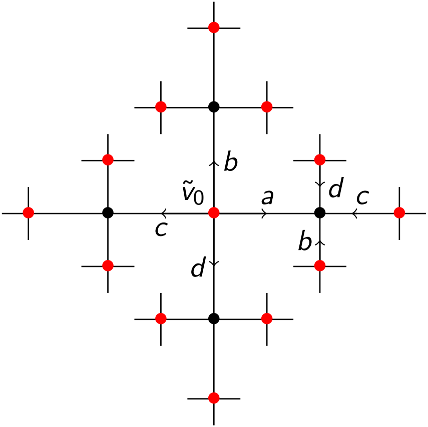
A finite connected graph X, with all vertices of degree > 2. The universal covering tree Δ has boundary $\partial\Delta$.





Example





The alphabet A is

$$\{a, \overline{a}, b, \overline{b}, c, \overline{c}, d, \overline{d}\}$$

$\Gamma = F_3$ acts freely on Δ and on $\partial\Delta$.

The decorating set D is $\{a, b, c, d\}$ and $\delta : D \to A$ is inclusion.
$M(y, x) = 1 \Leftrightarrow$ the path xy has length 2 in X (i.e. a simple loop).

Theorem

$\mathcal{A} = \mathcal{A}_D$ is a rank-1 CK-algebra and

$$K_0(\mathcal{A}) = \left\langle A \;\middle|\; a = \sum_{b \in A} M(a, b) b \right\rangle$$

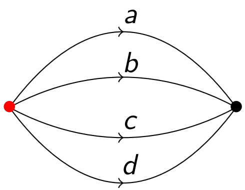
This is easily computed from the graph X.

Example: $\Gamma = F_3$

generators : $\{a, \overline{a}, b, \overline{b}, c, \overline{c}, d, \overline{d}\}$

8 relations :

$$\begin{aligned} a &= \overline{b} + \overline{c} + \overline{d} \\ \overline{a} &= b + c + d \\ etc & \quad \ldots \end{aligned}$$



Theorem

- $K_0(\mathcal{A}) = \mathbb{Z}^r \oplus \mathbb{Z}_{r-1}$, *where* r *is the rank of* Γ.
- *The class* $[\mathbf{1}] \in K_0(\mathcal{A})$ *has order* $r - 1$

REMARK

F_r can act in (many) different ways on a tree.

Acylindrical Groups

Suppose

- Δ is an infinite, locally finite tree
- $\partial\Delta$ is infinite
- $\Gamma < \mathrm{Aut}(\Delta)$ acts without inversion

Definition

- Γ is k-acylindrical if the stabilizer of any path of length k in Δ is trivial.
- Γ is acylindrical if it is k-acylindrical for some integer k.

(Z. Sela, 1997)
The name comes from the geometry of surfaces in 3-manifolds.

Examples

- Γ is 1-acylindrical if it acts freely on the set of edges.
- The action of $\Gamma_1 * \Gamma_2$ on its Bass-Serre tree is 1-acylindrical.
- The action of $\Gamma_1 *_{\Gamma_0} \Gamma_2$ on its Bass-Serre tree is 2-acylindrical if Γ_0 is malnormal in Γ_1.
- Every small splitting of a torsion free hyperbolic group Γ is 3-acylindrical.

small splitting: *action of Γ on a tree in which no edge stabilizer contains a nonabelian free group*

Theorem

Suppose that $\Gamma < \mathrm{Aut}(\Delta)$ *is acylindrical and* $\Gamma \backslash \Delta$ *is finite. Then* $\mathcal{A}_\Gamma = C(\partial\Delta) \rtimes \Gamma$ *is a simple (rank-1) C-K algebra.*

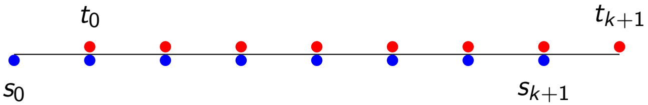
Construction

Suppose Γ is k-acylindrical.

