

# ASYMPTOTIC K-THEORY FOR GROUPS ACTING ON $\tilde{A}_2$ BUILDINGS

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ABSTRACT. Let  $\Gamma$  be a torsion free lattice in  $G = \mathrm{PGL}(3, \mathbb{F})$  where  $\mathbb{F}$  is a nonarchimedean local field. Then  $\Gamma$  acts freely on the affine Bruhat-Tits building  $\mathcal{B}$  of  $G$  and there is an induced action on the boundary  $\Omega$  of  $\mathcal{B}$ . The crossed product  $C^*$ -algebra  $\mathcal{A}(\Gamma) = C(\Omega) \rtimes \Gamma$  depends only on  $\Gamma$  and is classified by its K-theory. This article shows how to compute the K-theory of  $\mathcal{A}(\Gamma)$  and of the larger class of rank two Cuntz-Krieger algebras.

## 1. INTRODUCTION

Let  $\mathbb{F}$  be a nonarchimedean local field with residue field of order  $q$ . The Bruhat-Tits building  $\mathcal{B}$  of  $G = \mathrm{PGL}(n+1, \mathbb{F})$  is a building of type  $\tilde{A}_n$  and there is a natural action of  $G$  on  $\mathcal{B}$ . The vertex set of  $\mathcal{B}$  may be identified with the homogeneous space  $G/K$ , where  $K$  is an open maximal compact subgroup of  $G$ . The boundary  $\Omega$  of  $\mathcal{B}$  is the homogeneous space  $G/B$ , where  $B$  is the Borel subgroup of upper triangular matrices in  $G$ .

Let  $\Gamma$  be a torsion free lattice in  $G = \mathrm{PGL}(n+1, \mathbb{F})$ . Then  $\Gamma$  is automatically cocompact in  $G$  [Ser, Chapitre II.1.5, p116] and acts freely on  $\mathcal{B}$ . If  $n = 1$ , then  $\Gamma$  is a finitely generated free group [Ser],  $\mathcal{B}$  is a homogeneous tree, and the boundary  $\Omega$  is the projective line  $\mathbb{P}_1(\mathbb{F})$ . If  $n \geq 2$  then the group  $\Gamma$  and its action on  $\Omega$  are not so well understood. In contrast to the rank one case,  $\Gamma$  has Kazhdan's property (T) and by the Strong Rigidity Theorem of Margulis [Mar, Theorem VII.7.1], the lattice  $\Gamma$  determines the ambient Lie group  $G$ . Since the Borel subgroup  $B$  of  $G$  is unique, up to conjugacy, it follows that the action of  $\Gamma$  on  $\Omega$  is also unique, up to conjugacy. This action may be studied by means of the crossed product  $C^*$ -algebra  $C(\Omega) \rtimes \Gamma$ , which depends only on  $\Gamma$  and may conveniently be denoted by  $\mathcal{A}(\Gamma)$ .

Geometrically, a locally finite  $\tilde{A}_n$  building  $\mathcal{B}$  is an  $n$ -dimensional contractible simplicial complex in which each codimension one simplex lies on  $q+1$  maximal simplices, where  $q \geq 2$ . If  $n \geq 2$  then the number  $q$  is necessarily a prime power and is referred to as the order

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of the building. The building is the union of a distinguished family of  $n$ -dimensional subcomplexes, called apartments, and each apartment is a Coxeter complex of type  $\tilde{A}_n$ . If  $\mathcal{B}$  is a locally finite building of type  $\tilde{A}_n$ , where  $n \geq 3$ , then  $\mathcal{B}$  is the building of  $\mathrm{PGL}(n+1, \mathbb{F})$  for some (possibly non commutative) local field  $\mathbb{F}$  [Ron, p137]. The case of  $\tilde{A}_2$  buildings is somewhat different, because such a building might not be the Bruhat-Tits building of a linear group. In fact this is the case for the  $\tilde{A}_2$  buildings of many of the groups constructed in [CMSZ]. The boundary  $\Omega$  of  $\mathcal{B}$  is the set of chambers of the spherical building at infinity [Ron, Chapter 9], endowed with a natural totally disconnected compact Hausdorff topology [CMS], [Ca, Section 4].

Given an  $\tilde{A}_n$  building  $\mathcal{B}$  with vertex set  $\mathcal{B}^0$ , there is a type map  $\tau : \mathcal{B}^0 \rightarrow \mathbb{Z}/(n+1)\mathbb{Z}$  such that each maximal simplex (chamber) has exactly one vertex of each type. An automorphism  $\alpha$  of  $\Delta$  is said to be type-rotating if there exists  $i \in \mathbb{Z}/(n+1)\mathbb{Z}$  such that  $\tau(\alpha v) = \tau(v) + i$  for all vertices  $v \in \mathcal{B}^0$ . If  $\mathcal{B}$  is the Bruhat-Tits building of  $G = \mathrm{PGL}(n+1, \mathbb{F})$  then the action of  $G$  on  $\mathcal{B}$  is type rotating [St].

Now let  $\Gamma$  be a group of type rotating automorphisms of an  $\tilde{A}_n$  building  $\mathcal{B}$  and suppose that  $\Gamma$  acts freely on the vertex set  $\mathcal{B}^0$  with finitely many orbits. Then  $\Gamma$  acts on the boundary  $\Omega$  and the rigidity results of [KL] imply that, as in the linear case above, the action is unique up to conjugacy and the crossed product  $C^*$ -algebra  $\mathcal{A}(\Gamma) = C(\Omega) \rtimes \Gamma$  depends only on the group  $\Gamma$ .

The purpose of this paper is to compute the K-theory of the algebras  $\mathcal{A}(\Gamma)$  in the case  $n = 2$ . This is done by using the fact that the algebras are higher rank Cuntz-Krieger algebras, whose structure theory was developed in [RS2]. In particular they are purely infinite, simple and nuclear. It was proved in [RS2] that a higher rank Cuntz-Krieger algebra is stably isomorphic to a crossed product of an AF algebra by a free abelian group. The computation of the K-groups is therefore in principle completely routine: no new K-theoretic or geometric ideas are needed. Actually organizing and performing the computations is another matter. This paper does this in the case  $n = 2$ . The most precise results are obtained in Section 7 for the algebra  $\mathcal{A}(\Gamma)$  where  $\Gamma$  is an  $\tilde{A}_2$  group; that is  $\Gamma$  acts freely *and* transitively on the vertices of an  $\tilde{A}_2$  building. Such groups have been studied intensively in [CMSZ].

The detailed numerical results of our computations are available elsewhere, but we do present, in Example 7.3, the K-theory of  $\mathcal{A}(\Gamma)$  for two torsion free lattices  $\Gamma$  in  $\mathrm{PGL}(3, \mathbb{Q}_2)$ . The non isomorphism of these two groups is seen in the K-theory of  $\mathcal{A}(\Gamma)$  but not in the K-theory of the reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$ .

The article concludes with some results on the order of the class of the identity in  $K_0(\mathcal{A}(\Gamma))$

## 2. GROUPS ACTING ON $\tilde{A}_2$ BUILDINGS : STATEMENT OF THE MAIN RESULT

Let  $\mathcal{B}$  be a finite dimensional simplicial complex, whose maximal simplices we shall call chambers. All chambers are assumed to have the same dimension and adjacent chambers have a common codimension one face. A gallery is a sequence of adjacent chambers.  $\mathcal{B}$  is a chamber complex if any two chambers can be connected by a gallery.  $\mathcal{B}$  is said to be thin if every codimension one simplex is a face of precisely two chambers.  $\mathcal{B}$  is said to be thick if every codimension one simplex is a face of at least three chambers. A chamber complex  $\mathcal{B}$  is called a building if it is the union of a family of subcomplexes, called apartments, satisfying the following axioms [Br3].

- (B0): Each apartment  $\Sigma$  is a thin chamber complex with  $\dim \Sigma = \dim \mathcal{B}$ .
- (B1): Any two simplices lie in an apartment.
- (B2): Given apartments  $\Sigma, \Sigma'$  there exists an isomorphism  $\Sigma \rightarrow \Sigma'$  fixing  $\Sigma \cap \Sigma'$  pointwise.
- (B3):  $\mathcal{B}$  is thick.

A building of type  $\tilde{A}_2$  has apartments which are all Coxeter complexes of type  $\tilde{A}_2$ . Such a building is therefore a union of two dimensional apartments, each of which may be realized as a tiling of the Euclidean plane by equilateral triangles.

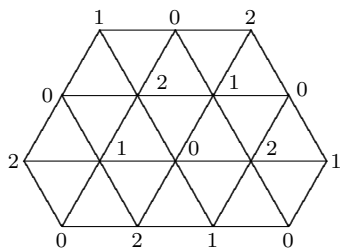
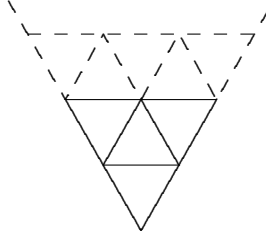


FIGURE 1. Part of an apartment in an  $\tilde{A}_2$  building, showing vertex types.

From now on we shall consider only locally finite buildings of type  $\tilde{A}_2$ . Each vertex  $v$  of  $\mathcal{B}$  is labeled with a type  $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$ , and each chamber has exactly one vertex of each type. An automorphism  $\alpha$  of  $\mathcal{B}$  is said to be type rotating if there exists  $i \in \mathbb{Z}/3\mathbb{Z}$  such that  $\tau(\alpha(v)) = \tau(v) + i$  for all vertices  $v \in \mathcal{B}$ .

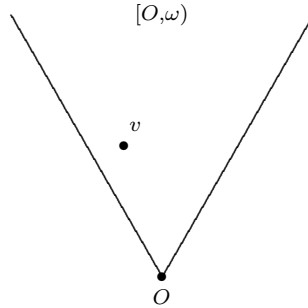
A sector is a  $\frac{\pi}{3}$ -angled sector made up of chambers in some apartment (Figure 2). Two sectors are equivalent if their intersection contains a sector.

FIGURE 2. A sector in a building  $\mathcal{B}$  of type  $\tilde{A}_2$ .

