

# Operator algebras, boundaries of buildings and K-theory

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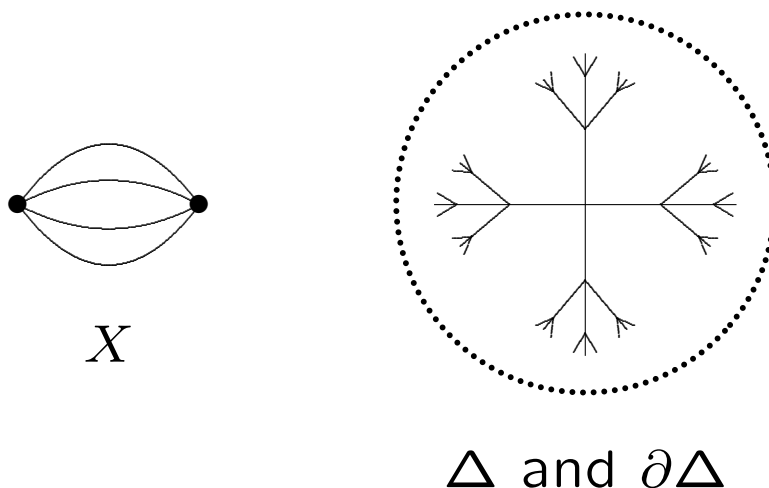
Spaces that arise in analysis are often pathological and cannot be studied by classical geometric methods. Consider instead an associated **algebraic** object.

For example, the topology of a “good” space  $S$  is completely determined by the commutative algebra

$$C(S) = \{f : S \rightarrow \mathbb{C} \mid f \text{ is continuous}\}.$$

If  $S$  is a “bad” space, replace  $C(S)$  by a **non-commutative** algebra.

**Example:** A finite connected graph  $X$ , with all vertices of degree  $> 2$ . The universal covering tree  $\Delta$  has boundary  $\partial\Delta$ .



Let  $\Gamma = \pi(X)$ , the fundamental group of  $X$ .  $\Gamma$  is a free group which acts freely on  $\Delta$  and

$$\Gamma \backslash \Delta = X$$

$\Gamma$  also acts on  $\partial\Delta$ , but this action is “bad”:  $\Gamma \backslash \partial\Delta$  is not Hausdorff.

$\Gamma$  acts on  $C(\partial\Delta)$  :

$$\gamma(f)(\omega) = f(\gamma^{-1}\omega)$$

Study the “bad” action by forming the **crossed product  $C^*$ -algebra**:

$$\begin{aligned}\mathcal{A}(\Gamma) &= C(\partial\Delta) \rtimes \Gamma \\ &= C^*\langle \Gamma \cup C(\partial\Delta) ; \gamma(f) = \gamma f \gamma^{-1} \rangle\end{aligned}$$

Here

$$\begin{array}{ll}C(\partial\Delta) \subset \mathcal{A}(\Gamma) & \text{an abelian subalgebra} \\ \Gamma \subset \mathcal{A}(\Gamma) & \text{a group of unitaries}\end{array}$$

$\mathcal{A}(\Gamma)$  is generated (as a  $C^*$ -algebra) by finite sums

$$\sum f_i \gamma_i, \quad f_i \in C(\partial\Delta), \gamma_i \in \Gamma$$

$$\begin{array}{ll}\text{product :} & f_1 \gamma_1 \cdot f_2 \gamma_2 = f_1 \gamma_1 (f_2) \gamma_1 \gamma_2 \\ \text{involution :} & (f \gamma)^* = \gamma^{-1} (f^*) \gamma^{-1}\end{array}$$

What about “good” actions?

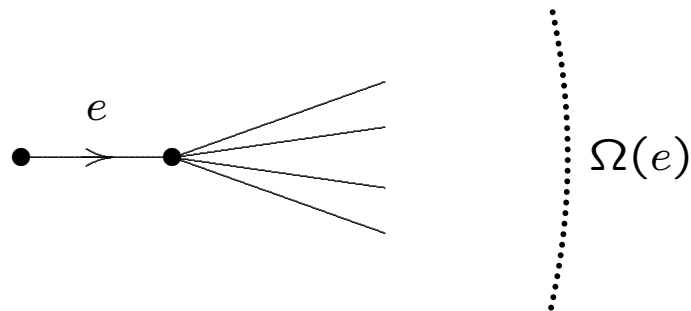
Example:  $X = \Gamma \backslash \Delta$ .

