AFFINE BUILDINGS, TILING SYSTEMS AND HIGHER RANK CUNTZ-KRIEGER ALGEBRAS

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ABSTRACT. To an r-dimensional subshift of finite type satisfying certain special properties we associate a C^*-algebra $\mathcal{A}$. This algebra is a higher rank version of a Cuntz-Krieger algebra. In particular, it is simple, purely infinite and nuclear. We study an example: if Γ is a group acting freely on the vertices of an $\widetilde{A}_2$ building, with finitely many orbits, and if Ω is the boundary of that building, then $C(\Omega) \rtimes \Gamma$ is the algebra associated to a certain two dimensional subshift.

INTRODUCTION

This paper falls into two parts. The self-contained first part develops the theory of a class of C^*-algebras which are higher rank generalizations of the Cuntz-Krieger algebras [CK, C1, C2]. We start with a set of r-dimensional words, based on an alphabet A, we define transition matrices M_j in each of r directions, satisfying certain conditions (H0)-(H3). The C^*-algebra $\mathcal{A}$ is then the unique C^*-algebra generated by a family of partial isometries $s_{u,v}$ indexed by compatible r-dimensional words u, v and satisfying relations (0.1) below. If $r = 1$ then $\mathcal{A}$ is a Cuntz–Krieger algebra. We prove that the algebra $\mathcal{A}$ is simple, purely infinite and stably isomorphic to the crossed product of an AF-algebra by a $\mathbb{Z}^r$-action.

The last part of the paper (Section 7) studies in detail one particularly interesting example. This example was the authors' motivation for introducing these algebras. Let $\mathcal{B}$ be an affine building of type $\widetilde{A}_2$. Let Γ be a group of type rotating automorphisms of $\mathcal{B}$ which acts freely on the vertex set with finitely many orbits. There is a natural action of Γ on the boundary Ω of $\mathcal{B}$, and we can form the universal crossed product algebra $C(\Omega) \rtimes \Gamma$. This algebra is isomorphic to an algebra of the form $\mathcal{A}$ obtained by the preceding construction. In the case where Γ also acts transitively on the vertices of $\mathcal{B}$ the algebra $C(\Omega) \rtimes \Gamma$ was previously studied in [RS1], where simplicity was proved. In a sequel to this paper [RS2] we explicitly compute the K-theory of some of these algebras.

In [Sp] J. SPIELBERG treated an analogous example in rank 1: Γ is a free group, the building is a tree, and $C(\Omega) \rtimes \Gamma$ is isomorphic to an ordinary Cuntz-Krieger algebra. In fact [Sp] deals more generally with the case where Γ is a free product of cyclic groups. The generalizations in this paper are motivated by Spielberg's work.

We now introduce some basic notation and terminology. Let $\mathbb{Z}_+$ denote the set of nonnegative integers. Let $[m, n]$ denote $\{m, m + 1, \ldots, n\}$, where $m \leq n$ are integers. If $m, n \in \mathbb{Z}^r$, say that $m \leq n$ if $m_j \leq n_j$ for $1 \leq j \leq r$, and when $m \leq n$, let $[m, n] = [m_1, n_1] \times \cdots \times [m_r, n_r]$. In $\mathbb{Z}^r$, let 0 denote the zero vector and let e_j denote the jth standard unit basis vector. We fix a finite set A (an "alphabet").

A $\{0, 1\}$-*matrix* is a matrix with entries in $\{0, 1\}$. Choose nonzero $\{0, 1\}$-matrices $M_1, M_2, \ldots, M_r$ and denote their elements by $M_j(b, a) \in \{0, 1\}$ for $a, b \in A$. If $m, n \in \mathbb{Z}^r$ with $m \leq n$, let

$$W_{[m,n]} = \{w : [m, n] \to A; \ M_j(w(l + e_j), w(l)) = 1 \text{ whenever } l, l + e_j \in [m, n]\}.$$

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Put $W_m = W_{[0,m]}$ if $m \geq 0$. Say that an element $w \in W_m$ has *shape m*, and write $\sigma(w) = m$. Thus W_m is the set of words of shape m, and we identify A with W_0 in the natural way. Define the initial and final maps $o : W_m \to A$ and $t : W_m \to A$ by $o(w) = w(0)$ and $t(w) = w(m)$. Fix a nonempty finite or countable set D (whose elements are "decorations"), and a map $\delta : D \to A$. Let $\overline{W}_m = \{(d,w) \in D \times W_m; \, o(w) = \delta(d)\}$, the set of "decorated words" of shape m, and identify D with $\overline{W}_0$ via the map $d \mapsto (d, \delta(d))$. Let $W = \bigcup_m W_m$ and $\overline{W} = \bigcup_m \overline{W}_m$, the sets of all words and all decorated words respectively. Define $o : \overline{W}_m \to D$ and $t : \overline{W}_m \to A$ by $o(d,w) = d$ and $t(d,w) = t(w)$. Likewise extend the definition of shape to $\overline{W}$ by setting $\sigma((d,w)) = \sigma(w)$.