- The alphabet A is $\Gamma\backslash\mathfrak{S}_{k+1}$, where $\mathfrak{S}_{k+1}$ is the set of directed paths of length $k+1$;

If $a, b \in A$, define $M(a, b) = 1$ if and only if

- $a = \Gamma\sigma$, where $\sigma = (s_0, s_1, \ldots, s_{k+1})$;
- $b = \Gamma\tau$, where $\tau = (t_0, t_1, \ldots, t_{k+1})$;
- $s_{j+1} = t_j$, $0 \leq j \leq k$.



Otherwise $M(a, b) = 0$.

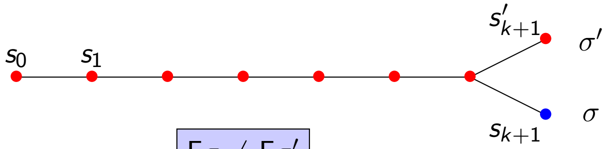
Let $\mathfrak{W}_m = \Gamma\backslash\mathfrak{S}_{m+k+1}$ and let $\mathfrak{W} = \bigcup_m \mathfrak{W}_m$. There is a bijection

$$\alpha : \mathfrak{W}_m \to W_m$$

defined by

$$\alpha(\Gamma(s_0, s_1, \ldots, s_{m+k+1})) = (\Gamma[s_0, s_{k+1}])(\Gamma[s_1, s_{k+2}])\ldots(\Gamma[s_m, s_{m+k+1}]).$$

Crucial Fact



$$\boxed{\Gamma\sigma \neq \Gamma\sigma'}$$

Decoration

Fix $P \in \Delta^0$.

- D is the set of directed segments of length $k+1$ which begin at P;
- $\overline{\mathfrak{W}}_m$ is the set of directed segments of length $m+k+1$ which begin at P.

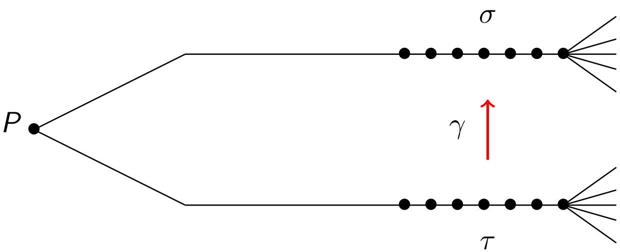
There is a natural bijection

$$\overline{\alpha} : \overline{\mathfrak{W}}_m \to \overline{W}_m.$$

Let $\sigma, \tau \in \mathfrak{S}_{k+1}$ be final segments of paths beginning at P.

If $\sigma = \gamma\tau$, where $\gamma \in \Gamma$, then define

$$\phi(s_{\overline{\alpha}(\sigma),\overline{\alpha}(\tau)}) = \gamma p_\tau$$



The map ϕ induces an isomorphism $\mathcal{A}_D \cong \mathcal{A}_\Gamma = C(\Omega) \rtimes \Gamma$.

Facts

- $\mathcal{A}_\Gamma$ depends only on Γ
- $K_0(\mathcal{A}_\Gamma) = \operatorname{coker}(I - M) = \left\langle A \,\middle|\, a = \sum_{b \in A} M(a,b)b \right\rangle$

Consequence

$K_0(\mathcal{A}_\Gamma)$ is Bowen-Franks invariant of the associated geodesic flow.

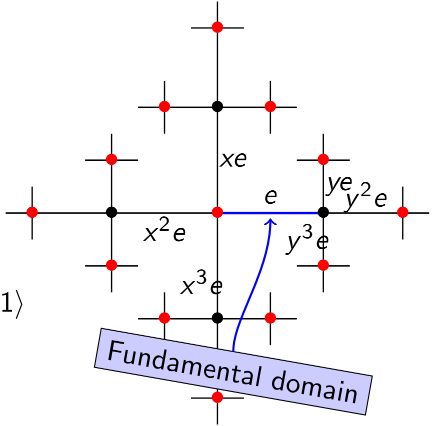
1-acylindrical example

$\Gamma = \mathbb{Z}_{\ell+1} * \mathbb{Z}_{m+1}$ acts on its Bass-Serre tree Δ: an $(\ell+1, m+1)$-semihomogeneous tree.

Γ acts freely and transitively on E.

