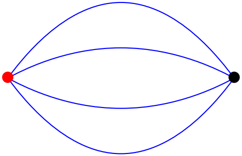
Building centralisers in $\widetilde{A}_2$ groups

Guyan Robertson

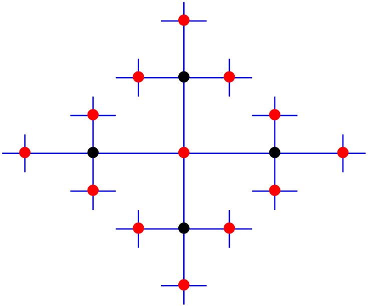
Newcastle University

Example

A finite connected graph X, with universal covering tree Δ.



X



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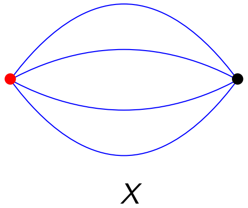

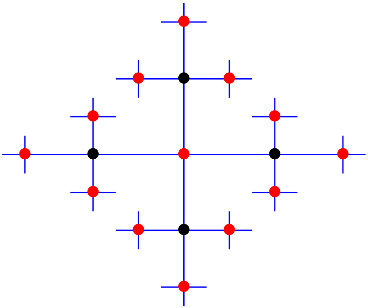

Let $\Gamma = \pi(X)$, the fundamental group of X.
Γ is a free group which acts freely on Δ and

$$\Gamma \backslash \Delta = X$$

Non-free actions - without inversion

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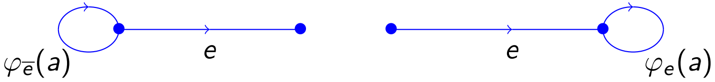
i.e. identify the composite "paths" from $o(e)$ to $t(e)$



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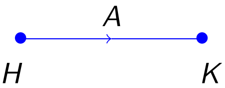

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$$\Gamma = H \underset{A}{*} K$$

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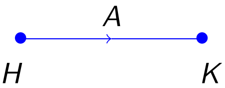

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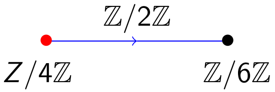
Γ acts on a tree Δ with

- edge stabilizers $\cong A$;
- vertex stabilizers $\cong H, K$.

Sub-Example: $SL_2(\mathbb{Z})$



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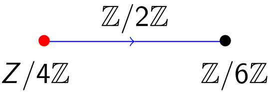
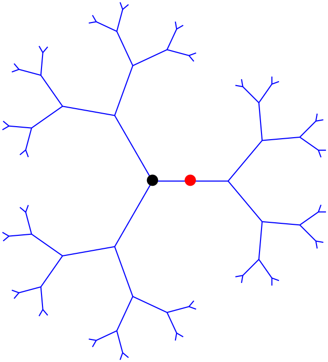


The fundamental group $\mathbb{Z}/4\mathbb{Z} \underset{\mathbb{Z}/2\mathbb{Z}}{*} \mathbb{Z}/6\mathbb{Z}$ acts on



The tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$

Let
$$G = \mathrm{PGL}_2(\mathbb{Q}_p) = \mathrm{GL}_2(\mathbb{Q}_p)/\mathbb{Q}_p^\times$$
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[Serre's proof of Ihara's theorem (1966).]

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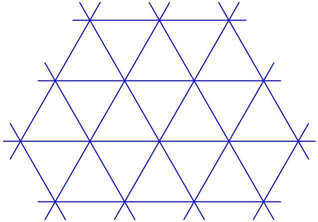
Δ is a building of type $\widetilde{A}_2$

$\widetilde{A}_2$ building

A contractible simplicial complex Δ, $\dim \Delta = 2$.



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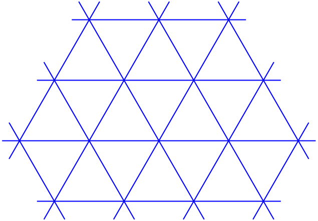

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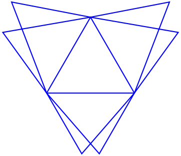
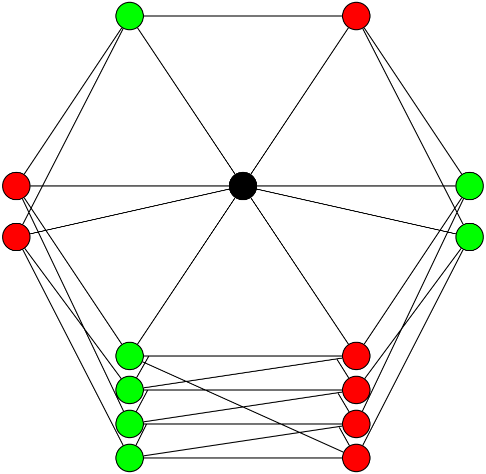


