Distances in Hyperbolic Spaces and Negative Definite Kernels

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"... at the end of the 17^{th} century, controversy raged between the followers of the physics of Descartes and of Newton. Descartes, with his whirlpools, hooked atoms, etc., explained everything and calculated nothing; Newton, with the law of gravitation, calculated everything and explained nothing."

René Thom, 1972.

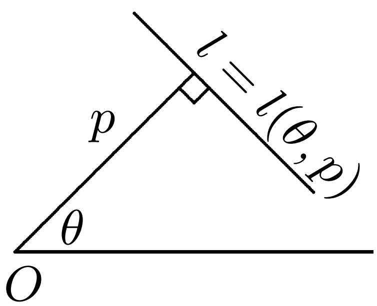
1 Crofton Formula

[M. W. Crofton, *Trans. Royal Soc. London,* **158** *(1868).*]

$$\mathcal{L} = \text{ space of lines } l \text{ in } \mathbb{R}^2,$$

$$G = \text{ group of rigid motions of } \mathbb{R}^2.$$

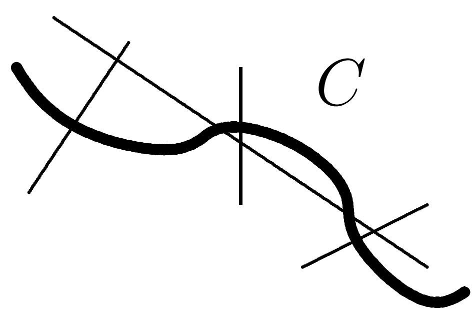
$\mathcal{L}$ has an essentially unique G-invariant measure μ :



$$d\mu = \frac{d\theta dp}{2}$$

The length of a rectifiable curve C is

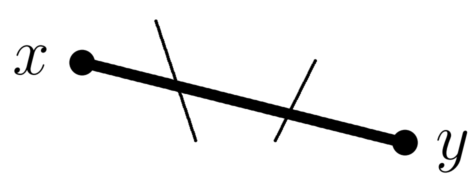
$$\int n(l) d\mu(l)$$



where $n(l) = \#(l \cap C)$.

Corollary.

$$d(x, y) = \mu\{l : l \cap [x, y] \neq \emptyset\}$$



2 Hyperbolic Spaces: $H^n_{\mathbb{F}}$, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

On $\mathbb{F}^{n+1}$ define

$$\langle x, y\rangle = -\overline{x_0}y_0 + \sum_{i=1}^{n} \overline{x_i}y_i$$

$$H^n_{\mathbb{F}} = \left\{x \in \mathbb{F}^{n+1} : \langle x, x\rangle < 0\right\} / \mathbb{F}^{\times}$$

Abuse notation : write x for $[x]$.

Distance in $H^n_{\mathbb{F}}$:

$$\cosh d(x, y) = \frac{|\langle x, y\rangle|}{(\langle x, x\rangle\langle y, y\rangle)^{\frac{1}{2}}}$$

Groups acting isometrically:

$$G = O(1, n);\ U(1, n);\ \mathrm{Sp}(1, n)\,.$$

Important Property:

$H^n_{\mathbb{F}}$ is *2-point homogeneous.*

That is, if $d(x, y) = d(x', y')$ then

$$\exists\, g \in G \quad \text{such that} \quad gx = x',\, gy = y'\,.$$

Other examples of 2-point homogeneous spaces:

$\mathbb{R}^m$, spheres, projective spaces.

3 Real Hyperbolic Space

There is a natural embedding

$$\begin{array}{ccc} & H_{\mathbb{R}}^{n-1} \hookrightarrow H_{\mathbb{R}}^{n} & \\ (x_0, \ldots, x_{n-1}) & \mapsto & (x_0, \ldots, x_{n-1}, 0) \end{array}$$

A **hyperplane** is a G-translate of $H_{\mathbb{R}}^{n-1}$ in $H_{\mathbb{R}}^{n}$.
($\equiv$ totally geodesic submanifold of codimension 1)

$$\begin{aligned} G_0 &:= \text{ stabilizer of } H_{\mathbb{R}}^{n-1} \\ &= O(1, n-1) \times O(1)\,. \\ \mathcal{S} &:= \text{ space of hyperplanes} \cong G/G_0\,. \end{aligned}$$