Given $j \leq k \leq l \leq m$ and a function $w : [j,m] \to A$, define $w|_{[k,l]} \in W_{l-k}$ by $w|_{[k,l]} = w'$ where $w'(i) = w(i+k)$ for $0 \leq i \leq l-k$. If $\overline{w} = (d,w) \in \overline{W}_m$, define

$$\overline{w}|_{[k,l]} = w|_{[k,l]} \in W_{l-k} \text{ if } k \neq 0,$$

$$\text{and} \qquad \overline{w}|_{[0,l]} = (d, w|_{[0,l]}) \in \overline{W}_l.$$

If $w \in W_l$ and $k \in \mathbb{Z}^r$, define $\tau_k w : [k, k+l] \to A$ by $(\tau_k w)(k+j) = w(j)$. If $w \in W_l$ where $l \geq 0$ and if $p \neq 0$, say that w is *p-periodic* if its p-translate, $\tau_p w$, satisfies $\tau_p w|_{[0,l] \cap [p,p+l]} = w|_{[0,l] \cap [p,p+l]}$.

Assume that the matrices M_i have been chosen so that the following conditions hold.

   (H0): Each M_i is a nonzero $\{0,1\}$-matrix.
   (H1): Let $u \in W_m$ and $v \in W_n$. If $t(u) = o(v)$ then there exists a unique $w \in W_{m+n}$ such that

$$w|_{[0,m]} = u \qquad \text{and} \qquad w|_{[m,m+n]} = v.$$

   (H2): Consider the directed graph which has a vertex for each $a \in A$ and a directed edge from a to b for each i such that $M_i(b,a) = 1$. This graph is irreducible.
   (H3): Let $p \in \mathbb{Z}^r$, $p \neq 0$. There exists some $w \in W$ which is not p-periodic.

Definition 0.1. In the situation of (H1) we write $w = uv$ and say that the product uv exists. This product is clearly associative.

The C^*-algebra $\mathcal{A}$ is defined as the universal C^*-algebra generated by a family of partial isometries $\{s_{u,v}; \, u,v \in \overline{W} \text{ and } t(u) = t(v)\}$ satisfying the relations

(0.1a) $$s_{u,v}{}^* = s_{v,u}$$

(0.1b) $$s_{u,v} s_{v,w} = s_{u,w}$$

(0.1c) $$s_{u,v} = \sum_{\substack{w \in W; \sigma(w) = e_j, \\ o(w) = t(u) = t(v)}} s_{uw,vw}, \quad \text{for } 1 \leq j \leq r$$

(0.1d) $$s_{u,u} s_{v,v} = 0, \quad \text{for } u,v \in \overline{W}_0, u \neq v.$$

1. Products of higher rank words

Condition (H1) is fundamental to all that follows. How then, does one verify (H1)? Given $\{0,1\}$-matrices M_i, $1 \leq i \leq r$, the following three simple conditions will be seen to be sufficient.

   (H1a): $M_i M_j = M_j M_i$.
   (H1b): For $i < j$, $M_i M_j$ is a $\{0,1\}$-matrix.
   (H1c): For $i < j < k$, $M_i M_j M_k$ is a $\{0,1\}$-matrix.

Indeed, the first two conditions are also necessary.

Lemma 1.1. *Fix $\{0,1\}$-matrices M_i, $1 \leq i \leq r$. Then* (H1) *implies* (H1a) *and* (H1b).

Proof. Suppose $(M_i M_j)(b,a) > 0$. Then there exists $c \in A$ so that $M_j(c,a) = 1 = M_i(b,c)$. Let $u \in W_{e_j}$ and $v \in W_{e_i}$ be given by

$$u(0) = a \qquad u(e_j) = c \qquad\qquad v(0) = c \qquad v(e_i) = b.$$

According to (H1) there is a unique $w \in W_{e_i+e_j}$ with $w(0) = a, w(e_j) = c, w(e_i + e_j) = b$. There must then be a unique $d \in A$ which can be used for the missing value of w, $w(e_i)$. That is, there must be a unique $d \in A$ satisfying $M_i(d, a) = 1 = M_j(b, d)$. Hence $(M_j M_i)(b, a) = 1$.