Answer:  $C_0(\Delta) \rtimes \Gamma \approx C(X)$ .

Let  $E = \{\text{oriented edges of } \Delta\}$ .

(Each geometric edge has two orientations.)

If  $e \in E$ , define a clopen subset  $\Omega(e)$  of  $\partial\Delta$



The indicator function  $p_e \in C(\partial\Delta) \subset \mathcal{A}(\Gamma)$ .

If  $e \in E$  and  $d = \gamma e$ , where  $\gamma \in \Gamma$ , then define a **partial isometry**

$$s_{d,e} = \gamma p_e .$$



$$\begin{aligned}
s_{d,e}^* s_{d,e} &= p_e && \text{initial projection} \\
s_{d,e} s_{d,e}^* &= \gamma p_e \gamma^{-1} \\
&= \gamma(p_e) \\
&= p_d && \text{final projection}
\end{aligned}$$

Therefore  $p_d \sim p_e$

(Murray–von Neumann equivalence)

## Facts :

(1)  $\mathcal{A}(\Gamma)$  is simple, purely infinite and generated by the operators  $s_{d,e}$ , where  $d \in \Gamma e$ .

**A Cuntz-Krieger algebra.**

(2)  $\mathcal{A}(\Gamma)$  is classified by the abelian group  $K_0(\mathcal{A}(\Gamma))$  and [1], where

$$K_0(\mathcal{A}) = \{[p] : p \text{ is a nonzero idempotent in } \mathcal{A}\}$$

addition :  $[p] + [q] = [p + q]$ , if  $pq = 0$ ,

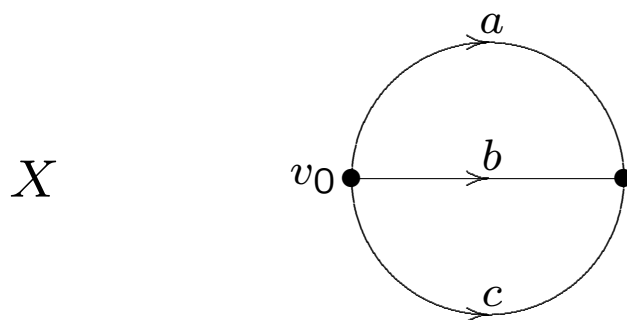
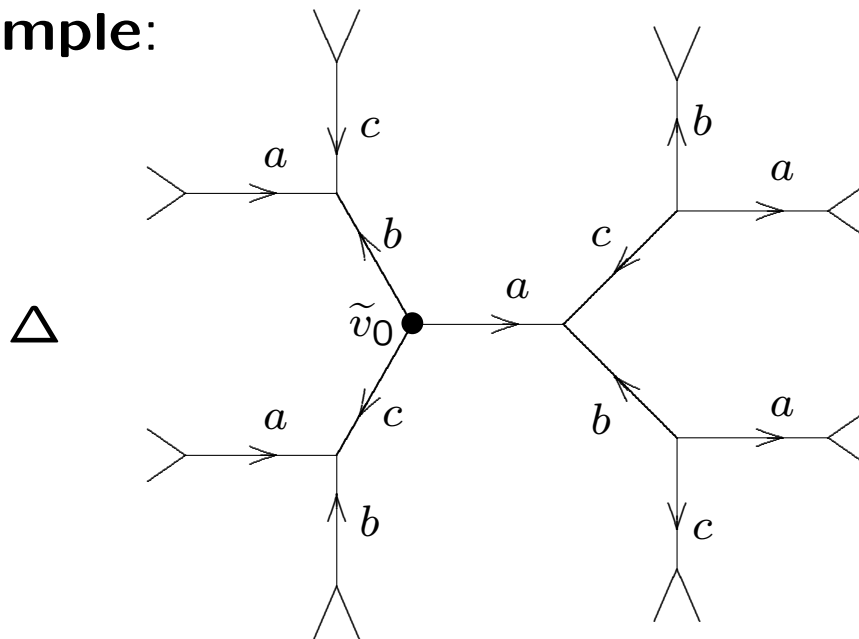
zero element :  $[p - p']$ , where  $p \sim p' < p$ .