If $m = n = 3$ then

$$\begin{aligned}\Gamma &= \mathbb{Z}_4 * \mathbb{Z}_4 \\ &= \langle x, y | x^4 = y^4 = 1\rangle\end{aligned}$$



Theorem

If $\Gamma = \mathbb{Z}_{\ell+1} * \mathbb{Z}_{m+1}$ *then* $K_0(\mathcal{A}_\Gamma) = \mathbb{Z}_{\ell m-1}$.

Proof.

If $\sigma \in \mathfrak{S}_2$, then σ is one of

This gives a partition $A = \Gamma\backslash\mathfrak{S}_2 = A_0 \sqcup A_1$, $|A_0| = \ell, |A_1| = m$.

If $a = \Gamma\sigma \in A_i$ then $a = \sum_{\substack{\Gamma\tau \in A_{i+1} \\ \tau_i = \sigma_i}} \Gamma\tau$, which depends only on σ_i.

Γ acts transitively on edges, so all elements of A_i are equal in $K_0(\mathcal{A}_\Gamma)$.
Therefore $K_0(\mathcal{A}_\Gamma)$ equals

$$\langle a_0, a_1 \mid a_0 = m a_1, a_1 = \ell a_0 \rangle = \langle a_0 \mid a_0 = \ell m a_1 \rangle = \mathbb{Z}_{\ell m-1}$$

□

Corollary

$K_1(\mathcal{A}_\Gamma) = 0$ and so $\mathcal{U}(\mathcal{A}_\Gamma)$ is connected.

Proof

This is because $K_1(\mathcal{A}_\Gamma)$ equals the torsion free part of $K_0(\mathcal{A}_\Gamma)$ and $K_1(\mathfrak{A}) = \mathcal{U}(\mathfrak{A})/\mathcal{U}_0(\mathfrak{A})$.

Part III

K-theory of rank 2 Cuntz-Krieger algebras

Let $\mathcal{A}$ be a rank two C-K algebra associated with an alphabet A and matrices M_1, M_2. [D is irrelevant for computing K_*.]
The matrices $(I - M_1, I - M_2)$ and $(I - M_1^t, I - M_2^t)$ define homomorphisms $\mathbb{Z}^A \oplus \mathbb{Z}^A \to \mathbb{Z}^A$.

Let

$$\begin{aligned} \text{coker } (I - M_1,\, I - M_2) &= \mathbb{Z}^r \oplus T \\ \text{coker } (I - M_1^t,\, I - M_2^t) &= \mathbb{Z}^s \oplus T' \end{aligned}$$

where T, T' are torsion groups. Then

Theorem

$$\begin{aligned} K_0(\mathcal{A}) &= \mathbb{Z}^{r+s} \oplus T \\ K_1(\mathcal{A}) &= \mathbb{Z}^{r+s} \oplus T'. \end{aligned}$$

Outline of proof

- $K_*(\mathcal{A}) = K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$, since $\mathcal{A}_D \otimes \mathcal{K} \cong \mathcal{F} \rtimes \mathbb{Z}^r$
- The Baum-Connes Conjecture with coefficients in an arbitrary C^*-algebra is true for the group $\mathbb{Z}^2$ (and much more generally). This implies that $K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$ coincides with its "γ-part".
- Therefore $K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$ may be computed as the limit of a Kasparov spectral sequence. The initial terms of the spectral sequence are $E^2_{p,q} = H_p(\mathbb{Z}^2, K_q(\mathcal{F}))$, the p^{th} homology of the group $\mathbb{Z}^2$ with coefficients in the module $K_q(\mathcal{F})$.
- Use the fact that $K_1(\mathcal{F}) = 0$.

Part IV

Boundary Algebras for $\widetilde{A}_2$ groups

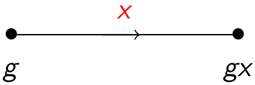
Boundary algebras provided the motivation for the general theory of higher rank CK-algebras.

Theorem

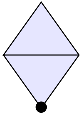
Let Δ be a building of type $\widetilde{A}_2$ with boundary Ω. Let Γ be a group of type rotating automorphisms of Δ that acts freely on the vertex set with finitely many orbits. Then $C(\Omega) \rtimes \Gamma$ is isomorphic to a rank 2 C-K algebra $\mathcal{A}$.