The boundary  $\Omega$  of  $\mathcal{B}$  is defined to be the set of equivalence classes of sectors in  $\mathcal{B}$ . In  $\mathcal{B}$  fix some vertex  $O$ . For any  $\omega \in \Omega$  there is a unique sector  $[O, \omega)$  in the class  $\omega$  having base vertex  $O$  [Ron, Theorem 9.6]. The boundary  $\Omega$  is a totally disconnected compact Hausdorff space with a base for the topology given by sets of the form

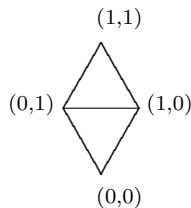
$$\Omega(v) = \{\omega \in \Omega : [O, \omega) \text{ contains } v\}$$

where  $v$  is a vertex of  $\mathcal{B}$  [CMS, Section 2]. If  $\mathcal{B}$  is the (type  $\tilde{A}_2$ ) Bruhat-Tits building of  $\mathrm{PGL}(3, \mathbb{F})$  where  $\mathbb{F}$  is a nonarchimedean local field then this definition of the boundary coincides with that given in the introduction [St].

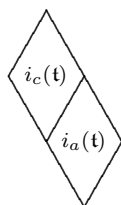
FIGURE 3. The sector  $[O, \omega)$ , where  $\omega \in \Omega(v)$ .

Let  $\mathcal{B}$  be a locally finite affine building of type  $\tilde{A}_2$ . Let  $\Gamma$  be a group of type rotating automorphisms of  $\mathcal{B}$  that acts freely on the vertex set with finitely many orbits. Let  $\mathfrak{t}$  be a model tile for  $\mathcal{B}$  consisting of two chambers with a common edge and with vertices coordinatized as shown in Figure 4. For definiteness, assume that the vertex  $(j, k)$  has type  $\tau(j, k) = j - k \pmod{2} \in \{0, 1, 2\}$ . Let  $\mathfrak{T}$  denote the set of type rotating isometries  $i : \mathfrak{t} \rightarrow \mathcal{B}$ , and let  $A = \Gamma \backslash \mathfrak{T}$ . Take  $A$  as an alphabet. Informally we think of elements of  $A$  as labeling the tiles of the building according to  $\Gamma$ -orbits.

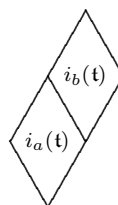
Two matrices  $M_1, M_2$  with entries in  $\{0, 1\}$  are defined as follows. If  $a, b \in A$ , say that  $M_1(b, a) = 1$  if and only if there are representative

FIGURE 4. The model tile  $\mathbf{t}$ 

isometries  $i_a, i_b$  in  $\mathfrak{T}$  whose ranges lie as shown in the diagram on the right of Figure 5. In that diagram, the tiles  $i_a(\mathbf{t}), i_b(\mathbf{t})$  have base vertices  $i_a(0, 0), i_b(0, 0)$  respectively and  $i_b(0, 0) = i_a(1, 0)$ . A similar definition applies for  $M_2(c, a) = 1$ .



$$M_2(c, a) = 1$$



$$M_1(b, a) = 1$$

FIGURE 5. Definition of the transition matrices.

In Section 5 below the following result is proved. It expresses the K-theory of  $\mathcal{A}(\Gamma)$  in terms of the cokernel of the homomorphism  $\mathbb{Z}^A \oplus \mathbb{Z}^A \rightarrow \mathbb{Z}^A$  defined by  $(I - M_1, I - M_2)$ .

**Theorem 2.1.** *Let  $\Gamma$  be a group of type rotating automorphisms of a building  $\mathcal{B}$  of type  $\tilde{A}_2$  which acts freely on the set of vertices of  $\mathcal{B}$  with finitely many orbits. Let  $\Omega$  be the boundary of the building and let  $\mathcal{A}(\Gamma) = C(\Omega) \rtimes \Gamma$ . Denote by  $M_1, M_2$  the associated transition matrices, as defined above. Let  $r$  be the rank, and  $T$  the torsion part, of the finitely generated abelian group  $\text{coker } (I - M_1, I - M_2)$ . Thus  $\text{coker } (I - M_1, I - M_2) = \mathbb{Z}^r \oplus T$ . Then*

$$K_0(\mathcal{A}(\Gamma)) = K_1(\mathcal{A}(\Gamma)) = \mathbb{Z}^{2r} \oplus T$$

**Remark 2.2.** The group  $\Gamma$  in Theorem 2.1 need not necessarily be torsion free. It may have 3-torsion and stabilize a chamber of the building.

### 3. RANK 2 CUNTZ-KRIEGER ALGEBRAS

In [RS2], the authors introduced a class of  $C^*$ -algebras which are higher rank analogues of the Cuntz-Krieger algebras [CK]. We shall

refer to the original algebras of [CK] as rank one Cuntz-Krieger algebras. The rank 2 case includes the algebras  $\mathcal{A}(\Gamma)$  arising from discrete group actions on the boundary of an  $\tilde{A}_2$  building as described in Section 2. In this section we shall compute the K-theory of a general rank 2 Cuntz-Krieger algebra  $\mathcal{A}$ .

We first outline how the algebra  $\mathcal{A}$  is defined. For our present investigation of the rank two case the assumptions in [RS2] can be somewhat simplified. Fix a finite set  $A$ , which is the “alphabet”. Start with a pair of nonzero matrices  $M_1, M_2$  with entries  $M_j(b, a) \in \{0, 1\}$  for  $a, b \in A$ . For an algebra of the form  $\mathcal{A}(\Gamma)$ , the alphabet  $A$  and the matrices  $M_1, M_2$  are defined in Section 2 above.

Let  $[m, n]$  denote  $\{m, m+1, \dots, n\}$ , where  $m \leq n$  are integers. If  $m, n \in \mathbb{Z}^2$ , say that  $m \leq n$  if  $m_j \leq n_j$  for  $j = 1, 2$ , and when  $m \leq n$ , let  $[m, n] = [m_1, n_1] \times [m_2, n_2]$ . In  $\mathbb{Z}^2$ , let  $0$  denote the zero vector and let  $e_j$  denote the  $j^{\text{th}}$  standard unit basis vector. If  $m \in \mathbb{Z}_+^2 = \{m \in \mathbb{Z}^2; m \geq 0\}$ , let

$$W_m = \{w : [0, m] \rightarrow A; M_j(w(l+e_j), w(l)) = 1 \text{ whenever } l, l+e_j \in [0, m]\}$$

and call the elements of  $W_m$  words. Let  $W = \bigcup_{m \in \mathbb{Z}_+^2} W_m$ . We say that a word  $w \in W_m$  has shape  $\sigma(w) = m$ , and we identify  $W_0$  with  $A$  in the natural way via the map  $w \mapsto w(0)$ . Define the initial and final maps  $o : W_m \rightarrow A$  and  $t : W_m \rightarrow A$  by  $o(w) = w(0)$  and  $t(w) = w(m)$ . We assume that the matrices  $M_1, M_2$  satisfy the following conditions.

- (H0):** Each  $M_i$  is a nonzero  $\{0, 1\}$ -matrix .
- (H1a):**  $M_1 M_2 = M_2 M_1$ .
- (H1b):**  $M_1 M_2$  is a  $\{0, 1\}$ -matrix .
- (H2):** The directed graph with vertices  $a \in A$  and directed edges  $(a, b)$  whenever  $M_i(b, a) = 1$  for some  $i$ , is irreducible.
- (H3):** For any nonzero  $p \in \mathbb{Z}^2$ , there exists a word  $w \in W$  which is not  $p$ -periodic, i.e. there exists  $l$  so that  $w(l)$  and  $w(l+p)$  are both defined but not equal.

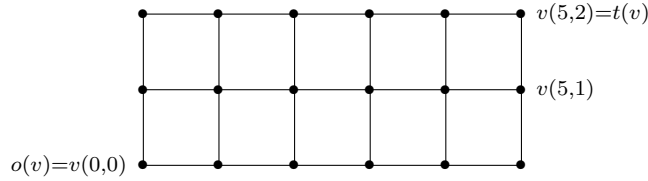
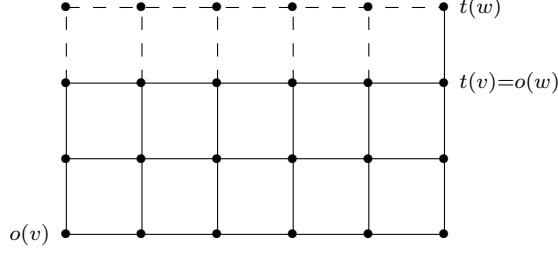


FIGURE 6. Representation of a two dimensional word  $v$  of shape  $m = (5, 2)$ .

If  $v \in W_m$  and  $w \in W_{e_j}$  with  $t(v) = o(w)$  then there exists a unique word  $vw \in W_{m+e_j}$  such that  $vw|_{[0, m]} = v$  and  $t(vw) = t(w)$  [RS2, Lemma 1.2]. The word  $vw$  is called the *product* of  $v$  and  $w$ .

The  $C^*$ -algebra  $\mathcal{A}$  is the universal  $C^*$ -algebra generated by a family of partial isometries  $\{s_{u,v}; u, v \in W \text{ and } t(u) = t(v)\}$  and satisfying

FIGURE 7. The product word  $vw$ , where  $w \in W_{e_2}$ .

the relations

$$(3.1a) \quad s_{u,v}^* = s_{v,u}$$

$$(3.1b) \quad s_{u,v}s_{v,w} = s_{u,w}$$

$$(3.1c) \quad s_{u,v} = \sum_{\substack{w \in W; \sigma(w)=e_j, \\ o(w)=t(u)=t(v)}} s_{uw,vw}$$

$$(3.1d) \quad s_{u,u}s_{v,v} = 0 \text{ for } u, v \in W_0, u \neq v.$$

It was shown in [RS2] that  $\mathcal{A}$  is simple, unital, nuclear and purely infinite, and that it is therefore classified by its K-theory. Moreover the algebra  $\mathcal{A}(\Gamma)$  arising from a discrete group action on the boundary of an  $\tilde{A}_2$  building is stably isomorphic to the corresponding algebra  $\mathcal{A}$ . See Section 5.