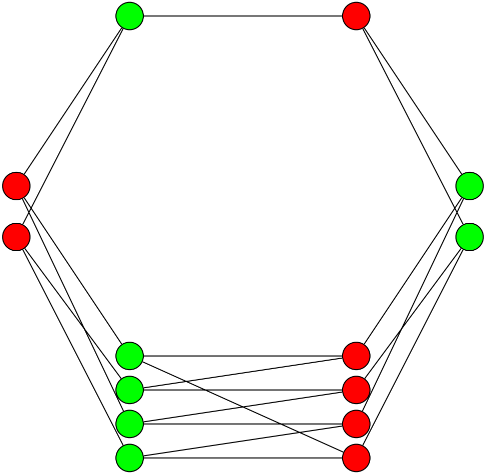
Each edge lies on $p + 1$ triangles.

The neighbours of a vertex ($p = 2$)



A ball of radius one.

THE NEIGHBOURS OF A VERTEX ($p = 2$)



The link of a vertex is a projective plane.

Mumford's group (1979)

D. Mumford constructed $\Gamma_M < \mathrm{PGL}_3(\mathbb{Q}_2)$:

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Γ_M has generators $x_0, x_1, \dots, x_6$, and relators

$$\begin{cases} x_0 x_0 x_6,\; x_0 x_2 x_3,\; x_1 x_2 x_6,\; x_1 x_3 x_5, \\ x_1 x_5 x_4,\; x_2 x_4 x_5,\; x_3 x_4 x_6. \end{cases}$$

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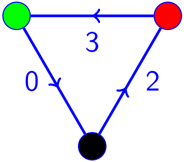
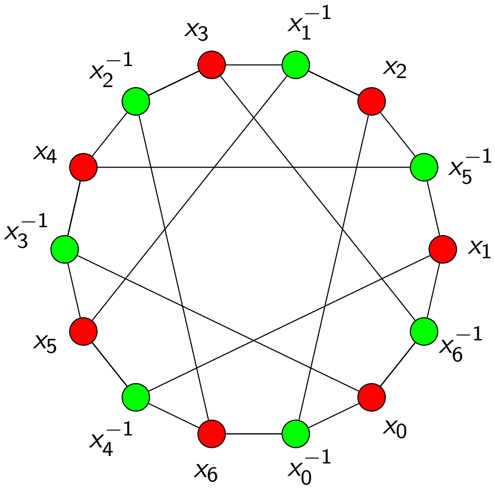
The 1-skeleton of Δ is the Cayley graph of Γ_M.

$x_0x_2x_3 = 1$



The projective plane of neighbours of 1



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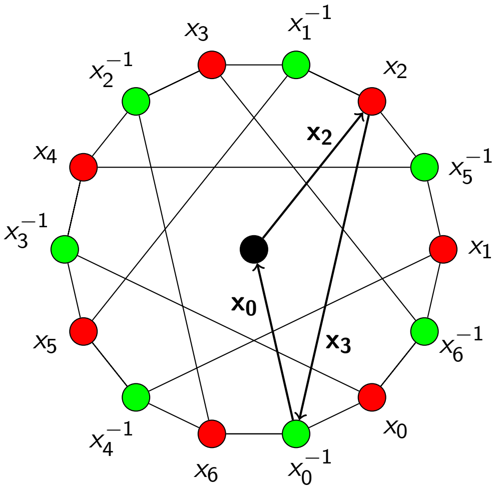

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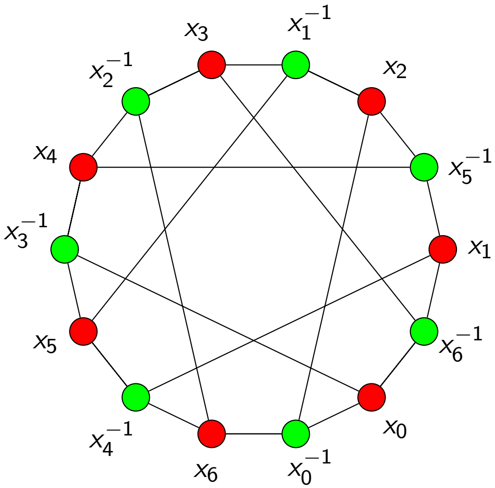

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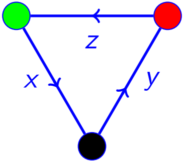

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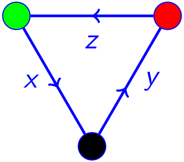

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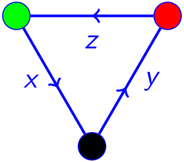

- $p = 2$: 8 groups.
- $p = 3$: 89 groups.