- Rigidity theorems imply that the action of Γ is unique up to conjugacy and so the crossed product C^*-algebra $\mathcal{A}(\Gamma) = C(\Omega) \rtimes \Gamma$ depends only on Γ. Write $\mathcal{A}_\Gamma = C(\Omega) \rtimes \Gamma$.
- For simplicity, assume that the action of Γ is also transitive on the vertex set. i.e. Γ is an $\widetilde{A}_2$ group. Then decorating set is trivial, i.e. $D = A$.

The 1-skeleton of Δ is the Cayley graph of Γ. Vertices of Δ are elements of Γ and a directed edge of the form $e = (g, gx)$ is labeled by a generator $x \in P$.



The alphabet A is defined to be the set of Γ-equivalence classes of basepointed parallelograms in Δ. Each $a \in A$ has a unique representative labelled parallelogram (*tile*) based at a fixed vertex.

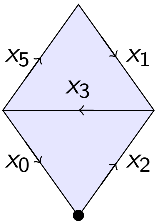


$\#A = q(q+1)(q^2+q+1)$. (Exercise.)

Mumford's group Γ_M has generators $x_0, x_1, \ldots, x_6$, and relators

$$\begin{cases} x_0x_0x_6,\ x_0x_2x_3,\ x_1x_2x_6,\ x_1x_3x_5, \\ x_1x_5x_4,\ x_2x_4x_5,\ x_3x_4x_6. \end{cases}$$

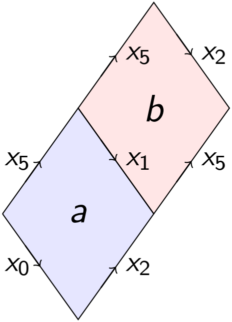
A tile for Mumford's group



Suppose that $\Gamma = \Gamma_M$. The transition matrices M_1, M_2 are defined as follows.

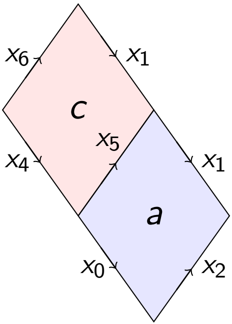
$M_1(b, a) = 1$ if and only if there are tiles representing a, b in the building Δ which lie as shown.

$$M_1(b, a) = 1$$



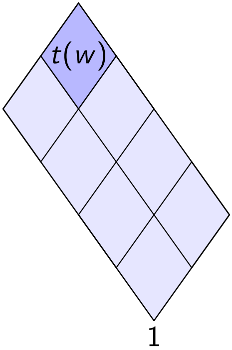
Otherwise $M_1(b, a) = 0$.

The diagram for M_2:



$M_2(c, a) = 1$

Let $\mathfrak{p}$ be a parallelogram based at 1 in some apartment of Δ.
$\mathfrak{p}$ is a union of tiles.

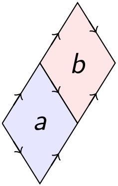


Associated to $\mathfrak{p}$ there is a two dimensional word $w = w(\mathfrak{p})$. The map $\mathfrak{p} \mapsto w(\mathfrak{p})$ is bijective, and we identify $\mathfrak{p}$ with $w(\mathfrak{p})$.

If $w = w(\mathfrak{p}) \in W$, the terminal letter $t(w) \in A$ is represented by the tile of $\mathfrak{p}$ farthest from 1.

Example

In the diagram below, the two letters a, b define a word $w \in W_{(1,0)}$, with $w(0,0) = a$ and $w(1,0) = b$.



Let $\Omega(w) = \{\omega \in \Omega\,;\, \mathfrak{p} \subset [1, \omega)\}$, the set of boundary points whose representative sectors based at 1 contain $\mathfrak{p}$.

We now describe the isomorphism $\phi : \mathcal{A} \to C(\Omega) \rtimes \Gamma$.