By [RS2, Section 6], the stabilized algebra  $\mathcal{A} \otimes \mathcal{K}$  can be constructed as a crossed product by  $\mathbb{Z}^2$ . The details are as follows. Let  $\mathcal{C} = \bigoplus_{a \in A} \mathcal{K}(\mathcal{H}_a)$ , where  $\mathcal{H}_a$  is a separable infinite dimensional Hilbert space. For each  $l \in \mathbb{Z}_+^2$  there is an endomorphism  $\alpha_l : \mathcal{C} \rightarrow \mathcal{C}$  defined by the equation

$$(3.2) \quad \alpha_l(x) = \sum_{w \in W_l} v_w x v_w^*$$

where  $v_w$  is a partial isometry with initial space  $\mathcal{H}_{o(w)}$  and final space lying inside  $\mathcal{H}_{t(w)}$ . For the precise definition we refer to [RS2], where  $v_w$  is denoted  $\psi(s'_{w,o(w)})$ . For each  $m \in \mathbb{Z}^2$  let  $\mathcal{C}^{(m)}$  be an isomorphic copy of  $\mathcal{C}$ , and for each  $l \in \mathbb{Z}_+^2$ , let

$$\alpha_l^{(m)} : \mathcal{C}^{(m)} \rightarrow \mathcal{C}^{(m+l)}$$

be a copy of  $\alpha_l$ . Let  $\mathcal{F} = \varinjlim \mathcal{C}^{(m)}$  be the direct limit of the category of  $C^*$ -algebras with objects  $\mathcal{C}^{(m)}$  and morphisms  $\alpha_l$ . Then  $\mathcal{F}$  is an AF-algebra. Since  $\alpha_l$  is an endomorphism, we may identify  $\mathcal{C}^{(m)}$  with its image in  $\mathcal{F}$  ([KR, Proposition 11.4.1]). If  $x \in \mathcal{C}$ , let  $x^{(m)}$  be the corresponding element of  $\mathcal{C}^{(m)}$ . Now  $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{F} \rtimes \mathbb{Z}^2$ , so that  $K_*(\mathcal{A}) = K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$ . The action of  $\mathbb{Z}^2$  on  $\mathcal{F}$  is defined by two commuting generators  $T_1, T_2$ , where  $T_j(x^{(m)}) = x^{(m-e_j)}$ ,  $j = 1, 2$ . We have

$K_0(\mathcal{F}) = \varinjlim K_0(\mathcal{C}^{(m)})$ . The maps  $T_j$  induce maps  $(T_j)_* : K_0(\mathcal{F}) \rightarrow K_0(\mathcal{F})$ ,  $j = 1, 2$ .

**Remark 3.1.** Note that in [RS2] we considered a more general algebra denoted  $\mathcal{A}_D$ , where  $D$  is a nonempty countable set (called the set of “decorations”) and there is an associated map  $\delta : D \rightarrow A$ . The algebra  $\mathcal{A}$  described above is simply the algebra  $\mathcal{A}_A$  where  $D = A$  and  $\delta$  is the identity map on  $A$ . It was shown in [RS2, Lemma 5.13, Corollary 5.15] that for any set  $D$  of decorations there exists an isomorphism  $\mathcal{A}_D \otimes \mathcal{K} \cong \mathcal{A} \otimes \mathcal{K}$ . The algebra  $\mathcal{A}_D$  therefore has the same K-theory as  $\mathcal{A}$ , namely the K-theory of the algebra  $\mathcal{F} \rtimes \mathbb{Z}^2$ .

#### 4. K-THEORY FOR RANK 2 CUNTZ-KRIEGER ALGEBRAS

Consider the chain complex

$$(4.1) \quad 0 \longleftarrow K_0(\mathcal{F}) \xleftarrow{(1-T_{2*}, T_{1*}-1)} K_0(\mathcal{F}) \oplus K_0(\mathcal{F}) \xleftarrow{\begin{pmatrix} 1-T_{1*} \\ 1-T_{2*} \end{pmatrix}} K_0(\mathcal{F}) \longleftarrow 0$$

For  $j \in \{0, 1, 2\}$ , denote by  $\mathfrak{H}_j$  the  $j^{\text{th}}$  homology group of the complex (4.1). In particular  $\mathfrak{H}_0 = \text{coker}(1 - T_{2*}, T_{1*} - 1)$  and  $\mathfrak{H}_2 = \ker \begin{pmatrix} 1-T_{1*} \\ 1-T_{2*} \end{pmatrix}$ .

**Proposition 4.1.** *If the group  $\mathfrak{H}_2$  is free abelian then*

$$\begin{aligned} K_0(\mathcal{A}) &\cong \mathfrak{H}_0 \oplus \mathfrak{H}_2, \\ K_1(\mathcal{A}) &\cong \mathfrak{H}_1. \end{aligned}$$

*Proof.* As observed in the introduction,  $K_*(\mathcal{A}) = K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$ . It is known that the Baum-Connes Conjecture with coefficients in an arbitrary  $C^*$ -algebra is true for the group  $\mathbb{Z}^2$  (and much more generally: [BBV] [Tu] [BCH, Section 9]). This implies that  $K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$  coincides with its “ $\gamma$ -part”, and  $K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$  may be computed as the limit of a Kasparov spectral sequence [Ka, p.199, Theorem]. The initial terms of the spectral sequence are  $E_{p,q}^2 = H_p(\mathbb{Z}^2, K_q(\mathcal{F}))$ , the  $p^{\text{th}}$  homology of the group  $\mathbb{Z}^2$  with coefficients in the module  $K_q(\mathcal{F})$ . (See [W, Chapter 5] for an explanation of spectral sequences and their convergence.) Noting that  $K_q(\mathcal{F}) = 0$  for  $q$  odd (since the algebra  $\mathcal{F}$  is  $AF$ ), it follows that  $E_{p,q}^2 = 0$  for  $q$  odd. Also  $E_{p,q}^2 = 0$  for  $p \notin \{0, 1, 2\}$ , and the differential  $d_2$  is zero. Thus

$$E_{p,q}^\infty = E_{p,q}^2 = \begin{cases} H_p(\mathbb{Z}^2, K_q(\mathcal{F})) & \text{if } p \in \{0, 1, 2\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$



To clarify notation, write  $G = \mathbb{Z}^2 = \langle s, t \mid st = ts \rangle$ . We have a free resolution  $F$  of  $\mathbb{Z}$  over  $\mathbb{Z}G$  given by

$$0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z}G \xleftarrow{(1-t, s-1)} \mathbb{Z}G \oplus \mathbb{Z}G \xleftarrow{\begin{pmatrix} 1-s \\ 1-t \end{pmatrix}} \mathbb{Z}G \longleftarrow 0$$

It follows [Br1, Chapter III.1] that  $H_*(G, K_0(\mathcal{F})) = H_*(F \otimes_G K_0(\mathcal{F}))$  is the homology of the complex (4.1). Therefore

$$E_{p,q}^\infty = \begin{cases} \mathfrak{H}_p & \text{if } p \in \{0, 1, 2\} \text{ and } q \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Convergence of the spectral sequence to  $K_*(\mathcal{F} \rtimes \mathbb{Z}^2)$  (see [W, Section 5.2]) means that

$$(4.2) \quad K_1(\mathcal{F} \rtimes \mathbb{Z}^2) = \mathfrak{H}_1$$

and that there is a short exact sequence

$$(4.3) \quad 0 \longrightarrow \mathfrak{H}_0 \longrightarrow K_0(\mathcal{F} \rtimes \mathbb{Z}^2) \longrightarrow \mathfrak{H}_2 \longrightarrow 0$$

The group  $\mathfrak{H}_2$  is free abelian. Therefore the exact sequence (4.3) splits. This proves the result  $\square$

**Remark 4.2.** Writing  $\mathcal{A} \otimes \mathcal{K} = \mathcal{F} \rtimes \mathbb{Z}^2 = (\mathcal{F} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$  and applying the PV-sequence of M. PIMSNER and D. VOICULESCU one obtains (4.3) without using the Kasparov spectral sequence. See [WO, Remarks 9.9.3] for a description of the PV-sequence and [WO, Exercise 9.K] for an outline of the proof.

**Remark 4.3.** From (4.1) one notes that  $\mathfrak{H}_0$  is none other than the  $\mathbb{Z}^2$ -coinvariants of  $K_0(\mathcal{F})$ . Hence the functorial map  $K_0(\mathcal{F}) \rightarrow K_0(\mathcal{F} \rtimes \mathbb{Z}^2)$  factors through  $\mathfrak{H}_0$  (Figure 8).

$$\begin{array}{ccc} K_0(\mathcal{F}) & \longrightarrow & K_0(\mathcal{F} \rtimes \mathbb{Z}^2) \\ \downarrow & \nearrow \text{dashed} & \\ \mathfrak{H}_0 & & \end{array}$$

FIGURE 8

It follows from the double application of the PV-sequence (Remark 4.2) that the maps  $\mathfrak{H}_0 \longrightarrow K_0(\mathcal{F} \rtimes \mathbb{Z}^2)$  of equation (4.3) and Figure 8 coincide. In particular, the map in Figure 8 is injective.

**Remark 4.4.** Double application of the PV-sequence is not sufficient to prove the formula (4.2). However if one generalizes [WO, Exercise 9.K] from  $\mathbb{Z}$  to  $\mathbb{Z}^2$  one obtains a proof of (4.2) at a (relatively) low level of K-sophistication.

Choose for each  $a \in A$ , a minimal projection  $P_a \in \mathcal{K}(\mathcal{H}_a)$ , and let  $[P_a]$  denote the corresponding class in  $K_0(\mathcal{K}(\mathcal{H}_a))$ . Then  $K_0(\mathcal{K}(\mathcal{H}_a)) \cong \mathbb{Z}$ , with generator  $[P_a]$ . Identify  $\mathbb{Z}^A$  with  $K_0(\mathcal{C}) = \bigoplus_a K_0(\mathcal{K}(\mathcal{H}_a))$  via the map  $(n_a)_{a \in A} \mapsto \sum_{a \in A} n_a [P_a]$ . The endomorphism  $\alpha_l$  induces a map  $(\alpha_l)_*$  on  $K_0$ . The following Lemma is crucial for the calculations which follow.