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- The union of all axes of g is a convex set $\mathrm{Min}(g)$.

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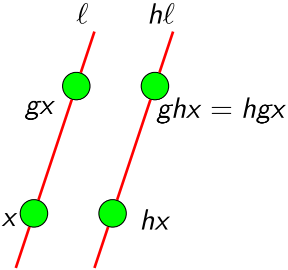
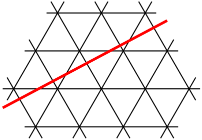


Regular elements

$g \in \Gamma$ is regular if no axis is parallel to a wall of Δ.



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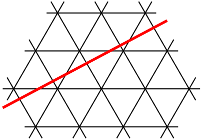

Suppose g is regular and ℓ is an axis for g.

- ℓ lies in a unique apartment $\mathcal{A}$
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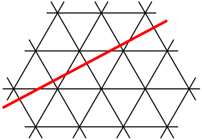

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Therefore $Z_\Gamma(g)$ is a Bieberbach group : virtually $\mathbb{Z}^2$.

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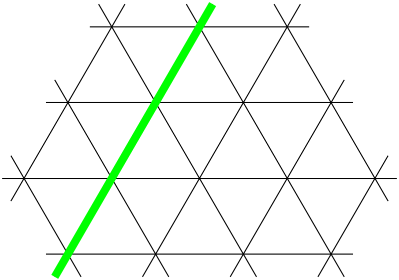

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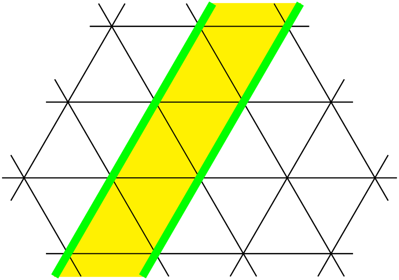
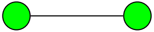


An edge in $\mathcal{T}_g$  is a strip (illustrated).

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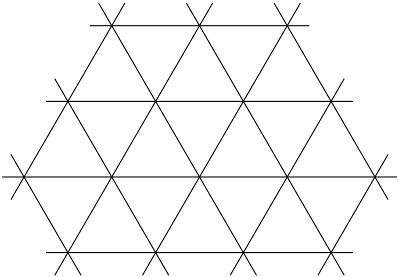

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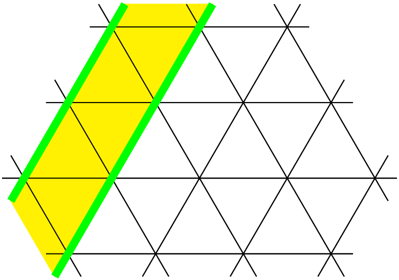

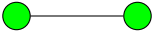

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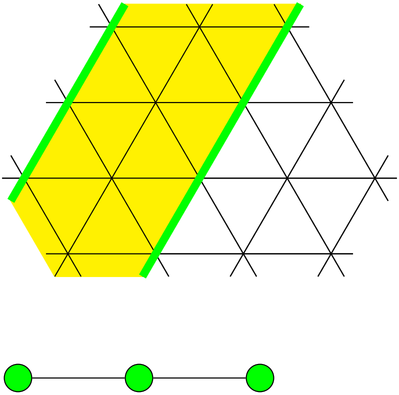

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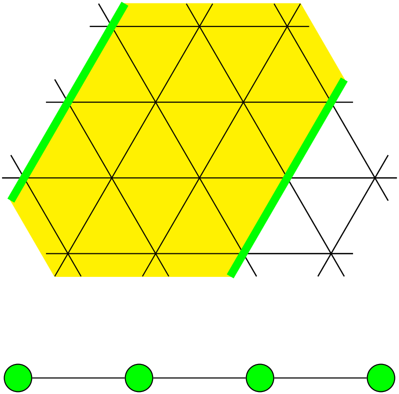

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- $Z_\Gamma(g)$ acts on $\mathcal{T}_g$
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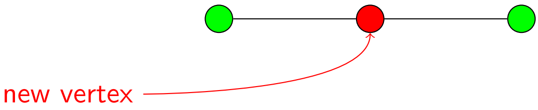
Note: G acts without inversion on barycentric subdivision of $\mathcal{T}_g$.