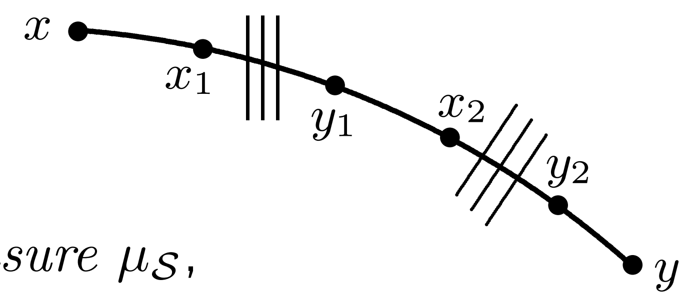
G, G_0 are unimodular locally compact groups.

$\therefore\ \exists$ G-invariant measure $\mu_{\mathcal{S}}$ on $\mathcal{S}$.

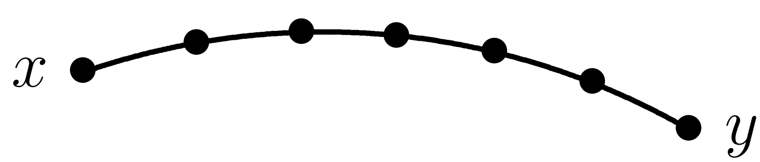
Proposition. There is a Crofton formula for distance in $H^n_{\mathbb{R}}$: $\exists\, k > 0$ such that

$$\mu_{\mathcal{S}}\{S \in \mathcal{S} : S \cap [x, y] \neq \emptyset\} = kd(x, y) \qquad (1)$$

Proof Idea.



By *2-point homogeneity*,
and *invariance of the measure* $\mu_{\mathcal{S}}$,
if $d(x_1, y_1) = d(x_2, y_2)$ then the set of hyperplanes meeting $[x_1, y_1]$ has the same measure as the set of hyperplanes meeting $[x_2, y_2]$.



Taking limits gives (1).

Easy to see $k \neq 0$. Why is $k < \infty$?

Because

$$\{S \in \mathcal{S} : S \cap [x, y] \neq \emptyset\}$$

is compact.

Reason for this: G_0 acts transitively on $H^{n-1}_{\mathbb{R}}$.

4 Half-Spaces

A half-space is a G-translate of the half-space

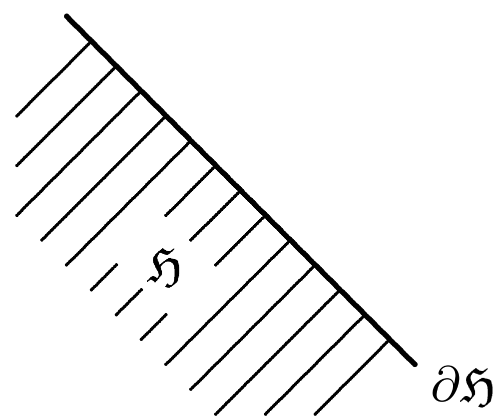
$$\mathfrak{H}_0 = \{(x_0, \ldots, x_n) \in H^n_{\mathbb{R}} : x_n > 0\}$$

$$\begin{aligned} H_0 :=& \text{ stabilizer of } \mathfrak{H}_0 \text{ in } G \\ &\cong O(1, n-1) \text{ is unimodular.} \end{aligned}$$

$\therefore \exists$ G-invariant measure $\mu_{\mathcal{H}}$ on the space $\mathcal{H} \cong G/H_0$ of half-spaces.

There is a natural double cover

$$\begin{array}{ccc} \mathfrak{H} & \to & \partial\mathfrak{H} \\ \mathcal{H} & \to & \mathcal{S} \\ \{\text{half-spaces}\} & & \{\text{hyperplanes}\} \end{array}$$

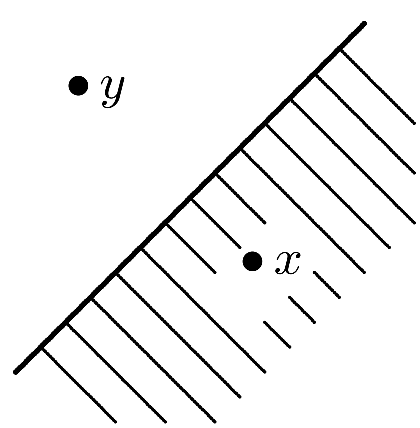


Define $\Sigma_x = \{\mathfrak{H} \in \mathcal{H} : x \in \mathfrak{H}\}$.