If  $e \in E$ ,  $[p_e] \in K_0(\mathcal{A}(\Gamma))$  depends only on  $\Gamma_e$ .

Let  $A := \{\Gamma_e : e \in E\}$ , (a finite alphabet).

$$A \approx \{\text{directed edges of } X\}$$

**Example:**



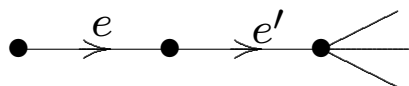
$$A = \{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$$

For  $a = \Gamma e \in A$ , let  $[a] := [p_e] \in K_0(\mathcal{A}(\Gamma))$ .

These are **all** the generators for  $K_0(\mathcal{A}(\Gamma))$ .

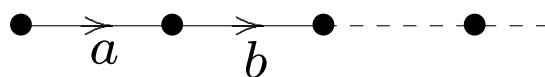
The idempotents  $p_e$  satisfy

$$p_e = \sum_{\substack{e' \in E \\ e \rightarrow e'}} p_{e'}$$



Define 0–1 matrix  $M$ , for  $a, b \in A$  by

$$M(a, b) = 1 \iff$$



Relations:

$$[a] = \sum_{b \in A} M(a, b)[b].$$

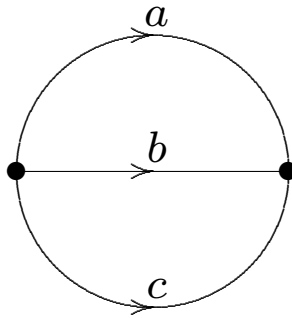
These are the **only** relations ...

## Theorem.

$$K_0(\mathcal{A}(\Gamma)) = \left\langle A \mid a = \sum_{b \in A} M(a, b)b \right\rangle.$$

This is easily computed from the graph  $X$ :

**Example:**



generators:  $\{a, \bar{a}, b, \bar{b}, c, \bar{c}\}$

relations:

$$\begin{aligned} a &= \bar{b} + \bar{c}, & b &= \bar{c} + \bar{a}, & c &= \bar{a} + \bar{b} \\ \bar{a} &= b + c, & \bar{b} &= c + a, & \bar{c} &= a + b \end{aligned}$$

**Result:**

$K_0(\mathcal{A}(\Gamma)) = \mathbb{Z}^r \oplus \mathbb{Z}/(r-1)\mathbb{Z}$ , where  $r$  is the rank of  $\Gamma$ . The class  $[1] \in K_0(\mathcal{A}(\Gamma))$  has order

$$r - 1 = -\chi(X)$$

(Euler characteristic)

**Remark:** It follows that  $\mathcal{A}(\Gamma)$  depends only on  $\Gamma$ .

**Special case:**

Let  $p$  be prime and  $\Gamma$  a torsion free lattice in

$$G = \mathrm{PGL}_2(\mathbb{Q}_p).$$

$G$  acts on  $\Delta$  (homogeneous tree of degree  $p+1$ ),

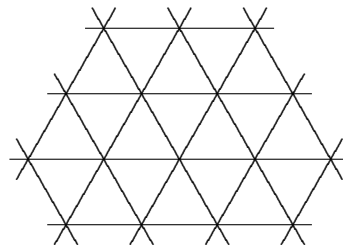
$X = \Gamma \backslash \Delta$  is a finite graph,  $\Gamma = \pi(X)$  and

$$\chi(X) = -\frac{(p-1)}{2} \cdot \#\{\text{vertices of } X\}$$

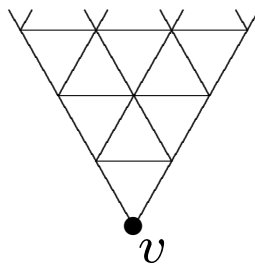
$$\boxed{\mathrm{PGL}_3(\mathbb{Q}_p)}$$

$G = \mathrm{PGL}_3(\mathbb{Q}_p)$  acts on its **building** of type  $\tilde{A}_2$ , which is a topologically contractible 2-dimensional complex  $\Delta$ .