**Lemma 4.5.** *The map  $(\alpha_{e_j})_* : K_0(\mathcal{C}) \rightarrow K_0(\mathcal{C})$  is given by the matrix  $M_j : \mathbb{Z}^A \mapsto \mathbb{Z}^A$ ,  $j = 1, 2$ .*

*Proof.* Note that  $(\alpha_{e_j})_*([P_a]) = [\alpha_{e_j}(P_a)]$ . Now

$$\alpha_{e_j}(P_a) = \sum_{w \in W_{e_j}} v_w P_a v_w^* = \sum_{w \in W_{e_j}; o(w)=a} v_w P_a v_w^*.$$

If  $t(w) = b$  then  $v_w P_a v_w^*$  is a minimal projection in  $\mathcal{K}(\mathcal{H}_b)$ , and so its class in  $K_0(\mathcal{K}(\mathcal{H}_b))$  equals  $[P_b]$ . Therefore

$$(\alpha_{e_j})_*([P_a]) = \sum_{w \in W_{e_j}; o(w)=a} [v_w P_a v_w^*] = \sum_{b; M_j(b,a)=1} [P_b].$$

Consequently

$$(\alpha_{e_j})_*\left(\sum_a n_a [P_a]\right) = \sum_a n_a \sum_b M_j(b, a) [P_b] = \sum_b \left(\sum_a M_j(b, a) n_a\right) [P_b].$$

This proves the result.  $\square$

Recall that  $M_1$  and  $M_2$  commute. If  $l = (l_1, l_2) \in \mathbb{Z}_+^2$  then we write  $(M_1, M_2)^l = M_1^{l_1} M_2^{l_2}$ .

**Remark 4.6.** The direct limit  $K_0(\mathcal{F})$  may be constructed explicitly as follows. Consider the set  $\mathcal{S}$  consisting of all elements  $(s_n)_{n \in \mathbb{Z}^2} \in \bigoplus_n K_0(\mathcal{C}^{(n)})$  such that there exists  $l \in \mathbb{Z}^2$ , for which  $s_{n+e_j} = M_j s_n$  for all  $n \geq l$ ,  $j = 1, 2$ . Say that two elements  $(s_n)_{n \in \mathbb{Z}^2}$ ,  $(t_n)_{n \in \mathbb{Z}^2}$  of  $\mathcal{S}$  are equivalent if there exists  $l \in \mathbb{Z}^2$ , for which  $s_n = t_n$  for all  $n \geq l$ . Then  $K_0(\mathcal{F}) = \varinjlim K_0(\mathcal{C}^{(n)})$  may be identified with  $\mathcal{S}$  modulo this equivalence relation. We refer to [Fu, Section 11] for more information about direct limits.

For  $c \in K_0(\mathcal{C}) = \mathbb{Z}^A$  let  $c^{(m)} \in K_0(\mathcal{C}^{(m)})$  be the corresponding element in  $K_0(\mathcal{F})$ , defined as follows.

$$c^{(m)} = (s_n)_{n \in \mathbb{Z}^2} \quad \text{where} \quad s_n = \begin{cases} (M_1, M_2)^{(n-m)} c & \text{if } n \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

In particular we identify  $c^{(m)}$  with  $((\alpha_l)_*(c))^{(m+l)}$  for  $l \in \mathbb{Z}_+^2$ .

Define  $\gamma_m : K_0(\mathcal{C}) \rightarrow K_0(\mathcal{F})$  by  $\gamma_m(c) = c^{(m)} \in K_0(\mathcal{F})$ .

**Remark 4.7.** It follows immediately from the definitions that

- (1)  $T_{j*}\gamma_m = \gamma_{m-e_j}$ .
- (2)  $\gamma_m(c) = \gamma_{m+l}((M_1, M_2)^l c)$  for  $l \in \mathbb{Z}_+^2$ .

**Lemma 4.8.** *The following assertions hold.*

- (1) *Any element in  $K_0(\mathcal{F})$  can be written as  $\gamma_m(c)$  for some  $m \in \mathbb{Z}_+^2$  and  $c \in K_0(\mathcal{C})$ .*
- (2) *If  $c \in K_0(\mathcal{C})$  and  $m \in \mathbb{Z}^2$  then  $\gamma_m(c) = 0$  if and only if  $(M_1, M_2)^l c = 0$  for some  $l \in \mathbb{Z}_+^2$ .*
- (3)  *$T_{j*}\gamma_m(c) = \gamma_m(M_j c)$  for  $j = 1, 2$ .*

*Proof.* Statements (1) and (2) follow from the definitions. To prove (3) note that  $((T_j)_*\gamma_m)(c) = (T_j)_*(c^{(m)}) = c^{(m-e_j)} = (M_j c)^{(m)} = (\gamma_m M_j)(c)$ .  $\square$

**Lemma 4.9.** *For  $j = 1, 2$  the map induced on the following complex by  $M_j$  acts as the identity on the homology groups.*

$$(4.4) \quad 0 \longleftarrow K_0(\mathcal{C}) \xleftarrow{(I-M_2, M_1-I)} K_0(\mathcal{C}) \oplus K_0(\mathcal{C}) \xleftarrow{\begin{pmatrix} I-M_1 \\ I-M_2 \end{pmatrix}} K_0(\mathcal{C}) \longleftarrow 0$$

*Proof.* Denote by  $[\cdot]$  the equivalence classes in the relevant homology groups.

**0-homology:** If  $c \in K_0(\mathcal{C})$ , then  $[c] - [M_j c] = [(I - M_j)c] = 0$ , the zero element in the 0-homology group.

**1-homology:** Let  $c_1, c_2 \in K_0(\mathcal{C})$  with  $(I - M_2)c_1 = (I - M_1)c_2$ . Then  $[(c_1, c_2)] - [(M_1 c_1, M_1 c_2)] = [((I - M_1)c_1, (I - M_2)c_1)] = 0$  in the 1-homology group. Likewise for  $M_2$ .

**2-homology:** Let  $c \in K_0(\mathcal{C})$  with  $c = M_1 c = M_2 c$ . Then  $[c] = [M_j c]$  in the 2-homology group.  $\square$

**Lemma 4.10.**

$$\varinjlim \left( K_0(\mathcal{C}^{(m)}) \xrightarrow{M_j} K_0(\mathcal{C}^{(m)}) \right) = K_0(\mathcal{F}) \xrightarrow{T_{j*}} K_0(\mathcal{F})$$

*Proof.* The direct limit of maps makes sense because the diagram

$$\begin{array}{ccc} K_0(\mathcal{C}^{(m)}) & \xrightarrow{(M_1, M_2)^l} & K_0(\mathcal{C}^{(m+l)}) \\ M_j \uparrow & & \uparrow M_j \\ K_0(\mathcal{C}^{(m)}) & \xrightarrow{(M_1, M_2)^l} & K_0(\mathcal{C}^{(m+l)}) \end{array}$$

commutes. The diagram

$$\begin{array}{ccc} K_0(\mathcal{C}^{(m)}) & \longrightarrow & K_0(\mathcal{F}) \\ M_j \uparrow & & \uparrow T_{j*} \\ K_0(\mathcal{C}^{(m)}) & \longrightarrow & K_0(\mathcal{F}) \end{array}$$

commutes by Lemma 4.8. Since  $K_0(\mathcal{F}) = \varinjlim K_0(\mathcal{C}^{(m)})$  the result follows from the uniqueness assertion in the universal property of direct limits [Fu, Theorem 11.2].  $\square$

By Lemma 4.10 we have

$$\begin{aligned} 0 &\longleftarrow K_0(\mathcal{F}) \xleftarrow{(1-T_{2*}, T_{1*}-1)} K_0(\mathcal{F}) \oplus K_0(\mathcal{F}) \xleftarrow{\begin{pmatrix} I-T_{1*} \\ I-T_{2*} \end{pmatrix}} K_0(\mathcal{F}) \longleftarrow 0 \\ &= \varinjlim (0 \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \oplus K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow 0) \end{aligned}$$

where the map  $K_0(\mathcal{C}^{(m)}) \rightarrow K_0(\mathcal{C}^{(m+l)})$  is given by  $(M_1, M_2)^l$ . Now homology is continuous with respect to direct limits [Sp, Theorems 5.19, 4.17]. Therefore it follows from Lemma 4.9 that

$$\begin{aligned} &\text{Hom} \left( \varinjlim (0 \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \oplus K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow 0) \right) \\ &= \varinjlim (\text{Hom} (0 \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \oplus K_0(\mathcal{C}^{(m)}) \longleftarrow K_0(\mathcal{C}^{(m)}) \longleftarrow 0)) \\ &= \text{Hom} (0 \longleftarrow K_0(\mathcal{C}) \longleftarrow K_0(\mathcal{C}) \oplus K_0(\mathcal{C}) \longleftarrow K_0(\mathcal{C}) \longleftarrow 0) \end{aligned}$$

where  $\text{Hom}$  denotes the homology of the complex. We have proved

**Lemma 4.11.** *The map of complexes in Figure 9 induces isomorphisms of the homology groups.*

$$\begin{array}{ccccccc} 0 & \longleftarrow & K_0(\mathcal{F}) & \xleftarrow{(1-T_{2*}, T_{1*}-1)} & K_0(\mathcal{F}) \oplus K_0(\mathcal{F}) & \xleftarrow{\begin{pmatrix} 1-T_{1*} \\ 1-T_{2*} \end{pmatrix}} & K_0(\mathcal{F}) \longleftarrow 0 \\ & & \uparrow \gamma_0 & & \uparrow \begin{pmatrix} \gamma_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} & & \uparrow \gamma_0 \\ 0 & \longleftarrow & K_0(\mathcal{C}) & \xleftarrow{(I-M_2, M_1-I)} & K_0(\mathcal{C}) \oplus K_0(\mathcal{C}) & \xleftarrow{\begin{pmatrix} I-M_1 \\ I-M_2 \end{pmatrix}} & K_0(\mathcal{C}) \longleftarrow 0 \end{array}$$

FIGURE 9

Recall that  $\mathfrak{H}_j$  denotes the  $j^{\text{th}}$  homology group of the complex (4.1). Lemma 4.11 shows that  $\mathfrak{H}_j$  is the  $j^{\text{th}}$  homology group of (4.4), i.e. the  $j^{\text{th}}$  homology group of the complex

$$(4.5) \quad 0 \longleftarrow \mathbb{Z}^A \xleftarrow{(I-M_2, M_1-I)} \mathbb{Z}^A \oplus \mathbb{Z}^A \xleftarrow{\begin{pmatrix} I-M_1 \\ I-M_2 \end{pmatrix}} \mathbb{Z}^A \longleftarrow 0$$

**Remark 4.12.** In particular,  $\mathfrak{H}_2$  is a free abelian group, and so Proposition 4.1 applies.

Let  $\text{tor}(G)$  denote the torsion part of the finitely generated abelian group  $G$ , and let  $\text{rank}(G)$  denote the rank of  $G$ ; that is the rank of the free abelian part of  $G$  (also sometimes called the torsion-free rank of  $G$ ).