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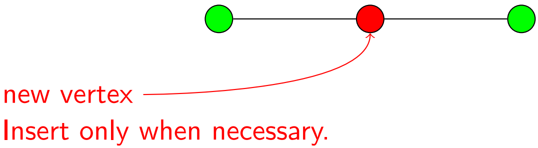

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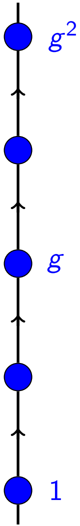

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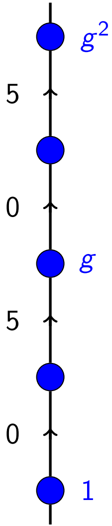
ℓ is a vertex of $\mathcal{T}_g$.



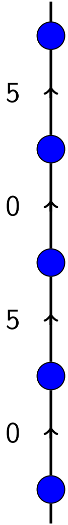
Example

$\Gamma = \Gamma_M$ (Mumford's group) and $g = x_0 x_5$.

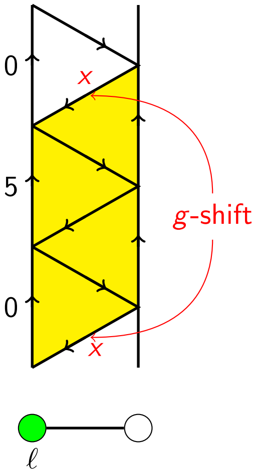
g has an axis ℓ passing through $1, g, g^2, \ldots$.

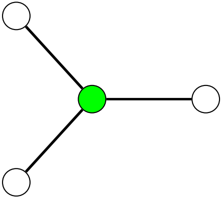
ℓ is a vertex of $\mathcal{T}_g$.

What are the vertices adjacent to ℓ?



Vertices adjacent to ℓ





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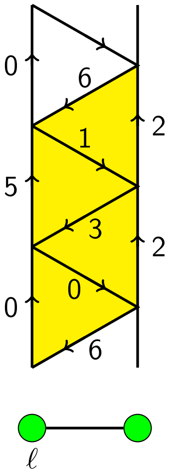
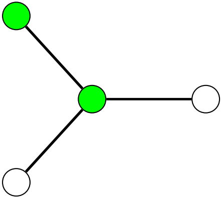
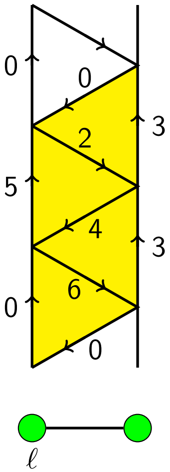
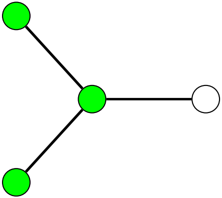