We have seen that if $(M_i M_j)(b, a) > 0$, then $(M_j M_i)(b, a) = 1$. Likewise, if $(M_j M_i)(b, a) > 0$, then $(M_i M_j)(b, a) = 1$. It follows that $M_i M_j$ and $M_j M_i$ are equal and have entries in $\{0, 1\}$. $\square$

Lemma 1.2. *Fix $\{0, 1\}$-matrices M_i satisfying* (H1a),(H1b), *and* (H1c). *Let $1 \leq j \leq r$. Let $w \in W_m$ and choose $a \in A$ so that $M_j(a, t(w)) = 1$. Then there exists a unique word $v \in W_{m+e_j}$ such that $v|_{[0,m]} = w$ and $t(v) = a$.*

Proof. In the case $r = 2$ this follows from conditions (H1a) and (H1b) alone. The situation is illustrated in Figure 1, for $j = 2$. The assertion is that there is a unique word v defined on the outer rectangle $[0, m + e_2]$ with final letter $v(m + e_2) = a$. The hypothesis is that there is a transition from $w(m)$ to a, in the sense that $M_2(a, w(m)) = 1$. Define $v(m + e_2) = a$. For notational convenience, let $n = m - e_1$. We have $M_1(w(m), w(n)) = 1$, and the product matrix $M_2 M_1$ defines a transition $w(n) \to w(m) \to a$. The conditions (H1a) and (H1b) assert that the product $M_1 M_2$ defines a unique transition $w(n) \to b \to a$, for some $b \in A$. Define $v(n + e_2) = b$. Continue the process inductively until v is defined uniquely on the whole of $[0, m + e_2]$. This completes the proof if $r = 2$.
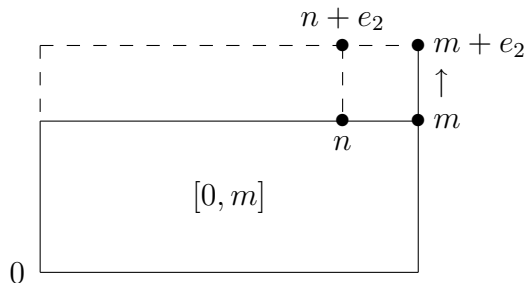

FIGURE 1. The case $r = 2$.

Now consider the case $r = 3$. Proceeding by induction as in the case $r = 2$, the extension problem reduces to that for a single cube. Consider therefore without loss of generality the unit cube based at 0 with $m = e_1 + e_2$ and $j = 3$, as illustrated in Figure 2. Then w is defined on the base of the cube $[0, m]$ and it is required to extend w to a function v on the whole cube taking the value a at $m + e_3$, under the assumption that there is a valid transition from $w(m)$ to a. Now use the case $r = 2$ on successive faces of the cube. Working on the right hand face there is a unique possible value b for $v(e_1 + e_3)$. Then, using this value for $v(e_1 + e_3)$ on the near face we obtain the value c for $v(e_3)$. Similarly, working respectively on the back and left faces we obtain a value $v(e_2 + e_3) = d$ and a second value, c', for $v(e_3)$.

Now suppose that $c \neq c'$. Working on the top face and using the values a, d, and c', we obtain another value, b' for $v(e_1 + e_3)$. There are two possible transitions along the directed path $0 \to e_3 \to e_1 + e_3 \to m + e_3$, namely

$$w(0) \to c' \to b' \to a \quad \text{and} \quad w(0) \to c \to b \to a.$$

This contradicts the assumption that $(M_2 M_1 M_3)(a, w(0)) \in \{0, 1\}$.

The proof for general r now follows by induction. The uniqueness of the extension follows from the two dimensional considerations embodied in Figure 1. All the compatibility conditions required for existence follow from the three dimensional considerations of Figure 2. $\square$

The next result follows by induction from Lemma 1.2.

