Trees, Buildings and Boundaries

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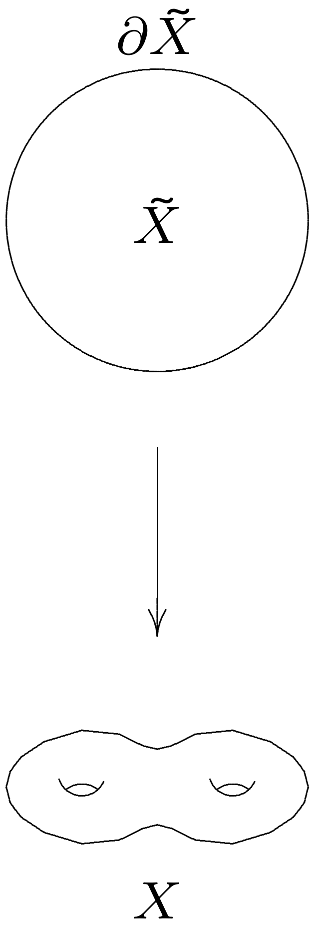
MOTIVATION



Γ acts on $\partial\tilde{X}$

ANALYSIS

$\Gamma = \pi(X)$

GEOMETRY

What does $(\Gamma, \partial\tilde{X})$ reveal about X?

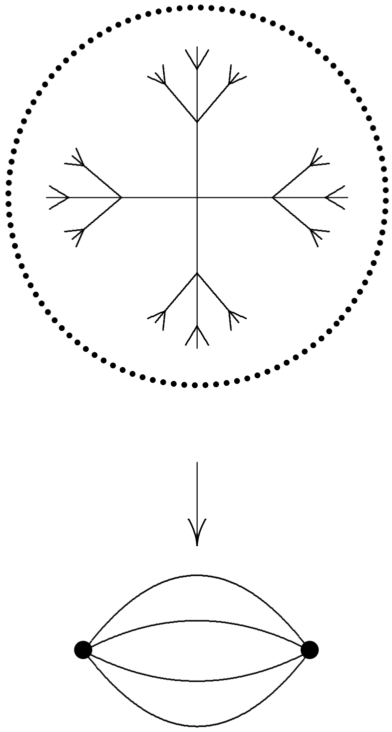
Example

X : A finite connected graph.

$\tilde{X}$: The universal covering space (a tree).

$\partial\tilde{X}$: The boundary of $\tilde{X}$.

$\tilde{X}$ and $\partial\tilde{X}$

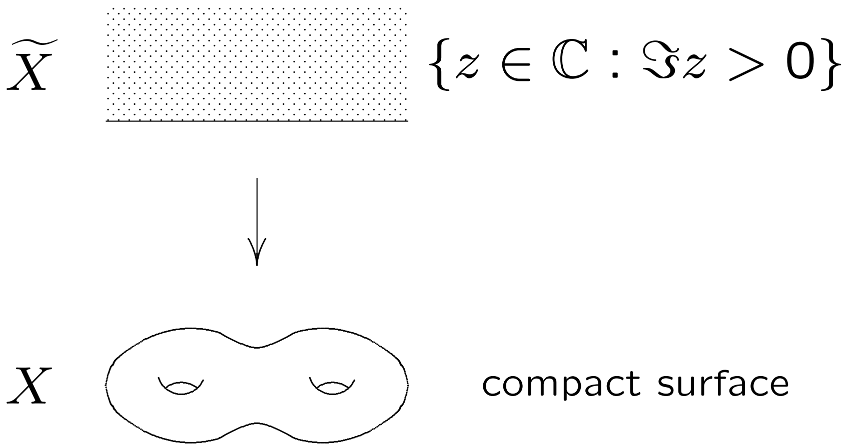


X

The **fundamental group** Γ of X is a **free group** which acts on $\widetilde{X}$ and

$$\Gamma\backslash\widetilde{X} = X.$$

Continuous Analogue



$\mathrm{PSL}_2(\mathbb{R})$ acts on $\widetilde{X}$ via

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}.$$

The fundamental group Γ of X embeds as a **lattice** in $\mathrm{PSL}_2(\mathbb{R})$: a discrete subgroup of finite covolume.