Crofton's formula (normalized)

gives

$$\boxed{\mu_{\mathcal{H}}(\Sigma_x \triangle \Sigma_y) = d(x, y)}$$



Consequence :

$$\sqrt{d} \text{ on } H_{\mathbb{R}}^n \text{ is a Hilbert space distance.} \tag{2}$$

Reason:

If $x \in H_{\mathbb{R}}^n$, let

$$\chi_x(\mathfrak{H}) = \begin{cases} 1 & \mathfrak{H} \in \Sigma_x \\ 0 & \mathfrak{H} \notin \Sigma_x \,. \end{cases}$$

Then

$$v_x := \chi_x - \chi_{x_0} \in \mathsf{L}^2(\mathcal{H}, \mu_{\mathcal{H}})\,,$$

where x_0 is fixed,

and

$$\mu_{\mathcal{H}}\left(\Sigma_x \triangle \Sigma_y\right) = \|v_x - v_y\|_{\mathsf{L}^2}^2\,.$$

J. Farault, K. Harzallah (1974) gave a different, analytic proof of (2). They also proved the same result for $H_{\mathbb{C}}^n$.

P. Julg (1998 Bourbaki Seminar) gave yet another proof.

5 Negative Definite Kernels

Let X be a set.

$f : X \times X \to \mathbb{R}$ is a **negative definite kernel** if

$$f(x,y) = f(y,x)\,, \qquad\qquad f(x,x) = 0\,,$$

and, for any finite set of points $x_j \in X$,

$$\sum \alpha_i \alpha_j f(x_i, x_j) \le 0 \qquad \text{if} \qquad \begin{cases} \alpha_j \in \mathbb{R}\,, \\ \sum \alpha_j = 0\,. \end{cases}$$

Example. H a real Hilbert space, $v_x \in H$, $x \in X$.

$$f(x,y) = \|v_x - v_y\|^2 \qquad\qquad \text{is a n.d.k.}$$

Reason:

$$\sum \alpha_i \alpha_j f(x_i, x_j) = -2 \left\| \sum_i \alpha_i v_{x_i} \right\|^2$$

if

$$\sum \alpha_i = 0\,. \qquad\qquad \text{[CHECK]}$$

All n.d.k.'s arise in this way!

Theorem. [I.J. Schoenberg, 1938] A metric space (X, d) embeds in a Hilbert space $\Leftrightarrow$ d^2 is a n.d.k.

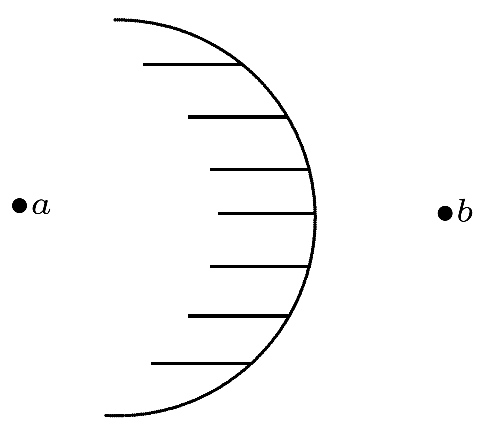
Consequence : d on $H^n_{\mathbb{R}}$ is a n.d.k.

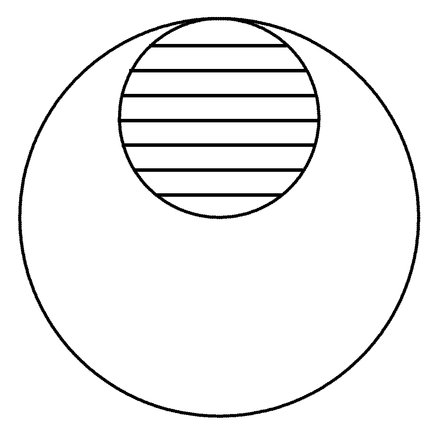
Can my proof work for $H^n_{\mathbb{C}}$?

Problem : Show $d(x, y)$ is a n.d.k. by the *method of half-spaces*? [True result, by Farault-Harzallah.]