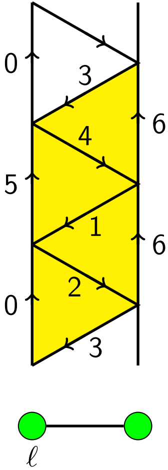


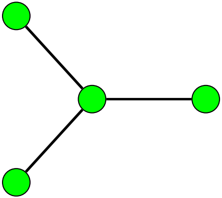
Vertices adjacent to ℓ

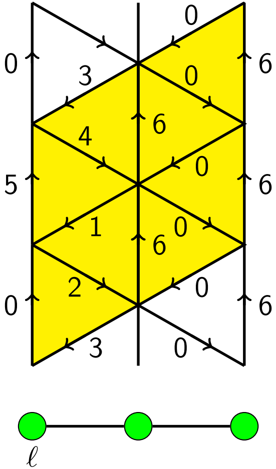


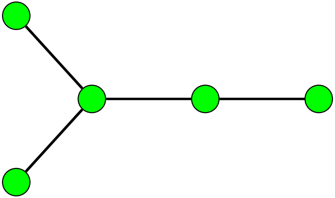


Vertices adjacent to ℓ

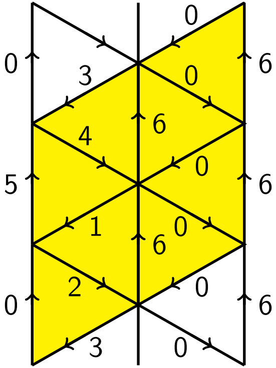


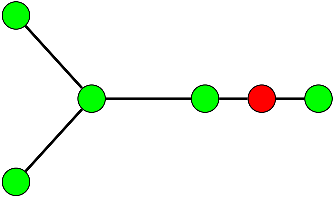


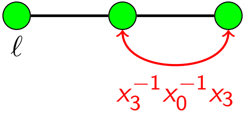




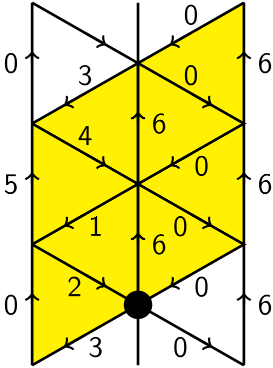
next vertex . . .

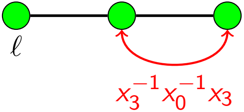


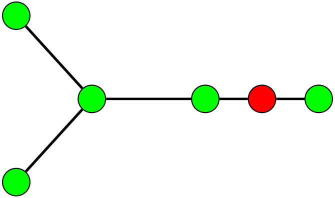




$$(x_3^{-1}x_0^{-1}x_3)^4 = x_0x_5$$







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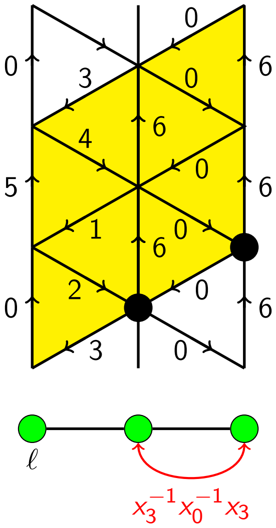

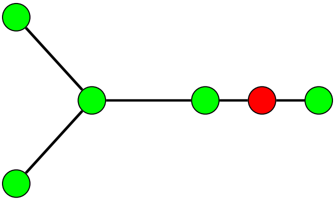

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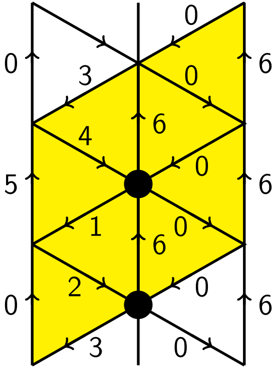

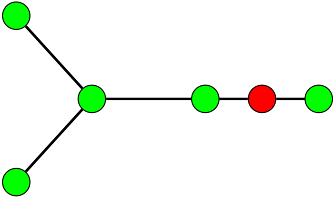

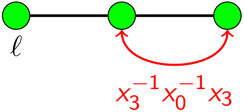

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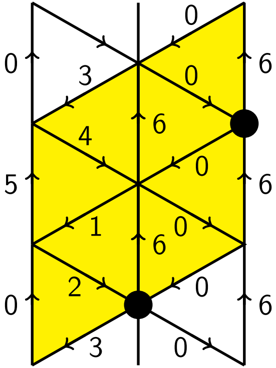

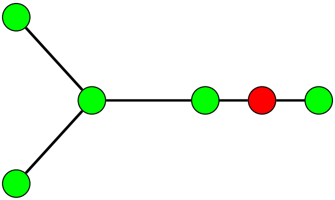

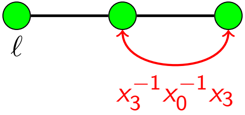

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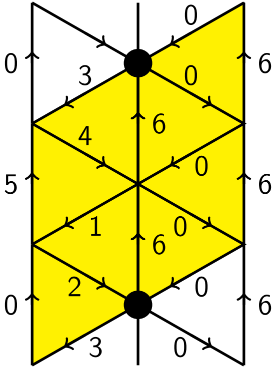