Replace $\mathbb{R}$ by a **local field** ...

For p prime, $\mathbb{Q}_p$ is the field of formal sums

$$x = a_j p^j + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots \,,$$

where each $a_i \in \{0, 1, \cdots, p-1\}$ and $a_j \neq 0$.

$$\begin{aligned} |x| &= p^{-j} \qquad \text{if } x \neq 0\,, \\ |0| &= 0 \end{aligned}$$

The p-adic integers

$$\begin{aligned} \mathbb{Z}_p &= \{x \in \mathbb{Q}_p : |x| \leq 1\} \\ &= \text{ set of sums with } j \geq 0 \\ &= \overline{\mathbb{Z}} \qquad\qquad \textit{a compact subring} \end{aligned}$$

Ultrametric Property:

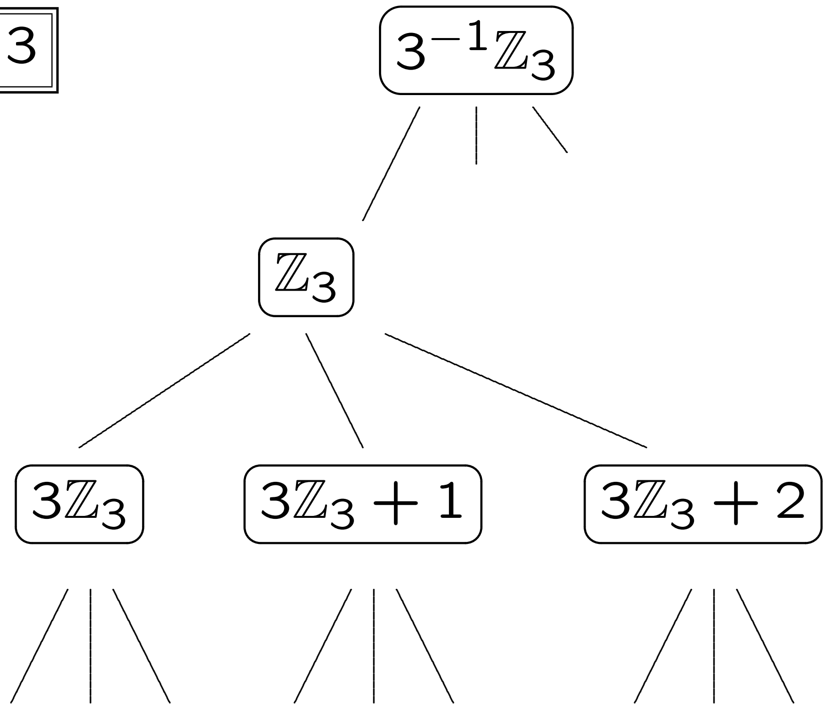
$$|x + y| \leq \max(|x|, |y|)$$

Two balls are either disjoint or one is contained in the other

⇓

Tree structure on the set of balls in $\mathbb{Q}_p$

$p = 3$



. . . a $(p+1)$-regular tree Δ.

The tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$

The group $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on its **building** Δ, which is a homogeneous tree of degree $p+1$.

A **vertex** is a maximal compact subgroup K of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

e.g. $K = \mathrm{PGL}_2(\mathbb{Z}_p)$.

An **edge** is (K, K') where $K \cap K'$ is a maximal proper subgroup of K and K'.

$\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on Δ via

$$K \mapsto g^{-1} K g$$

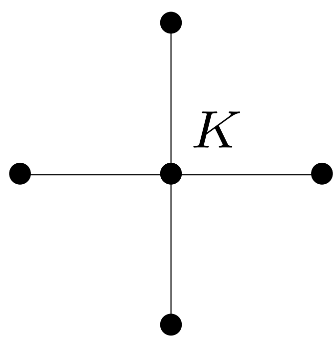
Example

In $\mathrm{PGL}_2(\mathbb{Q}_3)$, a vertex K has four neighbours

$$g^{-1}Kg$$

where

$$g = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

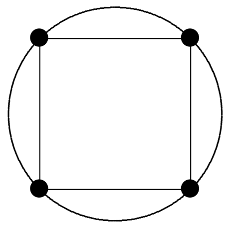


Non-rigidity

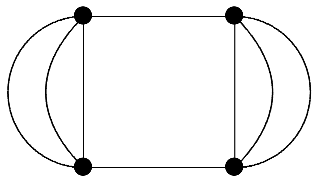
$\mathsf{PGL}_2(\mathbb{Q}_3)$ contains lattices