We have, by definition

- (1)  $\mathfrak{H}_0 = \text{coker } (I-M_2, M_1-I),$
- (2)  $\mathfrak{H}_2 = \ker \begin{pmatrix} I-M_1 \\ I-M_2 \end{pmatrix},$

$$(3) \mathfrak{H}_1 = \ker(I - M_2, M_1 - I) / \operatorname{im}\begin{pmatrix} I - M_1 \\ I - M_2 \end{pmatrix}.$$

The next result determines the K-theory of the algebra  $\mathcal{A}$  in terms of the matrices  $M_1$  and  $M_2$ .

**Proposition 4.13.** *The following equalities hold.*

$$\begin{aligned} \operatorname{rank}(K_0(\mathcal{A})) &= \operatorname{rank}(K_1(\mathcal{A})) = \operatorname{rank}(\operatorname{coker}(I - M_1, I - M_2)) + \operatorname{rank}(\operatorname{coker}(I - M_1^t, I - M_2^t)) \\ \operatorname{tor}(K_0(\mathcal{A})) &\cong \operatorname{tor}(\operatorname{coker}(I - M_1, I - M_2)) \\ \operatorname{tor}(K_1(\mathcal{A})) &\cong \operatorname{tor}(\operatorname{coker}(I - M_1^t, I - M_2^t)). \end{aligned}$$

*In particular  $K_0(\mathcal{A})$  and  $K_1(\mathcal{A})$  have the same torsion free parts.*

*Proof.* We have  $\operatorname{rank}(\ker\begin{pmatrix} I - M_1 \\ I - M_2 \end{pmatrix}) = \operatorname{rank}(\operatorname{coker}(I - M_1^t, I - M_2^t))$ . Hence, by Proposition 4.1 (and Remark 4.12),

$$\begin{aligned} \operatorname{rank}(K_0(\mathcal{A})) &= \operatorname{rank}(\mathfrak{H}_0) + \operatorname{rank}(\mathfrak{H}_2) \\ &= \operatorname{rank}(\operatorname{coker}(I - M_1, I - M_2)) + \operatorname{rank}(\operatorname{coker}(I - M_1^t, I - M_2^t)). \end{aligned}$$

Also

$$\begin{aligned} \operatorname{rank}(K_1(\mathcal{A})) &= \operatorname{rank}(\mathfrak{H}_1) \\ &= \operatorname{rank}(\ker(I - M_2, M_1 - I)) - \operatorname{rank}\begin{pmatrix} I - M_1 \\ I - M_2 \end{pmatrix} \\ &= 2n - \operatorname{rank}\begin{pmatrix} I - M_1 \\ I - M_2 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} I - M_1^t \\ I - M_2^t \end{pmatrix} \\ &= \operatorname{rank}(\operatorname{coker}(I - M_1, I - M_2)) + \operatorname{rank}(\operatorname{coker}(I - M_1^t, I - M_2^t)). \end{aligned}$$

Since  $\operatorname{tor}(\mathfrak{H}_0) = \operatorname{tor}(\operatorname{coker}(I - M_1, I - M_2))$  and  $\operatorname{tor}(\mathfrak{H}_2) = 0$ , it follows that

$$\operatorname{tor}(K_0(\mathcal{A})) = \operatorname{tor}(\operatorname{coker}(I - M_1, I - M_2)).$$

Finally

$$\operatorname{tor}(K_1(\mathcal{A})) = \operatorname{tor}(\mathfrak{H}_1) = \operatorname{tor}\begin{pmatrix} I - M_1 \\ I - M_2 \end{pmatrix} = \operatorname{tor}(\operatorname{coker}(I - M_1^t, I - M_2^t))$$

where the last equality follows from the Smith normal form for integer matrices.  $\square$

## 5. K-THEORY FOR BOUNDARY ALGEBRAS ASSOCIATED WITH $\tilde{A}_2$ BUILDINGS

Return now to the setup of Section 2. That is, let  $\mathcal{B}$  be a locally finite affine building of type  $\tilde{A}_2$ . Let  $\Gamma$  be a group of type rotating automorphisms of  $\mathcal{B}$  that acts freely on the vertex set with finitely many orbits. Let  $A$  denote the associated finite alphabet and let  $M_1, M_2$  be the transition matrices with entries indexed by elements of  $A$ .

It was shown in [RS2] that the conditions (H0), (H1a), (H1b) and (H3) of section 3 are satisfied by the matrices  $M_1, M_2$ . It was also proved in [RS2] that condition (H2) is satisfied if  $\Gamma$  is a lattice subgroup of  $\operatorname{PGL}_3(\mathbb{F})$ , where  $\mathbb{F}$  is a local field of characteristic zero. The proof uses the Howe-Moore Ergodicity Theorem. In forthcoming work of T.

Steger it is shown how to extend the methods of the proof of the Howe–Moore Theorem and so prove condition (H2) in the stated generality.

It follows from [RS2, Theorem 7.7] that the algebra  $\mathcal{A}(\Gamma)$  is stably isomorphic to the algebra  $\mathcal{A}$ . Moreover if the group  $\Gamma$  also acts transitively on the vertices of  $\mathcal{B}$  (which is the case in the examples of Section 7) then  $\mathcal{A}(\Gamma)$  is isomorphic to  $\mathcal{A}$ .

**Lemma 5.1.** *If  $M_1, M_2$  are associated with an  $\tilde{A}_2$  building as in Section 2, then there is a permutation matrix  $S : \mathbb{Z}^A \rightarrow \mathbb{Z}^A$  such that  $S^2 = I$  and  $SM_1S = M_2^t$ ,  $SM_2S = M_1^t$ . In particular  $\text{coker}(I - M_1, I - M_2) = \text{coker}(I - M_1^t, I - M_2^t)$*

*Proof.* Define  $s : \mathfrak{t} \rightarrow \mathfrak{t}$  by  $s(i)(j, k) = i(1 - k, 1 - j)$  for  $i \in \mathfrak{T}$  and  $0 \leq j, k \leq 1$ . Then  $s$  is the type preserving isometry of  $\mathfrak{t}$  given by reflection in the edge  $[(0, 1), (1, 0)]$ . (See Figure 10.) Now define a permutation

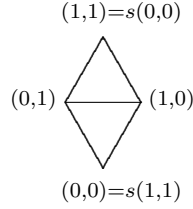


FIGURE 10. The reflection  $s$  of a tile  $\mathfrak{t}$

$s : A \rightarrow A$  by  $s(\Gamma i) = \Gamma s(i)$ . If  $a = \Gamma i_a, b = \Gamma i_b \in A$  then it is clear that  $M_1(b, a) = 1 \iff M_2(s(a), s(b)) = 1$ . The situation is illustrated, not too cryptically we hope, in Figure 11, where the tiles are located in the building  $\mathcal{B}$  and, for example, the tile labeled  $a$  is the range of a suitable isometry  $i_a : \mathfrak{t} \rightarrow \mathcal{B}$  with  $a = \Gamma i_a$ . Let  $S$  be the permutation matrix corresponding to  $s$ . Then  $M_1(b, a) = 1 \iff SM_2S^{-1}(a, b) = 1$ . Clearly  $S^2 = I$ . Therefore  $M_1^t = SM_2S$ . A similar argument proves the other equality.  $\square$

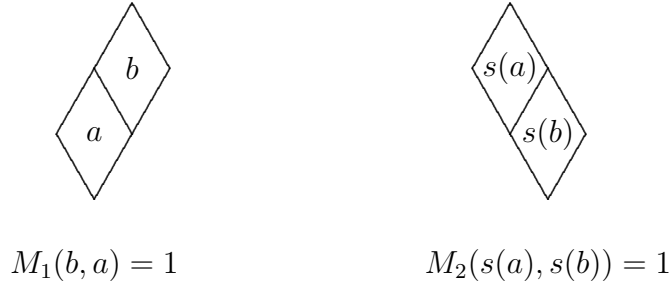


FIGURE 11. Reversing transitions between tiles

The proof of Theorem 2.1 now follows immediately from Proposition 4.13 and Lemma 5.1. The next result identifies which of the algebras  $\mathcal{A}(\Gamma)$  are rank one algebras.

**Corollary 5.2.** *Continue with the hypotheses of Theorem 2.1. The following are equivalent.*

- (1) *The algebra  $\mathcal{A}(\Gamma)$  is isomorphic to a rank one Cuntz-Krieger algebra;*
- (2) *the algebra  $\mathcal{A}(\Gamma)$  is stably isomorphic to a rank one Cuntz-Krieger algebra;*
- (3) *the group  $K_0(\mathcal{A}(\Gamma))$  is torsion free.*

*Proof.* The K-theory of a rank one Cuntz Krieger algebra  $\mathcal{O}_A$  can be characterized as follows (see [C1]):

$$K_0(\mathcal{O}_A) = (\text{finite group}) \oplus \mathbb{Z}^k; \quad K_1(\mathcal{O}_A) = \mathbb{Z}^k.$$

By Theorem 2.1, we have  $K_0 = K_1$  for the algebra  $\mathcal{A}(\Gamma)$ . Since stably isomorphic algebras have the same K-theory, it follows that if  $\mathcal{A}(\Gamma)$  is stably isomorphic to a rank one Cuntz-Krieger algebra then  $K_0(\mathcal{A}(\Gamma))$  is torsion free.

On the other hand, suppose that  $G_0 = K_0(\mathcal{A}(\Gamma))$  is torsion free. Let  $g_0 = [1] \in G_0$  be the class in  $K_0$  of the identity element of  $\mathcal{A}(\Gamma)$ . By a result of M. RORDAM [Ror, Proposition 6.6], there exists a simple rank one Cuntz-Krieger algebra  $\mathcal{O}_A$  such that  $K_0(\mathcal{O}_A) = G_0$  with the class of the identity in  $\mathcal{O}_A$  being  $g_0$ . Since  $G_0$  is torsion free we necessarily have  $K_1(\mathcal{O}_A) = G_0$  and by Theorem 2.1 we also have  $K_1(\mathcal{A}(\Gamma)) = G_0$ . Thus  $K_*(\mathcal{A}(\Gamma)) = K_*(\mathcal{O}_A)$  and the identity elements of the two algebras have the same image in  $K_0$ . Since the algebras  $\mathcal{A}(\Gamma)$  and  $\mathcal{O}_A$  are purely infinite, simple, nuclear and satisfy the Universal Coefficient Theorem, it now follows from the Classification Theorem of [Kir, Ph] that they are isomorphic.  $\square$

**Remark 5.3.** Corollary 5.2 can be used (see Remark 8.4) to verify that almost all the examples of rank 2 Cuntz-Krieger algebras described later are not stably isomorphic to ordinary (rank 1) Cuntz-Krieger algebras.