$\Delta$  is a union of **apartments**: flat subcomplexes isomorphic to a tessellation of  $\mathbb{R}^2$  by equilateral triangles.



The boundary  $\partial\Delta$  is a compact totally disconnected space whose points correspond to sectors in  $\Delta$  based at a fixed vertex  $v$ .



## The boundary algebra $\mathcal{A}(\Gamma)$ .

If  $\Gamma$  is a torsion free lattice in  $\mathrm{PGL}_3(\mathbb{Q}_p)$  then  $\Gamma$  acts freely on  $\Delta$ , the universal cover of the 2-dimensional complex  $X = \Gamma \backslash \Delta$ .

$X$  is determined by  $\Gamma$ , by Strong Rigidity.

$\Gamma$  acts on  $\Delta$ , and on  $\partial\Delta$ . Define

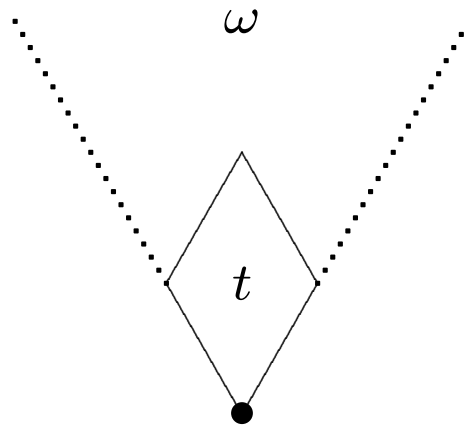
$$\mathcal{A}(\Gamma) := C(\partial\Delta) \rtimes \Gamma.$$

[Depends only on  $\Gamma$ .]

The algebras  $\mathcal{A}(\Gamma) = C(\partial\Delta) \rtimes \Gamma$  are examples of **higher rank Cuntz-Krieger algebras** whose structure theory has been developed by G. Robertson and T. Steger (1998-2001).

Given a basepointed tile  $t = \diamond$  in  $\Delta$ ,

let  $\Omega_t$  be the set of all  $\omega \in \partial\Delta$  such that



Let

$p_t = \text{char. function of } \Omega_t \in C(\partial\Delta) \subseteq \mathcal{A}(\Gamma).$

Then

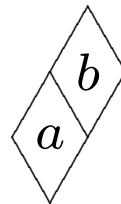
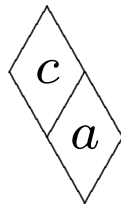
$$\gamma p_t \gamma^{-1} = p_{\gamma t}$$

so

$[p_t] \in K_0$  depends only on  $\Gamma t$ .

Let  $A := \{\Gamma t : t \text{ a tile}\}$ , (*a finite alphabet*).

Define 0–1 matrices  $M_1, M_2$ , for  $a, b \in A$  by



$$M_2(a, c) = 1$$

$$M_1(a, b) = 1$$

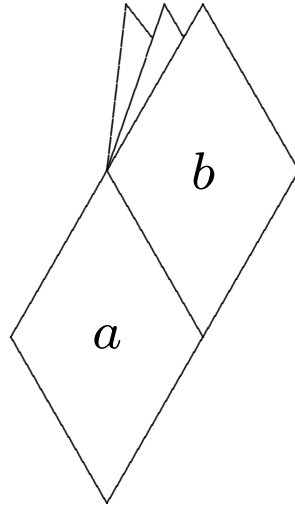
For  $a = \Gamma t \in A$ , let  $[a] := [p_t] \in K_0(\mathcal{A}(\Gamma))$ .

Relations:

- $[a] = \sum_{b \in A} M_1(a, b)[b]$  ;

- $[a] = \sum_{b \in A} M_2(a, b)[b]$  .