$$\Gamma_X \cong \Gamma_Y \cong F_5$$

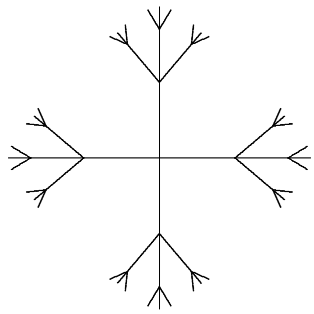
with

$\Delta/\Gamma_X =$ 

X

$\Delta/\Gamma_Y =$ 

Y

where $\Delta =$ 

Γ_X, Γ_Y are not conjugate in $\mathsf{PGL}_2(\mathbb{Q}_3)$.

The boundary action

The action of Γ on Δ extends to $\partial\Delta$.

Let $\Gamma < \mathrm{PGL}_2(\mathbb{Q}_p)$ be a torsion free lattice.

Then

$$\Gamma \cong F_r$$

a free group of rank r.

Study the action of Γ on $\partial\Delta$ by forming the **crossed product C^*-algebra**

$$\mathcal{A} = C(\partial\Delta) \rtimes \Gamma$$

generated by

$C(\partial\Delta) \subset \mathcal{A}(\Gamma)$ an abelian subalgebra

$\Gamma \subset \mathcal{A}(\Gamma)$ a group of unitaries

such that

$$f(\gamma^{-1}t) = (\gamma f \gamma^{-1})(t),$$

$$\gamma \in \Gamma, f \in C(\partial\Delta).$$

$\mathcal{A}$ is classified by the group $K_0(\mathcal{A})$.

Theorem. $K_0(\mathcal{A}) = \mathbb{Z}^r \oplus \mathbb{Z}/(r-1)\mathbb{Z}$

... depends only on Γ (weak rigidity).

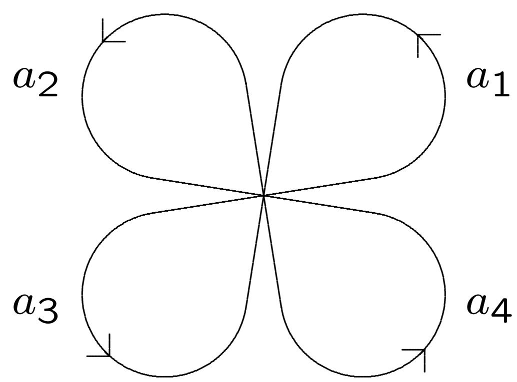
$$K_0(\mathcal{A}) = \{[e] : e \neq 0,\ e^2 = e = e^* \in \mathcal{A}\}$$

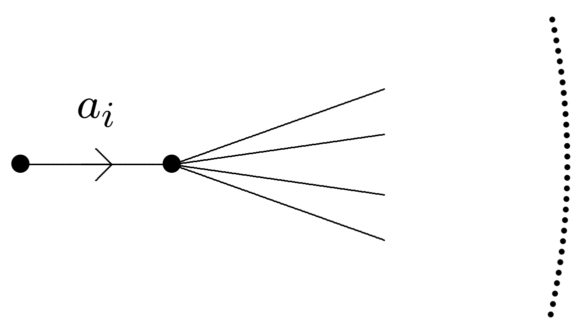
where $e_1 \sim e_2 \iff e_1 = x^*x,\ e_2 = xx^*$.

Note. $K_0(\mathcal{A})$ determines Γ.

In this case:

- the class of $\mathbf{1}$ generates $\mathbb{Z}/(r-1)\mathbb{Z}$;
- each factor of $\mathbb{Z}^r$ corresponds to a generator a_i of F_r.