## 6. REDUCTION OF ORDER

Continue with the assumptions of Section 5. The following lemma will simplify the calculation of the K-groups, by reducing the order of the matrices involved.

**Lemma 6.1.** *Suppose that  $M_1, M_2$  are  $\{0, 1\}$ -matrices acting on  $\mathbb{Z}^A$ .*

(i) *Let  $\hat{A}$  be a set and let  $\hat{\pi} : A \rightarrow \hat{A}$  be a surjection. Suppose that  $M_j(b, a) = M_j(b, a')$  if  $\hat{\pi}(a) = \hat{\pi}(a')$ . Let the matrix  $\hat{M}_j$  acting on  $\mathbb{Z}^{\hat{A}}$  be given by  $\hat{M}_j(\hat{b}, \hat{\pi}(a)) = \sum_{\hat{\pi}(b)=\hat{b}} M_j(b, a)$ . Then the canonical map from  $\mathbb{Z}^A$  onto  $\mathbb{Z}^{\hat{A}}$  which sends generators to generators induces an isomorphism from  $\text{coker}(I - M_1, I - M_2)$  onto  $\text{coker}(I - \hat{M}_1, I - \hat{M}_2)$ .*

(ii) *Let  $\check{A}$  be a set and let  $\check{\pi} : A \rightarrow \check{A}$  be a surjection. Suppose that  $M_j(b, a) = M_j(b', a)$  if  $\check{\pi}(b) = \check{\pi}(b')$ . Let the matrix  $\check{M}_j$  acting on*

$\mathbb{Z}^{\hat{A}}$  be given by  $\check{M}_j(\check{\pi}(b), \check{a}) = \sum_{\check{\pi}(a)=\check{a}} M_j(b, a)$ . Then the canonical map from  $\mathbb{Z}^A$  onto  $\mathbb{Z}^{\hat{A}}$  which sends generators to generators induces an isomorphism from  $\text{coker}(I - M_1, I - M_2)$  onto  $\text{coker}(I - \check{M}_1, I - \check{M}_2)$ .

*Proof.* (i) Let  $(e_a)_{a \in A}$  be the standard set of generators for the free abelian group  $\mathbb{Z}^A$  and  $(e_{\hat{a}})_{\hat{a} \in \hat{A}}$  that of  $\mathbb{Z}^{\hat{A}}$ . Define the map  $\pi : \mathbb{Z}^A \rightarrow \mathbb{Z}^{\hat{A}}$  by  $\pi(e_a) = e_{\hat{\pi}(a)}$ . Observe that the diagram

$$\begin{array}{ccc} \mathbb{Z}^A & \xrightarrow{M_j} & \mathbb{Z}^A \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{Z}^{\hat{A}} & \xrightarrow{\hat{M}_j} & \mathbb{Z}^{\hat{A}} \end{array}$$

commutes. Therefore so does the diagram

$$\begin{array}{ccc} \mathbb{Z}^A \oplus \mathbb{Z}^A & \xrightarrow{(I - M_1, I - M_2)} & \mathbb{Z}^A \\ \pi \oplus \pi \downarrow & & \downarrow \pi \\ \mathbb{Z}^{\hat{A}} \oplus \mathbb{Z}^{\hat{A}} & \xrightarrow{(I - \hat{M}_1, I - \hat{M}_2)} & \mathbb{Z}^{\hat{A}} \end{array}$$

Hence there is a well defined map of cokernels, which is surjective because  $\pi$  is. The kernel of  $\pi$  is generated by  $\{e_a - e_{a'}; \hat{\pi}(a) = \hat{\pi}(a')\}$ . Now if  $\hat{\pi}(a) = \hat{\pi}(a')$  then according to the hypothesis of the lemma,  $M_j e_a = M_j e_{a'}$  and so  $(I - M_j)(e_a - e_{a'}) = e_a - e_{a'}$ . Hence the kernel of  $\pi$  is contained in the image of the map  $(I - M_1, I - M_2)$ . It follows by diagram chasing that the map on cokernels is injective.

(ii) The argument in this case is a little harder but similar. Note that the vertical maps in the diagrams go up rather than down.  $\square$

We now explain how Lemma 6.1 is used in our calculations to reduce calculations based on rhomboid tiles to calculations based on triangles. Let  $\hat{\mathbf{t}}$  be the model triangle with vertices  $\{(1, 1), (0, 1), (1, 0)\}$ , which is the upper half of the model tile  $\mathbf{t}$ . Let  $\hat{\mathfrak{T}}$  denote the set of type rotating isometries  $i : \hat{\mathbf{t}} \rightarrow \mathcal{B}$ , and let  $\hat{A} = \Gamma \backslash \hat{\mathfrak{T}}$ . We think of  $\hat{A}$  as labels for triangles in  $\mathcal{B}$ , just as  $A$  is thought of as labels for parallelograms. Each type rotating isometry  $i : \mathbf{t} \rightarrow \mathcal{B}$  restricts to a type rotating isometry  $\hat{i} = i|_{\hat{\mathbf{t}}} : \hat{\mathbf{t}} \rightarrow \mathcal{B}$ . Define  $\hat{\pi} : A \rightarrow \hat{A}$  by  $\hat{\pi}(a) = \Gamma \hat{i}_a$  where  $a = \Gamma i_a$ . It

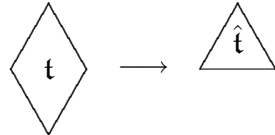


FIGURE 12. The restriction  $\mathbf{t} \rightarrow \hat{\mathbf{t}}$

is clear that  $M_j(b, a) = M_j(b, a')$  if  $\hat{\pi}(a) = \hat{\pi}(a')$ . This is illustrated in



Figure 13 for the case  $M_1(b, a) = 1$ . Thus the hypotheses of Lemma 6.1(i) are satisfied.

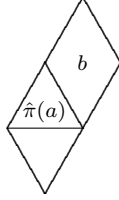


FIGURE 13.  $M_1(b, a) = 1$

Each matrix  $\hat{M}_j$  has entries in  $\{0, 1\}$ . For example Figure 14 illustrates the configuration for  $\hat{M}_1(\hat{b}, \hat{a}) = 1$ .

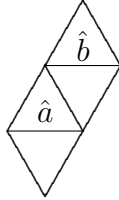


FIGURE 14.  $\hat{M}_1(\hat{b}, \hat{a}) = 1$

Note that although the matrices  $\hat{M}_1$  and  $\hat{M}_2$  are used to simplify the final computation, they could not be used to define the algebra  $\mathcal{A}$  because their product  $\hat{M}_1\hat{M}_2$  need not have entries in  $\{0, 1\}$ . In fact in the gallery of Figure 15 the triangle labels  $\hat{a}$  and  $\hat{c}$  do not uniquely determine the triangle label  $\hat{b}$ . In other words there is more than one such two step transition from  $\hat{a}$  to  $\hat{c}$ .

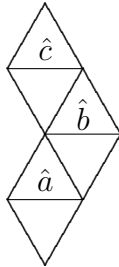


FIGURE 15. Non uniqueness of two-step transition

Similar arguments apply to the set  $\check{A}$  obtained by considering the model triangle  $\check{\mathbf{t}}$  with vertices  $\{(0, 0), (0, 1), (1, 0)\}$ , which is the lower half of the model tile  $\mathbf{t}$  (Figure 16).

The map  $\tilde{\pi} : A \rightarrow \tilde{A}$  is induced by the restriction of Figure 16 and one applies Lemma 6.1(ii) to the resulting matrices  $\tilde{M}_1, \tilde{M}_2$ . Figure 17 illustrates the configuration for  $\tilde{M}_1(\tilde{b}, \tilde{a}) = 1$ .

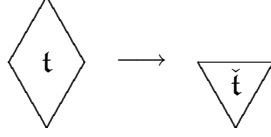


FIGURE 16. The restriction  $\mathbf{t} \rightarrow \tilde{\mathbf{t}}$

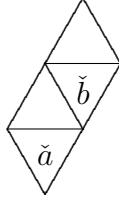


FIGURE 17.  $\tilde{M}_1(\tilde{b}, \tilde{a}) = 1$

The same argument as in Lemma 5.1 shows that there is an isomorphism  $V : \mathbb{Z}^{\tilde{A}} \rightarrow \mathbb{Z}^{\tilde{A}}$  such that  $\hat{M}_1 = V^{-1} \tilde{M}_2 V$  and vice versa.

We may summarize the preceding discussion as follows.

**Corollary 6.2.** *Assume the notation and hypotheses of Theorem 2.1. Let  $\hat{M}_j, \tilde{M}_j$  ( $j = 1, 2$ ) be the matrices defined as above. Then*

$$\begin{aligned} K_0(\mathcal{A}_D) &= K_1(\mathcal{A}_D) = \mathbb{Z}^{2r} \oplus \text{tor}(\text{coker } (I - \hat{M}_1, I - \hat{M}_2)) \\ &= \mathbb{Z}^{2r} \oplus \text{tor}(\text{coker } (I - \tilde{M}_1, I - \tilde{M}_2)) \end{aligned}$$

where  $r = \text{rank}(\text{coker } (I - \hat{M}_1, I - \hat{M}_2)) = \text{rank}(\text{coker } (I - \tilde{M}_1, I - \tilde{M}_2))$ .

## 7. K-THEORY FOR THE BOUNDARY ALGEBRA OF AN $\tilde{A}_2$ GROUP

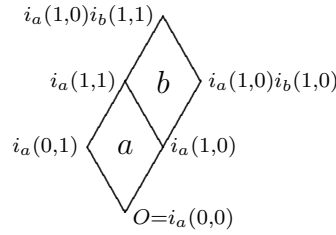
Now suppose that  $\Gamma$  is an  $\tilde{A}_2$  group. This means that  $\Gamma$  is a group of automorphisms of the  $\tilde{A}_2$  building  $\mathcal{B}$  which acts freely and transitively in a type rotating manner on the vertex set of  $\mathcal{B}$ . If  $\Omega$  is the boundary of  $\mathcal{B}$  then the algebra  $\mathcal{A} = \mathcal{A}(\Gamma)$  was studied in [RS1], [RS2, Section 7]. Suppose that the building  $\mathcal{B}$  has order  $q$ . If  $q = 2$  there are eight  $\tilde{A}_2$  groups  $\Gamma$ , all of which embed as lattices in a linear group  $\text{PGL}(3, \mathbb{F})$  over a local field  $\mathbb{F}$ . If  $q = 3$  there are 89 possible  $\tilde{A}_2$  groups, of which 65 do not embed naturally in linear groups.