If $w_1, w_2 \in W$ with $t(w_1) = t(w_2) = a \in A$, let $\gamma \in \Gamma$ be the unique element such that $\gamma t(w_1) = t(w_2)$. Then

$$\phi(s_{w_2,w_1}) = \gamma \mathbf{1}_{\Omega(w_1)} = \mathbf{1}_{\Omega(w_2)} \gamma. \tag{3}$$

This defines a *-homomorphism of because the operators of the form $\phi(s_{w_2,w_1})$ satisfy the relevant relations.

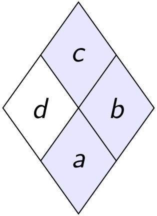
Exercise

Verify this.

Since $\mathcal{A}$ is simple, ϕ is injective. The proof of surjectivity is omitted.

A key step is the verification of the conditions (H0-H3).

- (H0) is obvious.
- (H1) follows from the fact that in the configuration illustrated by the Figure below, the tiles a, b, c determine a unique tile d lying in an apartment of Δ containing a, b, c.



- (H3) follows from thickness of the building, which allows words to be extended so as to lack periodicities.

The hardest condition to prove is (H2), i.e. irreducibility of the associated directed graph. This can be done by a direct combinatorial argument for $\widetilde{A}_2$ groups.

K-theory

Lemma

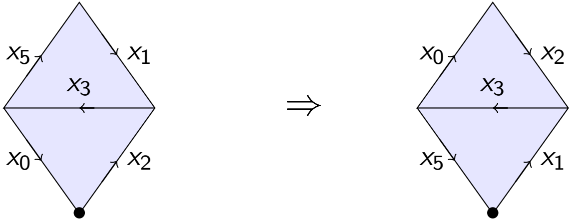
There is a permutation matrix $S : \mathbb{Z}^A \to \mathbb{Z}^A$ such that $S^2 = I$ and $SM_1S = M_2^t$, $SM_2S = M_1^t$. In particular

$$\text{coker } (I - M_1,\ I - M_2) = \text{coker } (I - M_1^t,\ I - M_2^t)$$

Proof

S is defined by vertical flip.

S:



Let r be the rank, and T the torsion part, of the finitely generated abelian group $C(\Gamma) = \operatorname{coker}(I-M_1, I-M_2)$. Thus $C(\Gamma) \cong \mathbb{Z}^r \oplus T$.

$C(\Gamma)$ is the abelian group with generating set A and relations

$$a = \sum_{b \in A} M_j(b,a)b, \quad a \in A, \qquad j = 1, 2.$$

Theorem

Let Γ be an $\widetilde{A}_2$ group. Then

$$K_0(\mathcal{A}_\Gamma) = K_1(\mathcal{A}_\Gamma) = \mathbb{Z}^{2r} \oplus T. \tag{4}$$

Proof.

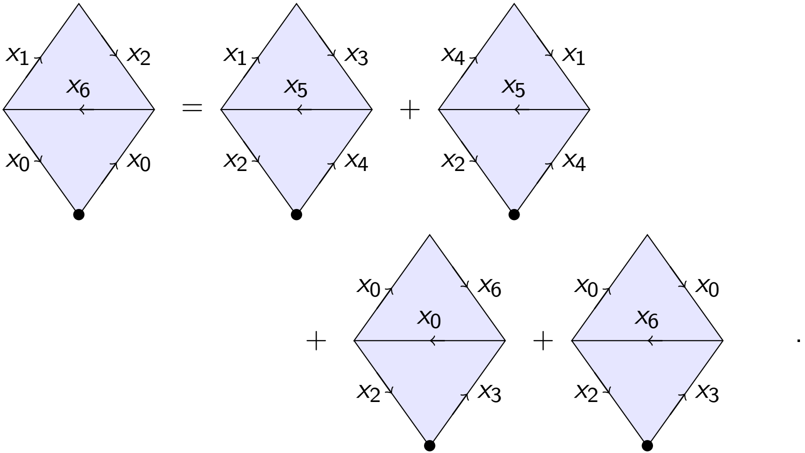
The general K-theory result simplifies by the preceding lemma. $\square$

Conjecture

If Γ is torsion free, then $r = \operatorname{rank} H_2(\Gamma, \mathbb{Z})$. [Known: $\geq$.]