The building of $\mathsf{PGL}_3(\mathbb{Q}_p)$

. . . a simply connected simplicial complex Δ, with $\dim \Delta = 2$.

A **vertex** is a maximal compact subgroup K of $\mathsf{PGL}_3(\mathbb{Q}_p)$.

e.g. $\quad K = \mathsf{PGL}_3(\mathbb{Z}_p)$.

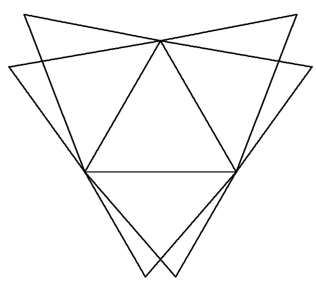
An **edge** is (K, K') where $K \cap K'$ is a maximal proper subgroup of K and K'.

$\mathsf{PGL}_3(\mathbb{Q}_p)$ acts on Δ via

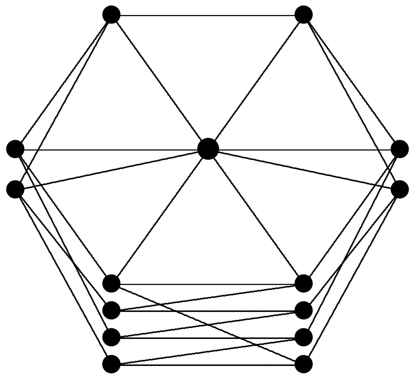
$$K \mapsto g^{-1} K g$$

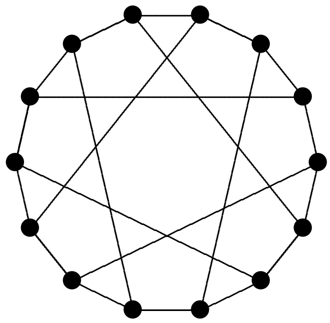
Each edge lies on $p + 1$ triangles.

$p = 2$:



Neighbours of a vertex ($p = 2$)



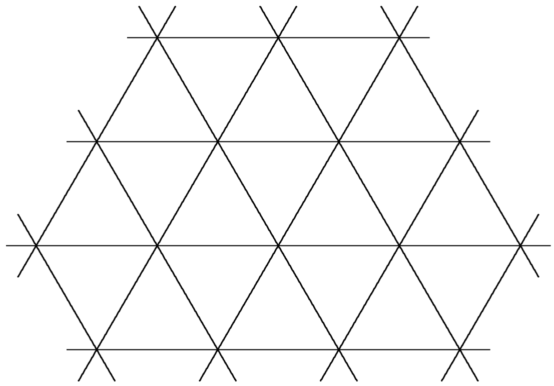


7 point
projective plane

On the left: a ball of radius one.

On the right: a sphere of radius one.

Δ is a union of apartments : flat subcomplexes isomorphic to a tessellation of $\mathbb{R}^2$ by equilateral triangles.



$\widetilde{A}_2$ Coxeter complex

The boundary $\partial\Delta$

The boundary of Δ is

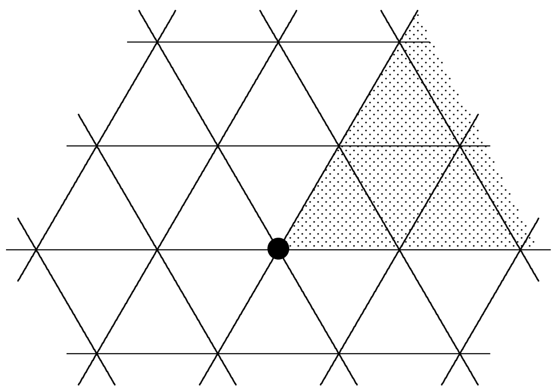
$$\partial\Delta \cong G/B$$

where $G = \mathrm{PGL}_3(\mathbb{Q}_p)$ and

$$B = \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix} \cap G.$$

$\partial\Delta$ is compact and totally disconnected.

Points of $\partial\Delta$ correspond to sectors based at a fixed vertex of Δ.