The 1-skeleton of  $\mathcal{B}$  is the Cayley graph of the group  $\Gamma$  with respect to its canonical set  $P$  of  $(q^2 + q + 1)$  generators. The set  $P$  is identified

with the set of points of a finite projective plane  $(P, L)$  and the set of lines  $L$  is identified with  $\{x^{-1}; x \in P\}$ . The relations satisfied by the elements of  $P$  are of the form  $xyz = 1$ . There is such a relation if and only if  $y \in x^{-1}$ , that is the point  $y$  is incident with the line  $x^{-1}$  in the projective plane  $(P, L)$ . See [CMSZ] for details.

Since  $\Gamma$  acts freely and transitively on the vertices of  $\mathcal{B}$ , each element  $a \in A$  has a unique representative isometry  $i_a : \mathfrak{t} \rightarrow \mathcal{B}$  such that  $i_a(0, 0) = O$ , the fixed base vertex of  $\mathcal{B}$ . We assume for definiteness that the vertex  $O$  has type 0. It then follows that the vertex  $i_a(1, 0)$  has type 1,  $i_a(0, 1)$  has type 2 and  $i_a(1, 1)$  has type 0. The combinatorics of the finite projective plane  $(P, L)$  shows that there are precisely  $q(q+1)(q^2+q+1)$  possible choices for  $i_a$ . That is  $\#(A) = q(q+1)(q^2+q+1)$ . Thinking of the 1-skeleton of  $\mathcal{B}$  as the Cayley graph of the group  $\Gamma$  with  $O = e$ , we shall identify elements of  $\Gamma$  with vertices of  $\mathcal{B}$  via  $\gamma \mapsto \gamma(O)$ .

We now examine the transition matrices  $M_1, M_2$  in this situation. If  $a, b \in A$ , we have  $M_1(b, a) = 1$  if and only if there are representative isometries in  $a$  and  $b$  respectively whose ranges are tiles which lie as shown in Figure 18. More precisely this means that the ranges  $i_a(\mathfrak{t})$  and  $i_a(1, 0)b(\mathfrak{t})$  lie in the building as shown in Figure 18, where they are labeled  $a$  and  $b$  respectively.



$$M_1(b, a) = 1$$

FIGURE 18. Transitions between tiles

**Lemma 7.1.** *The  $\{0, 1\}$ -matrices  $M_j$  ( $j = 1, 2$ ) have order  $\#(A) = q(q+1)(q^2+q+1)$  and each row or column has precisely  $q^2$  nonzero entries.*

*Proof.* Suppose that  $a \in A$  has been chosen. Refer to Figure 18. In the link of the vertex  $i_a(1, 1)$ , let the vertices of type 1 correspond to points in  $P$  and the vertices of type 2 correspond to lines in  $L$ . There are then  $q+1$  choices for a line incident with the point  $i_a(0, 1)$ ; therefore there are  $q$  choices for  $i_a(1, 0)i_b(1, 0)$ . After choosing  $i_a(1, 0)i_b(1, 0)$  there are  $q$  choices for the point  $i_a(1, 0)i_b(1, 1)$ . That choice determines  $b$ . There are therefore  $q^2$  choices for  $b$ . This proves that for each  $a \in A$ , there are  $q^2$  choices for  $b \in A$  such that  $M_1(b, a) = 1$ . That is, each column of the matrix  $M_1$  has precisely  $q^2$  nonzero entries. A similar argument applies to rows.  $\square$

In order to compute the K-theory of  $\mathcal{A} = \mathcal{A}(\Gamma)$ , it follows from Section 6 that we need only compute  $\text{coker}({}_I\tilde{M}_1, {}_I\tilde{M}_2)$  or equivalently  $\text{coker}({}_I\tilde{M}_1, {}_I\tilde{M}_2)$ . For definiteness we deal in detail with the former. We shall see that this reduces the order of the matrices by a factor of  $q$ . Since  $\Gamma$  acts freely and transitively on the vertices of  $\mathcal{B}$ , each class  $\hat{a} \in \hat{A}$  contains a unique representative isometry  $\hat{i}_a : \mathfrak{t} \rightarrow \mathcal{B}$  such that  $\hat{i}_a(1, 1) = O$ , the fixed base vertex of  $\mathcal{B}$ . The isometry  $\hat{i}_a$  is completely determined by its range which is a triangle in  $\mathcal{B}$  whose edges are labeled by generators in  $P$ , according to the structure of the 1-skeleton of  $\mathcal{B}$  as a Cayley graph. In this way the element  $\hat{a} \in \hat{A}$  may be identified with an ordered triple  $(a_0, a_1, a_2)$ , where  $a_0, a_1, a_2 \in P$  and  $a_0 a_1 a_2 = 1$ . See Figure 19.

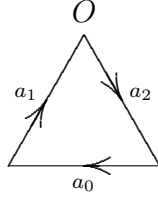


FIGURE 19. Representation of  $\hat{a}$

Note that in this representation of  $\hat{a}$  there are  $(q^2 + q + 1)$  choices for  $a_0$ . Having chosen  $a_0$ , there are  $q + 1$  choices for  $a_1$ , since  $a_1$  is incident with  $a_0^{-1}$ . The element  $a_2$  is then uniquely determined. This shows that  $\#(\hat{A}) = (q + 1)(q^2 + q + 1)$ .

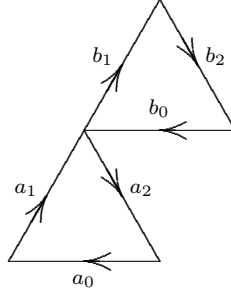
Given  $\hat{a} \in \hat{A}$ , an element  $\hat{b} \in \hat{A}$  satisfies  $\hat{M}_1(\hat{b}, \hat{a}) = 1$  if and only if the 1-skeleton of  $\mathcal{B}$  contains a diagram of the form shown in Figure 20. In terms of the projective plane  $(P, L)$ , this diagram is possible if and only if  $b_1 \notin a_1^{-1}$  ( $q^2$  choices for  $b_1$ ). Then  $b_0$  is uniquely specified by  $b_0^{-1} = b_1 \vee a_2$ , the line containing the points  $b_1$  and  $a_2$ . This determines  $b_2$  and hence  $\hat{b}$ . Thus  $\hat{M}_1$  is a  $\{0, 1\}$ -matrix of order  $(q + 1)(q^2 + q + 1)$ , whose entries are specified by

$$(7.1) \quad \hat{M}_1(\hat{b}, \hat{a}) = 1 \iff b_1 \notin a_1^{-1}, \quad b_0^{-1} = b_1 \vee a_2.$$

In particular for a fixed  $\hat{a} \in \hat{A}$  we have  $\hat{M}_1(\hat{b}, \hat{a}) = 1$  for precisely  $q^2$  choices of  $\hat{b} \in \hat{A}$ . That is, each column of the matrix  $\hat{M}_1$  has precisely  $q^2$  nonzero entries. Analogously, for a fixed  $\hat{b} \in \hat{A}$  we have  $\hat{M}_1(\hat{b}, \hat{a}) = 1$  for precisely  $q^2$  choices of  $\hat{a} \in \hat{A}$ . Again refer to Figure 20. There are precisely  $q^2$  choices of the line  $a_1^{-1}$  such that  $b_1 \notin a_1^{-1}$ . Then  $a_2$  is specified by  $a_2 = a_1^{-1} \wedge b_0^{-1}$  and this determines  $\hat{a}$  completely.

A similar argument shows that the  $\{0, 1\}$ -matrix  $\hat{M}_2$  is specified by

$$(7.2) \quad \hat{M}_2(\hat{b}, \hat{a}) = 1 \iff a_2 \notin b_2^{-1}, \quad b_0 = a_1^{-1} \wedge b_2^{-1}.$$

FIGURE 20.  $\hat{M}_1(\hat{b}, \hat{a}) = 1$ 

Using the preceding discussion and the explicit triangle presentations for  $\tilde{A}_2$  groups given in [CMSZ], we may now proceed to compute the K-theory of the algebra  $\mathcal{A}$  by means of Corollary 6.2, with “upward pointing” triangles. The authors have done extensive computations for more than 100 different groups with  $2 \leq q \leq 11$ , including all possible  $\tilde{A}_2$  groups for  $q = 2, 3$ . The complete results are available at

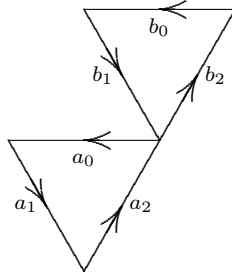
<http://maths.newcastle.edu.au/~guyan/Kcomp.ps.gz>

or from either of the authors.

Everything above applies mutatis mutandis for “downward pointing” triangles. In an obvious notation, illustrated in Figure 21, we have

$$(7.3) \quad \check{M}_1(\check{b}, \check{a}) = 1 \iff b_2 \notin a_2^{-1}, \quad a_0 = b_1^{-1} \wedge a_2^{-1}.$$

The accuracy of our computations was confirmed by repeating them with “downward pointing” triangles.

FIGURE 21.  $\check{M}_1(\check{b}, \check{a}) = 1$ 

It is convenient to summarize the general structure of the matrices we are considering.

**Lemma 7.2.** *The  $\{0, 1\}$ -matrices  $\hat{M}_j$ ,  $\check{M}_j$  ( $j = 1, 2$ ) have order  $\#(\hat{A}) = (q + 1)(q^2 + q + 1)$  and each row or column has precisely  $q^2$  nonzero entries.*

**Example 7.3.** Consider the following two  $\tilde{A}_2$  groups, which are both torsion free lattices in  $\mathrm{PGL}(3, \mathbb{Q}_2)$ , where  $\mathbb{Q}_2$  is the field of 2-adic numbers [CMSZ].