Lemma 1.3. *Fix $\{0, 1\}$-matrices M_i, $1 \leq i \leq r$, satisfying* (H1a), (H1b), *and* (H1c). *Let $1 \leq j_1, \ldots, j_p \leq r$, let $a_0, \ldots, a_p \in A$ and suppose that $M_{j_i}(a_i, a_{i-1}) = 1$ for $1 \leq i \leq p$. Then there exists a unique word $w \in W$ with $\sigma(w) = e_{j_1} + \cdots + e_{j_p}$, such that $w(0) = a_0$ and $w(e_{j_1} + \cdots + e_{j_i}) = a_i$ for $1 \leq i \leq p$.*
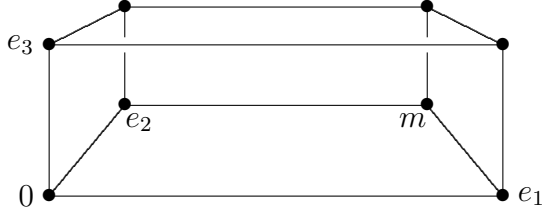
FIGURE 2. The case $r = 3$.

As a consequence we have

Lemma 1.4. *Fix $\{0,1\}$-matrices M_i, $1 \le i \le r$. If* (H1a), (H1b)*, and* (H1c) *hold, then* (H1) *holds.*

Proof. Let $u \in W_m$ and $v \in W_n$. Suppose $t(u) = o(v)$. Choose $j_1, \ldots, j_p$ as in Lemma 1.3 so that

$$e_{j_1} + \cdots + e_{j_q} = m \qquad e_{j_{q+1}} + \cdots + e_{j_p} = n$$

for some q, $0 \le q \le p$. Choose a_i, $0 \le i \le q$ so as to force the w of Lemma 1.3 to satisfy $w|_{[0,m]} = u$. Thus $a_q = t(u) = o(v)$. Choose a_i, $q \le i \le p$ so as to force w to satisfy $w|_{[m,m+n]} = v$. The existence and uniqueness in (H1) follow from the existence and uniqueness in Lemma 1.3. $\square$

Corollary 1.5. *If $\overline{u} = (d, u) \in \overline{W}_m$ and $v \in W_n$ with $t(\overline{u}) = o(v)$, then there exists a unique $\overline{w} \in \overline{W}_{m+n}$ such that*

$$\overline{w}|_{[0,m]} = \overline{u} \ and \ \overline{w}|_{[m,m+n]} = v.$$

In these circumstances we write $\overline{w} = \overline{u}v$, and say that the product $\overline{u}v$ exists.

Proof. This is immediate, with $\overline{u}v = (d, uv)$. $\square$

For the next two lemmas, and for the rest of the paper, suppose that matrices M_i have been chosen so that (H0)–(H2) hold.

Lemma 1.6. *Let $a, b \in A$ and $n \in \mathbb{Z}_+^r$. There exists $w \in W$ with $\sigma(w) \ge n$ such that $o(w) = a$ and $t(w) = b$.*

Proof. By condition (H0), the matrix M_j is nonzero, so there exists at least one word of shape e_j. Using this, choose words $w_1, \ldots, w_q$ so that $\sigma(w_1) + \cdots + \sigma(w_q) \ge n$. Using conditions (H1) and (H2), one can always find a word with a given origin and terminus. So choose $s_0 \in W$ with $o(s_0) = a$, $t(s_0) = o(w_1)$, choose $s_k \in W$ with $o(s_k) = t(w_k)$, $t(s_k) = o(w_{k+1})$ for $1 \le k \le q - 1$, and choose $s_q \in W$ with $o(s_q) = t(w_q)$, $t(s_q) = b$. Let $w = s_0 w_1 s_1 w_2 \ldots w_{q-1} s_{q-1} w_q s_q$. $\square$

Lemma 1.7. *Given $\overline{u} \in \overline{W}$ and $b \in A$, there exists $v \in W$ such that $\overline{u}v$ exists, $\sigma(v) \ne 0$ and*

$$t(v) = t(\overline{u}v) = b.$$

Proof. This follows immediately Lemma 1.6. $\square$

2. NONPERIODICITY

Assume that M_i, $1 \le i \le r$, have been chosen and that (H0)–(H2) hold. In the large class of examples associated to affine buildings it is fairly easy to verify the nonperiodicity condition, (H3). However, in general it is hard to see how one can start with the matrices M_i and check (H3). In this section we present a condition which implies (H3), and show how it can in principle be checked. This material is not used in the remainder of the paper.

(**H3***): Fix j, $1 \le j \le r$. Let $m \in \mathbb{Z}_+^r$ with $m_j = 0$. Let $w \in W_m$. Then there exist $u, u' \in W_{m+e_j}$ such that $u|_{[0,m]} = u'|_{[0,m]} = w$ but $u(e_j) \ne u'(e_j)$.