Rigidity

Let $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_p)$ be a lattice.

The embedding of Γ in $\mathrm{PGL}_3(\mathbb{Q}_p)$ is unique.
(.... *up to conjugacy*)

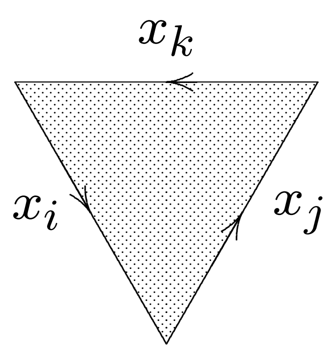
Also Γ does not embed as a lattice in any other $\mathrm{PGL}_n(\mathbb{F})$, where $\mathbb{F}$ is a local field, $n \geq 2$.

Example (D. Mumford 1979)

$\Gamma < \mathrm{PGL}_3(\mathbb{Q}_2)$, with generators $x_0, x_1, \ldots, x_6$, and relators

$$\begin{cases} x_0x_0x_6, x_0x_2x_3, x_1x_2x_6, x_1x_3x_5, \\ x_1x_5x_4, x_2x_4x_5, x_3x_4x_6. \end{cases}$$

$X = \Delta/\Gamma$ is obtained by glueing 7 triangles



where $x_ix_jx_k$ is a relator.

X has Euler Characteristic $\chi(\Gamma) = 1$.

There are **three** torsion-free lattices $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_2)$ with $\chi(\Gamma) = 1$.

The boundary action

Let $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_p)$ be a torsion-free lattice.

The crossed product C^*-algebra

$$\mathcal{A} = C(\partial\Delta) \rtimes \Gamma$$

is classified by $K_0(\mathcal{A})$.

Example. For Mumford's group,

$$K_0(\mathcal{A}) = (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/3\mathbb{Z}\,.$$

The torsion-free lattices $\Gamma < \mathrm{PGL}_3(\mathbb{Q}_2)$ with $\chi(\Gamma) = 1$ are distinguished from each other by $K_0(\mathcal{A})$.

How is $K_0(\mathcal{A})$ related to Γ?

Theorem. $\chi(\Gamma) = 1 + \frac{1}{2}\,\mathrm{rank}\,K_0(\mathcal{A})$.

Remark: If $p > 2$ then $\chi(\Gamma) > 1$.

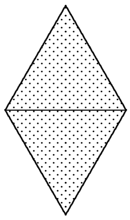
Consequence: $K_0(\mathcal{A})$ distinguishes Mumford's group from **all** other torsion free lattices

$$\Gamma < \mathrm{PGL}_3(\mathbb{Q}_p), \quad p > 1\,.$$

How to compute $K_0(\mathcal{A})$

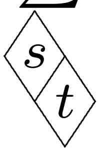
Consider

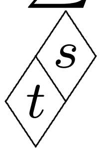
$\mathfrak{D}$: the set of Γ-orbits of oriented tiles in Δ



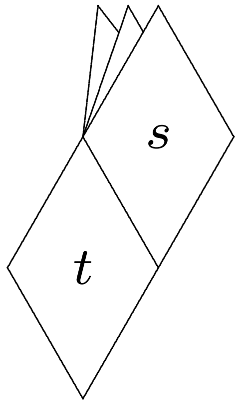
$\mathfrak{C}$: the abelian group with generating set $\mathfrak{D}$ and relations

$$t = \sum s, \qquad t = \sum s\,.$$





Geometrically :

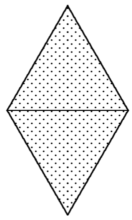


each tile t is a formal sum of tiles s.