These are the **only** relations ...



$$[a] = \sum_{b \in A} M_1(a, b)[b];$$

Let

$$C = \left\langle A \left| a = \sum_{b \in A} M_j(a, b)b, j = 1, 2 \right. \right\rangle .$$

**Theorem.** [G. Robertson, T. Steger, 2001]

$$K_0(\mathcal{A}(\Gamma)) = C \oplus \mathbb{Z}^{\text{rank}(C)}.$$

**Note :** There are generators of  $K_0$  **not** of the form  $[a] = [p_t]$ .

**Theorem.**  $m \cdot [1] = 0$  in  $K_0(\mathcal{A}(\Gamma))$ , where

$$m = \gcd(3, p - 1) \cdot \frac{(p^2 - 1)}{3} \cdot \#\{\text{vertices of } \Gamma \setminus \Delta\}$$

Strong numerical evidence suggests that the order of  $[1]$  is actually :

$$\frac{(p - 1)}{\gcd(3, p - 1)} \cdot \#\{\text{vertices of } \Gamma \setminus \Delta\}$$

**Note:**

$$\chi(\Gamma \setminus \Delta) = \frac{(p - 1)(p^2 - 1)}{3} \cdot \#\{\text{vertices of } \Gamma \setminus \Delta\}.$$

**Example:** The simplest possible  $\Gamma$  has generators  $x_0, x_1, \dots, x_6$ , and relations

$$\begin{cases} x_0x_1x_4, x_0x_2x_1, x_0x_4x_2, x_1x_5x_5, \\ x_2x_3x_3, x_3x_5x_6, x_4x_6x_6. \end{cases}$$

- $\Gamma$  is a torsion free lattice in  $\mathrm{PGL}_3(\mathbb{Q}_2)$ .
- $\Gamma$  acts transitively on vertices of  $\Delta$ .

$$\begin{aligned} K_0(\mathcal{A}(\Gamma)) &= (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z}, \\ [1] &= 0 \end{aligned}$$

$\exists$  exactly 3 such  $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_2)$ .

(Cartwright, Mantero, Steger, Zappa, 1993)

3 different groups  $K_0(\mathcal{A}(\Gamma))$ :

$$\mathbb{Z}/3\mathbb{Z} \quad (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z} \quad (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/3\mathbb{Z}$$

**Other affine buildings:** the boundary algebras  $\mathcal{A}(\Gamma)$  are again simple and purely infinite, but  $K_0(\mathcal{A}(\Gamma))$  is harder to compute. However

**Theorem.** Let  $G$  be a semisimple Chevalley group over  $\mathbb{Q}_p$ . Let  $\Gamma$  be a lattice in  $G$ . Then  $[1]$  has finite order in  $K_0(\mathcal{A}(\Gamma))$ .

If  $G$  is not type  $\tilde{E}_8$  or  $\tilde{F}_4$ , and  $\Gamma$  is torsion free, then

$$\text{order of } [1] < \#\{\text{faces of } \Gamma \backslash \Delta\}.$$

**Continuous Analogue:**  $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ , the fundamental group of a Riemann surface  $M$ .

$\Gamma$  acts on the Poincaré upper half-plane

$$\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}.$$

and on  $\partial\mathfrak{H} = \mathbb{R} \cup \{\infty\} = \mathbb{S}^1$ . Let

$$\mathcal{A}(\Gamma) = C(\mathbb{S}^1) \rtimes \Gamma.$$

**Fact:** The class  $[1] \in K_0(\mathcal{A}(\Gamma))$  has order  $-\chi(M)$ . (A. Connes; T. Natsume)

**Question :** Is  $[1]$  always torsion for geometric boundary algebras?

**Example:**  $G = \mathrm{PSL}_2(\mathbb{C})$  acts on hyperbolic 3-space and its boundary  $\mathbb{S}^2$ .

If  $\Gamma < G$  is a countable discrete subgroup then  $[1]$  is **not** torsion in  $K_0(\mathcal{A}(\Gamma))$  (A. Connes).