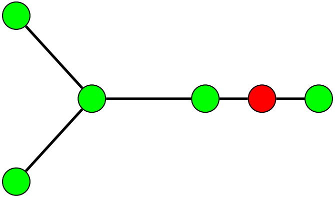

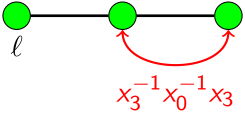

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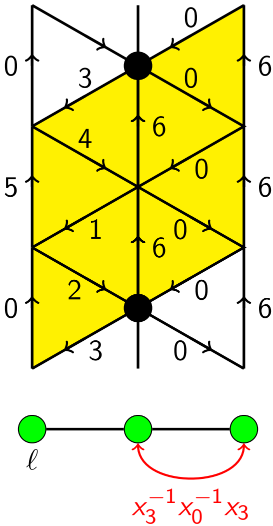

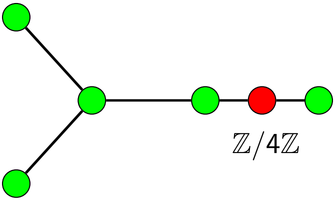

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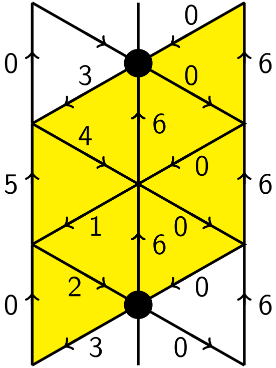
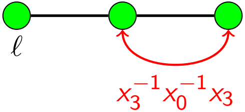
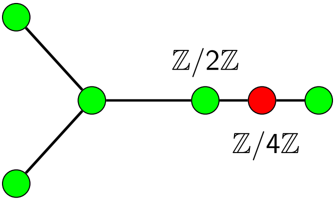


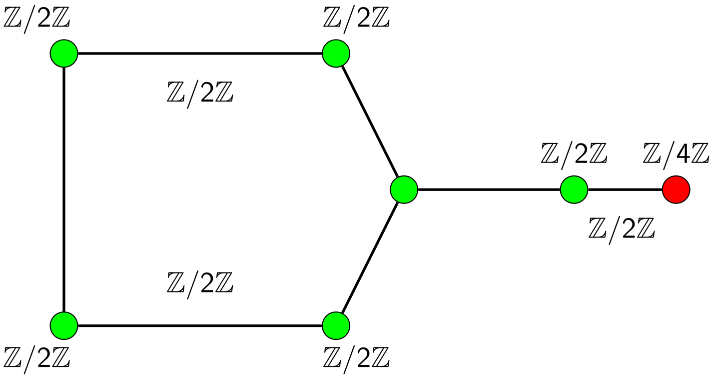




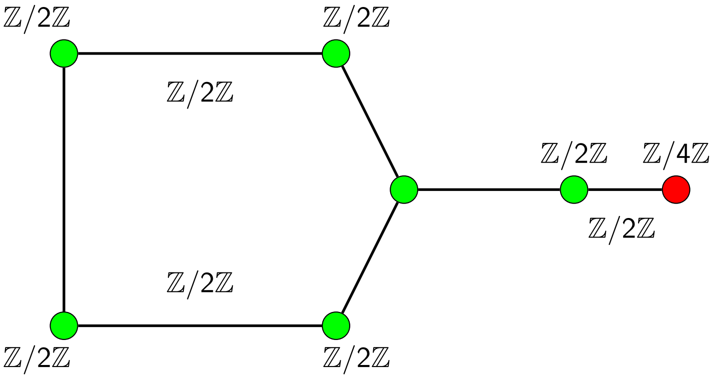
$$(x_3^{-1}x_6x_3)^2 = x_0x_5$$

$$(x_3^{-1}x_0^{-1}x_3)^4 = x_0x_5$$

$G = Z_\Gamma(g)/\langle g \rangle$ acts on $\mathcal{T}_g$, with quotient graph of groups:



$G = Z_\Gamma(g)/\langle g\rangle$ acts on $\mathcal{T}_g$, with quotient graph of groups:



$Z_\Gamma(g)/\langle g\rangle$ is isomorphic to $\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^{*2} * (\mathbb{Z}/4\mathbb{Z})$

References

- D. I. Cartwright, A. M. Mantero, T. Steger and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, I and II, *Geom. Ded.* **47** (1993), 143–166 and 167–223.
- D. I. Cartwright and T. Steger, Enumeration of the 50 fake projective planes, *preprint*, 2009.
- D. Mumford, An algebraic surface with K ample, $(K^2) = 9$, $p_g = q = 0$, *Amer. J. Math.* **101** (1979), 233–244.
- G. Robertson, Centralizers in $\widetilde{A}_2$ groups, *preprint*.