Obvious candidates for half-spaces are

1. EQUIDISTANT HALF-SPACES $\{x : d(x, a) \le d(x, b)\}$
2. HOROBALLS (*balls centred at* ∞)





These do **not** work : $\mu\left(\Sigma_x \triangle \Sigma_y\right) = \infty$.

6 Hypermetrics

Return to what we proved for $H^n_{\mathbb{R}}$.

Here is an abstract necessary condition for a distance to arise from a measure on a space of "half-spaces".

Let X be a set, and $d : X \times X \to \mathbb{R}_+$.

Proposition.[J. B. Kelly, 1970] Suppose there is a measure space (Ω, μ) containing measurable sets Σ_x, $x \in X$, such that $d(x, y) = \mu(\Sigma_x \triangle \Sigma_y)$. Then d is **hypermetric**.

That is,

$$\sum t_i t_j d(x_i, x_j) \leq 0\,,$$

if $x_1, \ldots, x_k \in X,\ t_1, \ldots, t_k \in \mathbb{Z}$ and $\sum t_j = 1$.

Proof.

$$d(x,y) = \int_\Omega \Big| \chi_{\Sigma_x}(\omega) - \chi_{\Sigma_y}(\omega) \Big| \, d\mu(\omega).$$

Therefore

$$\sum t_i t_j d(x_i, x_j) = \int_\Omega \sum_{i,j} t_i t_j \Big| \chi_{\Sigma_{x_i}}(\omega) - \chi_{\Sigma_{x_j}}(\omega) \Big| \, d\mu(\omega).$$

We show the integrand is ≤ 0.

Fix $\omega \in \Omega$. Define $\delta : X \to \{0,1\}$ by

$$\delta(x) = \chi_{\Sigma_x}(\omega)\,.$$

$$\begin{aligned} \sum t_i t_j \,|\delta(x_i) - \delta(x_j)| &= 2 \left(\sum_{\delta(x_i)=1} t_i \right) \left(\sum_{\delta(x_j)=0} t_j \right) \\ &= 2PQ \quad \text{where} \begin{cases} P,\, Q \in \mathbb{Z}, \\ P + Q = 1 \end{cases} \\ &\leq 0\,. \end{aligned}$$

What does the hypermetric property mean?

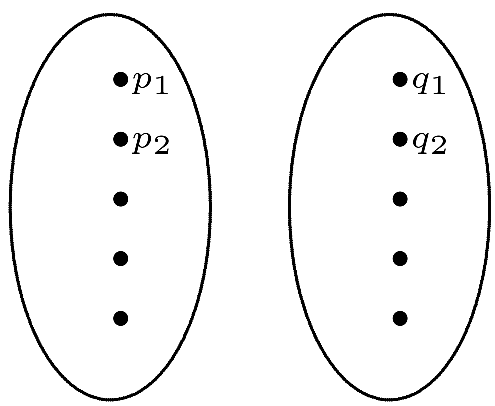
Let

$$\{x_1, \ldots, x_{2n+1}\} = \{p_1, \ldots, p_n\} \cup \{q_1, \ldots, q_{n+1}\}$$

$$t_1 = \cdots = t_n = -1\,; \qquad t_{n+1} = \cdots = t_{2n+1} = +1\,.$$

$$\implies \qquad \sum(-1)^2 d(p_i, p_j) + \sum(1)^2 d(q_i, q_j) + \sum(-1)(1) d(p_i, q_j) \leq 0$$

$$\implies \sum d(p_i, p_j) + \sum d(q_i, q_j) \leq \sum d(p_i, q_j)$$



$n = 1 \Rightarrow$ triangle inequality.

$d(x, y) = \mu(\Sigma_x \triangle \Sigma_y) \Rightarrow d$ hypermetric

$\Rightarrow d$ negative definite.

Is d on $H^n_{\mathbb{C}}$ hypermetric?

Computations with $\sim 10^4$ points seem to show the question is finely balanced.

7 Quaternionic Hyperbolic Space

$G = \mathrm{Sp}(1,n)$ acts isometrically on $H^n_{\mathbb{H}}$.

B. Kostant (1969) proved a "rigidity" result :

G has **Kazhdan's property (T)**. That is,

$$\boxed{\text{every negative definite } f : G \to \mathbb{R} \text{ is bounded.}}$$

[f *negative definite* means $(g,h) \mapsto f(g^{-1}h)$ is n.d.k.]