The group B.2 of [CMSZ], which we shall denote  $\Gamma_{B.2}$  has presentation

$$\langle x_i, 0 \leq i \leq 6 \mid x_0x_1x_4, x_0x_2x_1, x_0x_4x_2, x_1x_5x_5, x_2x_3x_3, x_3x_5x_6, x_4x_6x_6 \rangle.$$

The group C.1 of [CMSZ], which we shall denote  $\Gamma_{C.1}$  has presentation

$$\langle x_i, 0 \leq i \leq 6 \mid x_0x_0x_6, x_0x_2x_3, x_1x_2x_6, x_1x_3x_5, x_1x_5x_4, x_2x_4x_5, x_3x_4x_6 \rangle.$$

These groups are not isomorphic. Indeed the MAGMA computer algebra package shows that  $\Gamma_{B.2}$  has a subgroup of index 5, whereas  $\Gamma_{C.1}$  does not. This non isomorphism is revealed by the K-theory of the boundary algebras. Performing the computations above shows that

$$K_0(\mathcal{A}(\Gamma_{B.2})) = K_1(\mathcal{A}(\Gamma_{B.2})) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z},$$

$$K_0(\mathcal{A}(\Gamma_{C.1})) = K_1(\mathcal{A}(\Gamma_{C.1})) = (\mathbb{Z}/2\mathbb{Z})^4 \oplus \mathbb{Z}/3\mathbb{Z}.$$

These examples are not typical in that  $K_*$  of a boundary algebra usually has a free abelian component. Note also that in both these cases  $[1] = 0$  in  $K_0(\mathcal{A}(\Gamma))$ . See Remark 8.4.

On the other hand, using the results of V. Lafforgue [La], the K-theory of the reduced group  $C^*$ -algebras of these groups can easily be computed. The result is the same for these two groups:

$$K_0(C_r^*(\Gamma_{B.2})) = K_0(C_r^*(\Gamma_{C.1})) = \mathbb{Z},$$

$$K_1(C_r^*(\Gamma_{B.2})) = K_1(C_r^*(\Gamma_{C.1})) = (\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z}.$$

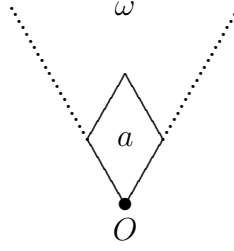
## 8. THE CLASS OF THE IDENTITY IN K-THEORY

Continue with the assumptions of Section 7; that is  $\Gamma$  is an  $\tilde{A}_2$  group. Since the algebras  $\mathcal{A}(\Gamma)$  are purely infinite, simple, nuclear and satisfy the Universal Coefficient Theorem [RS2, Remark 6.5], it follows from the Classification Theorem of [Kir, Ph] that they are classified by their K-groups together with the class  $[1]$  in  $K_0$  of the identity element  $1$  of  $\mathcal{A}(\Gamma)$ . It is therefore important to identify this class. We prove that  $[1]$  is a torsion element of  $K_0$ .

Let  $i \in \mathfrak{T}$ , that is, suppose that  $i : \mathfrak{t} \rightarrow \mathcal{B}$  is a type rotating isometry. Let  $\Omega(i)$  be the subset of  $\Omega$  consisting of those boundary points represented by sectors which originate at  $i(0, 0)$  and contain  $i(\mathfrak{t})$ . Clearly  $\Omega(\gamma i) = \gamma \Omega(i)$  for  $\gamma \in \Gamma$ . For each  $i \in \mathfrak{t}$  let  $\mathbf{1}_i$  be the characteristic function of the set  $\Omega(i)$ .

**Lemma 8.1.** *If  $i_1, i_2 \in \mathfrak{T}$  with  $\Gamma i_1 = \Gamma i_2$  then  $[\mathbf{1}_{i_1}] = [\mathbf{1}_{i_2}]$ .*

*Proof.* If  $i_1 = \gamma i_2$  with  $\gamma \in \Gamma$  then the covariance condition for the action of  $\Gamma$  on  $C(\Omega)$  implies that  $\mathbf{1}_{i_1} = \gamma \mathbf{1}_{i_2} \gamma^{-1}$ . The result now follows because equivalent idempotents belong to the same class in  $K_0$ .  $\square$

FIGURE 22.  $\mathbf{1}_a(\omega) = 1$ 

For each  $a \in A$  let  $\mathbf{1}_a = \mathbf{1}_{i_a}$ . See Figure 22. It follows from the discussion in [CMS, Section 2] that the identity function in  $C(\Omega)$  may be expressed as  $\mathbf{1} = \sum_{a \in A} \mathbf{1}_a$ .

**Proposition 8.2.** *In the group  $K_0(\mathcal{A}(\Gamma))$  we have  $(q^2 - 1)[\mathbf{1}] = 0$ .*

*Proof.* Referring to Figure 18, we have for each  $a \in A$ ,  $\mathbf{1}_a = \sum \mathbf{1}_{i_a(1,0)i_b}$ , where the sum is over all  $b \in A$  such that  $i_b(\mathbf{t})$  lies as shown in Figure 18; that is the sum is over all  $b \in A$  such that  $M_1(b, a) = 1$ . Now by Lemma 8.1,  $[\mathbf{1}_{i_a(1,0)i_b}] = [\mathbf{1}_{i_b}] = [\mathbf{1}_b]$  and so

$$[\mathbf{1}_a] = \sum_{b \in A} M_1(b, a)[\mathbf{1}_b].$$

It follows that

$$[\mathbf{1}] = \sum_{a \in A} [\mathbf{1}_a] = \sum_{a \in A} \sum_{b \in A} M_1(b, a)[\mathbf{1}_b].$$

By Lemma 7.1, there are  $q^3(q+1)(q^2+q+1)$  nonzero terms in this double sum and each term  $[\mathbf{1}_b]$  occurs exactly  $q^2$  times. Thus  $[\mathbf{1}] = q^2 \sum_{a \in A} [\mathbf{1}_a] = q^2[\mathbf{1}]$ , which proves the result.  $\square$

**Proposition 8.3.** *For  $q \not\equiv 1 \pmod{3}$ ,  $q - 1$  divides the order of  $[\mathbf{1}]$ . For  $q \equiv 1 \pmod{3}$ ,  $(q - 1)/3$  divides the order of  $[\mathbf{1}]$ .*

*Proof.* By [RS2, Theorem 7.7], the algebra  $\mathcal{A}(\Gamma)$  is isomorphic to the algebra  $\mathcal{A}$ , which is in turn stably isomorphic to the algebra  $\mathcal{F} \rtimes \mathbb{Z}^2$  [RS2, Theorem 6.2]. We refer to Section 1 for notation and terminology. Recall that  $\mathcal{F} = \varinjlim \mathcal{C}^{(m)}$  where  $\mathcal{C}^{(m)} \cong \bigoplus_{a \in A} \mathcal{K}(\mathcal{H}_a)$ . The isomorphism  $\mathcal{A}(\Gamma) \rightarrow \mathcal{A}$  has the effect  $\mathbf{1}_a \mapsto s_{a,a}$  and the isomorphism  $\mathcal{A} \otimes \mathcal{K} \rightarrow \mathcal{F} \rtimes \mathbb{Z}^2$  sends  $s_{a,a} \otimes E_{1,1}$  to a minimal projection  $P_a \in \mathcal{K}(\mathcal{H}_a) \subset \mathcal{C}^{(0)} \subset \mathcal{F}$ .

As an abelian group in terms of generators and relations, we have

$$(8.1) \quad \text{coker}(I - M_1, I - M_2) = \langle e_a; e_a = \sum_b M_j(b, a)e_b, j = 1, 2 \rangle.$$

By Lemma 4.11  $\text{coker}(I - M_1, I - M_2)$  is isomorphic to  $\mathfrak{H}_0 = \text{coker}(1 - T_{2*}, T_{1*} - 1)$ . Under this identification,  $\sum_{a \in A} e_a$  maps to the coset of  $[\mathbf{1}] \in K_0(\mathcal{F})$ . By Remark 4.3 that coset maps to  $[\mathbf{1}] \in K_0(\mathcal{F} \rtimes \mathbb{Z}^2)$  under the injection of (4.3). Thus the order of  $[\mathbf{1}] \in K_0(\mathcal{F} \rtimes \mathbb{Z}^2)$  is equal to the order of  $\sum_{a \in A} e_a$  in  $\text{coker}(I - M_1, I - M_2)$ .

Each of the relations in the equation (8.1) expresses a generator  $e_a$  as the sum of exactly  $q^2$  generators. It follows that there exists a homomorphism  $\psi$  from  $\text{coker}(I-M_1, I-M_2)$  to  $\mathbb{Z}/(q^2 - 1)$  which sends each generator to  $1 + (q^2 - 1)\mathbb{Z}$ . As  $\sum_{a \in A} e_a$  has  $q(q+1)(q^2 + q + 1)$  terms,

$$\psi\left(\sum_{a \in A} e_a\right) \equiv q(q+1)(q^2 + q + 1) \equiv 3(q+1) \pmod{q^2 - 1}$$

Consequently, the order of  $\psi(\sum_{a \in A} e_a)$  is

$$\begin{aligned} \frac{q^2 - 1}{(q^2 - 1, 3(q+1))} &= \frac{q^2 - 1}{(q+1)(q-1, 3)} = \frac{q-1}{(q-1, 3)} \\ &= \begin{cases} q-1 & \text{if } q \not\equiv 1 \pmod{3}, \\ (q-1)/3 & \text{if } q \equiv 1 \pmod{3}. \end{cases} \end{aligned}$$

The result follows since the order of  $\sum_{a \in A} e_a$  is necessarily a multiple of the order of  $\psi(\sum_{a \in A} e_a)$ .  $\square$

**Remark 8.4.** Propositions 8.2 and 8.3 give upper and lower bounds for the order of  $[\mathbf{1}]$  in  $K_0$ . The authors have computed the K-groups for the boundary algebras associated with more one hundred different  $\tilde{A}_2$  groups, for  $2 \leq q \leq 11$ . These numerical results strongly suggest that if  $q \not\equiv 1 \pmod{3}$  [respectively  $q \equiv 1 \pmod{3}$ ] then the order of  $[\mathbf{1}]$  is precisely  $q-1$  [respectively  $(q-1)/3$ ]. Our computational results are complete in two cases: if  $q = 2$  then  $[\mathbf{1}] = 0$  and if  $q = 3$  then  $[\mathbf{1}]$  has order 2.

Propositions 8.2 and 8.3 show that if  $q \neq 2, 4$  then  $[\mathbf{1}]$  is a nonzero torsion element in  $K_0(\mathcal{A}(\Gamma))$ . It follows from Corollary 5.2 that for  $q \neq 2, 4$  the corresponding algebras are not isomorphic to any rank one Cuntz-Krieger algebra. The only group  $\Gamma$  among the eight groups for  $q = 2$ , for which  $K_0(\mathcal{A}(\Gamma))$  is torsion free is the group B.3. We do not know if such a group exists for  $q = 4$ .

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