Example: Mumford's group

For Mumford's group Γ, $C(\Gamma)$ has $7 \times 3 \times 2 = 42$ generators and $2 \times 42 = 84$ relations of the form



Computation gives $C(\Gamma_M) = (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/3\mathbb{Z}$.

Consequence

$$K_0(\mathcal{A}_{\Gamma_M}) = (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/3\mathbb{Z}.$$

If $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_p)$ is a torsion-free lattice then Γ has Euler characteristic

$$\chi(\Gamma) = \frac{n_0}{3}(p-1)(p^2-1).$$

where n_0 is the number of vertex orbits of Γ acting on Δ.

There are **three** torsion-free lattices $\Gamma_1, \Gamma_2, \Gamma_3 < \mathrm{PGL}_3(\mathbb{Q}_2)$ with $\chi(\Gamma_j) = 1$. (One of them is Mumford's group.)

$\Gamma_1, \Gamma_2, \Gamma_3$ are distinguished from each other by $K_0(\mathcal{A}_\Gamma)$.

Typical computational result

$\Gamma < \mathrm{PGL}_3(\mathbb{Q}_7)$ an $\widetilde{A}_2$ group of H. Voskuil.

$$K_0(\mathcal{A}_\Gamma) = \mathbb{Z}^{190} \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})^{21}$$

Note that

$$H_1(\Gamma, \mathbb{Z}) = (\mathbb{Z}/3\mathbb{Z})^7$$
$$H_2(\Gamma, \mathbb{Z}) = \mathbb{Z}^{95}$$

and $[\mathbf{1}]$ has order

$$\frac{(p-1)}{\gcd(3, p-1)} = 2\,.$$

There is a simple rank one C-K algebra $\mathcal{A}_0$ such that
$K_*(\mathcal{A}_0) = (\mathbb{Z}, \mathbb{Z})$.

Theorem

Let Γ be an $\widetilde{A}_2$ group. There is a simple rank one C-K algebra $\mathcal{A}_1$ such that $\mathcal{A}_\Gamma$ is stably isomorphic to $\mathcal{A}_0 \otimes \mathcal{A}_1$.

Proof.

There is a simple rank one C-K algebra $\mathcal{A}_1$ such that
$K_*(\mathcal{A}_1) = (\mathbb{Z}^r \oplus T, \mathbb{Z}^r)$.
By the Künneth Theorem for tensor products,

$$K_*(\mathcal{A}_0 \otimes \mathcal{A}_1) == (\mathbb{Z}^r \oplus T, \mathbb{Z}^r) \otimes (\mathbb{Z}, \mathbb{Z}) = (\mathbb{Z}^{2r} \oplus T, \mathbb{Z}^{2r} \oplus T).$$

The result follows from the Classification Theorem. □

Triangle groups

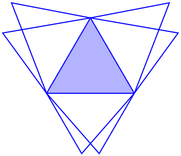
Definition

A triangle group Γ is a group of type preserving automorphisms of Δ which acts regularly on the set of chambers of Δ.

Example: $q = 2$

$$\left\langle s_i, i \in \mathbb{Z}_3 \mid s_i^3 = 1, s_i s_{i+1} = (s_{i+1}s_i)^2 \right\rangle \tag{5}$$

acts on the euclidean building Δ of $\mathrm{SL}_3(\mathbb{Q}_2)$.



- $q = 2$: 4 groups.
- $q = 8$: 44 groups.

Theorem

If Γ is a triangle group then

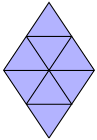
- $\mathcal{A}_\Gamma = C(\Omega) \rtimes \Gamma$ *is a rank 2 Cuntz-Krieger algebra;*
- $K_0(\mathcal{A}_\Gamma) = K_1(\mathcal{A}_\Gamma) = \mathbb{Z}_{q^2-1}$.

Corollary

$\mathcal{A}_\Gamma$ is stably isomorphic to $\mathcal{O}_{q^2} \otimes \mathcal{O}_{q^2}$.

The plan of the proof is similar to the case of $\widetilde{A}_2$ groups.

Tiles have the form



2-dimensional words are formed using two transition matrices, with overlapping tiles.