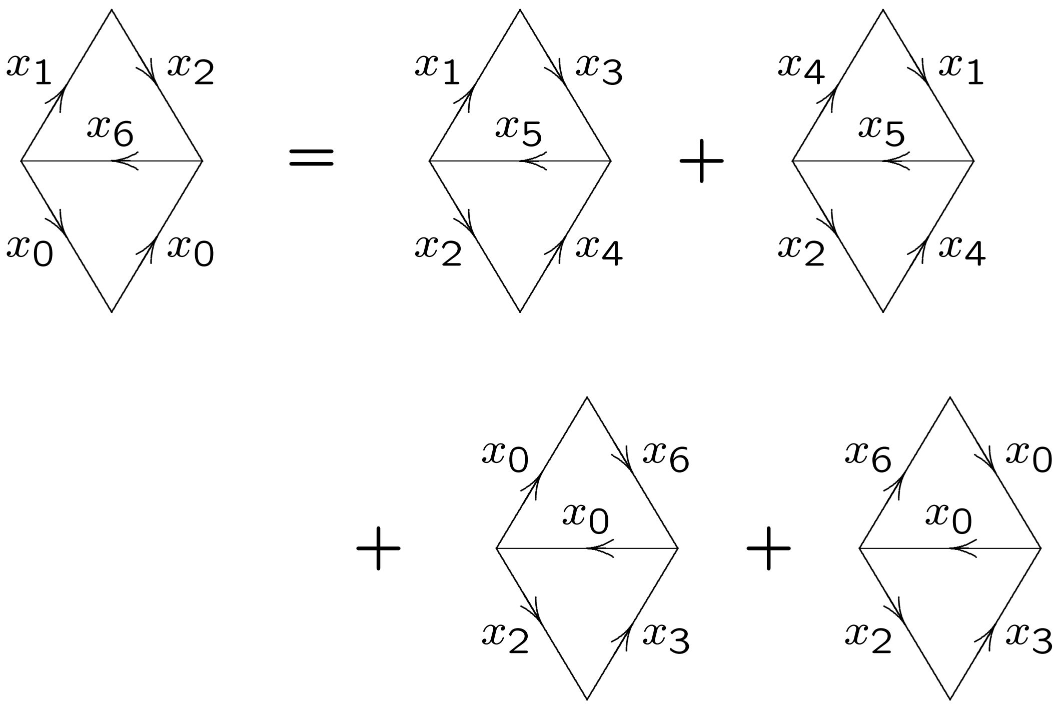
Theorem. $K_0(\mathcal{A}) = \mathfrak{C} \oplus \mathbb{Z}^{\mathrm{rank}(\mathfrak{C})}$.

Example (Mumford's group)

$\mathfrak{C}$ has $7 \times 3 \times 2 = 42$ generators



and $2 \times 42 = 84$ relations of the form



giving

$$\mathfrak{C} = (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/3\mathbb{Z}\,.$$

Typical computation

$\Gamma < \mathrm{PGL}_3(\mathbb{Q}_7)$, the lattice of H. Voskuil satisfying

$$H_1(\Gamma, \mathbb{Z}) = (\mathbb{Z}/3\mathbb{Z})^7$$
$$H_2(\Gamma, \mathbb{Z}) = \mathbb{Z}^{95}$$

Direct computation gives

$$K_0(\mathcal{A}) = \mathbb{Z}^{190} \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})^{21}$$

and $[\mathbf{1}] = \sum_{s \in \mathfrak{D}} s$ has order

$$\frac{(p-1)}{\gcd(3, p-1)} \cdot \mathrm{vol}(G/\Gamma) = 2\,.$$

Strong numerical evidence suggests that the order of $[\mathbf{1}]$ is always given by this formula.

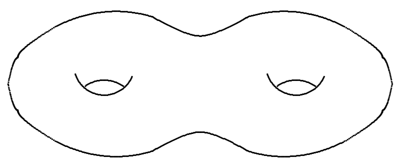
Continuous analogue

The tree is a combinatorial analogue of the Poincaré disc $\mathbb{D}$.

A torsion free cocompact Fuchsian group

$$\Gamma < \mathsf{PSL}_2(\mathbb{R})$$

acts on $\mathbb{D}$ and is the fundamental group of the Riemann surface of genus g.



Γ act on $\partial\mathbb{D} = \mathbb{S}^1$ and the boundary algebra $\mathcal{A}$ satisfies

$$K_0(\mathcal{A}) = \mathbb{Z}^{2g+1} \oplus \mathbb{Z}/(2g-2)\mathbb{Z}\,.$$