## Proof of tree case

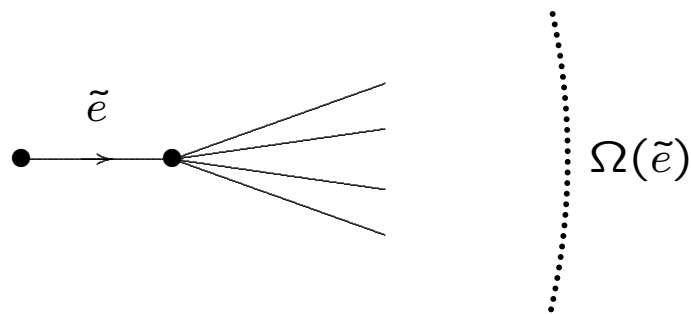
$$-\chi(\Gamma).[1] = 0 \text{ in } K_0(\mathcal{A}(\Gamma))$$

$$\tilde{E} = \tilde{E}_+ \sqcup \tilde{E}_- = \{\text{oriented edges of } \Delta\}$$

$$\tilde{V} = \{\text{vertices of } \Delta\}$$

$E, V$  : oriented edges, vertices of  $X = \Gamma \setminus \Delta$

If  $\tilde{e} \in \tilde{E}$ ,  $\Omega(\tilde{e})$  is a clopen subset of  $\partial\Delta$ :



The indicator function  $p_{\tilde{e}} \in C(\partial\Delta) \subset \mathcal{A}(\Gamma)$ .

$[p_{\tilde{e}}] \in K_0(\mathcal{A}(\Gamma))$  depends only on  $e = \Gamma\tilde{e} \in E$ .

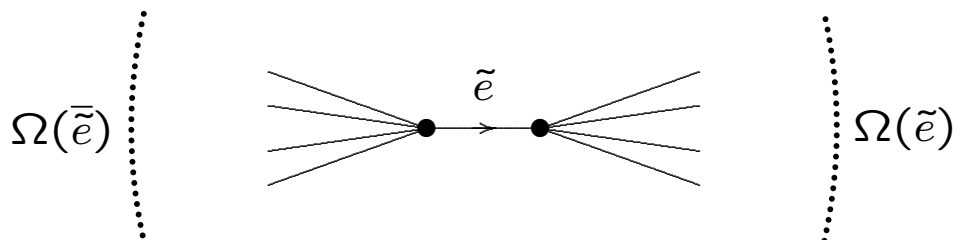
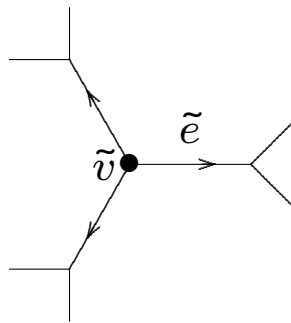
Reason:  $p_{\gamma\tilde{e}} = \gamma \cdot p_{\tilde{e}} = \gamma p_{\tilde{e}} \gamma^{-1}$ .

Therefore write  $[e] = [p_{\tilde{e}}] \in K_0(\mathcal{A}(\Gamma))$ .

The idempotents  $p_{\tilde{e}}$  satisfy the following relations

$$\sum_{\substack{\tilde{e} \in \tilde{E} \\ o(\tilde{e}) = \tilde{v}}} p_{\tilde{e}} = \mathbf{1}, \quad \text{for } \tilde{v} \in \tilde{V}; \quad (1a)$$

$$p_{\tilde{e}} + p_{\bar{\tilde{e}}} = \mathbf{1}, \quad \text{for } \tilde{e} \in \tilde{E}. \quad (1b)$$



The relations (1) project to the following relations in  $K_0(\mathcal{A}(\Gamma))$ .

$$\sum_{\substack{e \in E \\ o(e)=v}} [e] = [\mathbf{1}], \quad \text{for } v \in V; \quad (2a)$$

$$[e] + [\bar{e}] = [\mathbf{1}], \quad \text{for } e \in E. \quad (2b)$$

Since the map  $e \mapsto o(e) : E \rightarrow V$  is surjective, the relations (2) imply that

$$\begin{aligned} n_V[\mathbf{1}] &= \sum_{v \in V} \sum_{\substack{e \in E \\ o(e)=v}} [e] = \sum_{e \in E} [e] \\ &= \sum_{e \in E_+} ([e] + [\bar{e}]) = \sum_{e \in E_+} [\mathbf{1}] \\ &= n_{E_+}[\mathbf{1}]. \end{aligned}$$

Therefore  $(n_V - n_{E_+}).[\mathbf{1}] = 0$ .

i.e  $\chi(X).[1] = 0$ .