Now $g \mapsto d(gx_0, x_0)$ is unbounded. Therefore

$$\boxed{d(x,y) \text{ is } \mathbf{not} \text{ negative definite.}}$$

Direct Proof :

$$\{p_1, \ldots, p_{12}\} = \{\, (3, \pm 2 \pm 2\varepsilon, 0)\,;\ \varepsilon = i, j, k\,\}$$

$$\{p_{13}, \ldots, p_{24}\} = \{\, (3, 0, \pm 2 \pm 2\varepsilon)\,;\ \varepsilon = i, j, k\,\}$$

$$t_1 = \cdots = t_{12} = -1\,;$$

$$t_{13} = \cdots = t_{24} = +1$$

$$\sum t_i t_j d(p_i, p_j) = 417.031 - 415.767$$

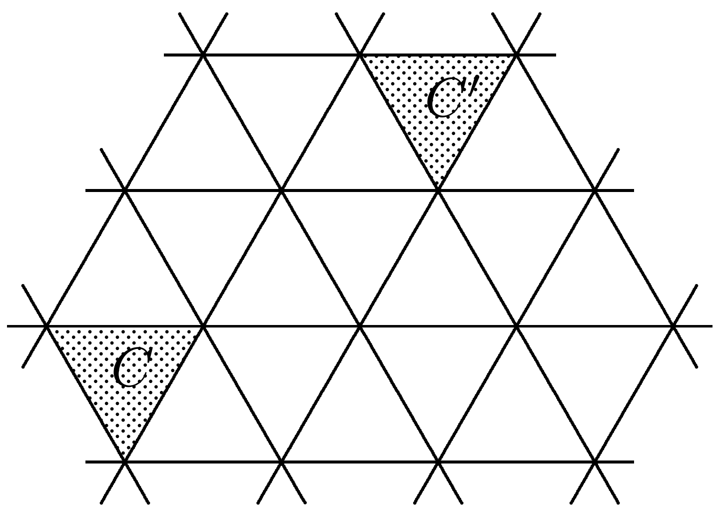
$$> 0.$$

To show that a group G does **not** have property (T), find a geometrical object X on which G acts, and use the method of $1/2$-spaces to construct an unbounded G-invariant negative definite kernel.

E.g. W an infinite Coxeter group.

$X =$ Coxeter complex.

There is a "Crofton formula" for distance between maximal simplices.



$$d(C, C') = 6$$

$d(C, C') =$ no of walls separating C and C'.

$\Longrightarrow W$ does not have property (T), if W is infinite.

This method seems to be almost universally applicable.

Does it really fail for $U(1, n)$?

2-Point Homogeneous Riemannian Manifolds

	d n.d.k.?	d hypermetric?
$\mathbb{R}^n$	Y	Y
S^n	Y	Y
$H^n_{\mathbb{R}}$	Y	Y
$H^n_{\mathbb{C}}$	Y	?
$H^n_{\mathbb{H}}$	N	N
$H^2_{\mathbb{O}}$	N	N
$P^n_{\mathbb{F}}$	N[a]	N

a : Fails for 6 points

The projective plane $P^2_{\mathbb{R}} = \mathbb{R}^3 \big/ \mathbb{R}^\times$

Distance : $\cos d(x,y) = \frac{|(x\cdot y)|}{((x\cdot x)(y\cdot y))^{\frac{1}{2}}}$.

To show d is not n.d.k., choose points so that

$$\sum d(p_i, p_j) + \sum d(q_i, q_j) > \sum d(p_i, q_j).$$

Let

$p_1 = (1,0,1)$, $p_2 = (1,0,-1)$, $p_3 = (0,1,0)$,

and

$q_1 = (0,1,1)$, $q_2 = (0,1,-1)$, $q_3 = (1,0,0)$.

Then

$d(p_i, p_j) = d(q_i, q_j) = \pi/2 \qquad$ (*the diameter of* $P^2_{\mathbb{R}}$).

$\sum d(p_i, p_j) + \sum d(q_i, q_j) = 3\pi/2 + 3\pi/2 = 3\pi$

and

$\sum d(p_i, q_j) = 4\pi/3 + 4\pi/4 + \pi/2 = 17\pi/6$

NOTE. There is a Crofton formula for geodesic distance d on the space $P^n_{\mathbb{R}}$. The method of half-spaces does not work: a hyperplane does not have two sides.