For $l, m \in \mathbb{Z}^r$, define

$$l \wedge m = (l_1 \wedge m_1, \ldots, l_r \wedge m_r),$$
$$l \vee m = (l_1 \vee m_1, \ldots, l_r \vee m_r),$$
$$|l| = l \vee (-l).$$

If $w \in W_l$ where $l \geq 0$ and if $p \neq 0$, recall that w is p-*periodic* if its p-translate, $\tau_p w$, satisfies $\tau_p w|_{[0,l] \cap [p, p+l]} = w|_{[0,l] \cap [p, p+l]}$.
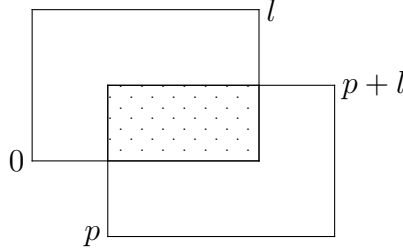

FIGURE 3. The region $[0, l] \cap [p, p+l]$.

Lemma 2.1. *Conditions* (H0)–(H2) *and* (H3*) *imply condition* (H3).

Proof. Observe that p-periodicity is invariant under replacement of p by $-p$. We may therefore assume that p has at least one positive component which we may take to be p_1. Let $p_+ = p \vee 0$ and $p_- = (-p) \vee 0$. We will construct $w \in W_{|p|}$. In this case w is defined on $[0, |p|]$, $\tau_p w$ is defined on $[p, |p|+p]$, and $\tau_p w$ and w are both defined on $[p \vee 0, |p| \wedge (|p|+p)] = [p_+, p_+]$, a single point. The word w is p-periodic if and only if $w(p_+) = (\tau_p w)(p_+) = w(p_+ - p) = w(p_-)$. The situation is illustrated in Figure 4. Choose any v in $W_{|p|-e_1}$ and let $u = v|_{[p_+ - e_1, |p|-e_1]} \in W_{p_-}$. By condition (H3*), there exist two different words $x, x' \in W_{p_- + e_1}$ such that $x|_{[0,p_-]} = x'|_{[0,p_-]}$ but $x(e_1) \neq x'(e_1)$. At least one of $x(e_1)$ and $x'(e_1)$ differs from $v(p_-)$; we may assume that $x(e_1) \neq v(p_-)$. Let w be defined by $w|_{[0,|p|-e_1]} = u$, $w|_{[p_+ - e_1, |p|]} = x$. Then $w(p_+) = x(e_1) \neq v(p_-) = w(p_-)$, so w is not p-periodic. $\square$
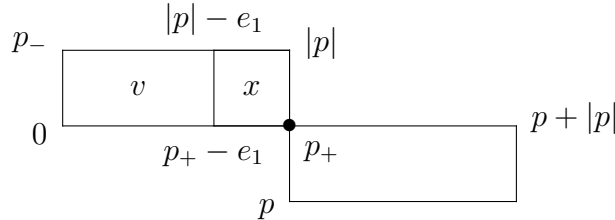

FIGURE 4. Proof of Lemma 2.1 in the case $r = 2$.

Now we discuss the checkability of (H3*). Fix j, $1 \leq j \leq r$. For $w \in W_m$ with $\sigma(w)_j = 0$ let

$$A(j, w) = \{u(e_j); u \in W_{m+e_j} \quad \text{and} \quad u|_{[0,m]} = w\}.$$

Given w, one can calculate $A(j, w)$ by considering, one at a time, the possible values of $u(m + e_j)$, and working back to find the possible values of $u(e_j)$ as in the proof of Lemma 1.2. The assertion of (H3*) is that $\#A(j, w) \geq 2$ for any w with $\sigma(w)_j = 0$.

Let $v, w \in W$ with $\sigma(v) = e_k$, $k \neq j$, $\sigma(w)_j = 0$, and suppose that vw is defined. Then

$$A(j, vw) = \{a \in A; M_j(a, o(v)) = 1 \text{ and } M_k(b, a) = 1 \text{ for some } b \in A(j, w)\}.$$

Thus one can calculate $A(j, vw)$ from the knowledge of v and $A(j, w)$.

To check (H3*) for the fixed value of j, one proceeds to construct, for each $c \in A$ a complete list of possibilities for $A(j, w)$ with $o(w) = c$. The first step in the algorithm is to insert in the lists all $A(j, w)$ for $w \in W_0$. Then proceeding cyclically through all words v with $\sigma(v) = e_k$, for all $k \neq j$, the algorithm adds to the lists all possible values of $A(j, vw)$ corresponding to values

$A(j, w)$ already on the lists. The algorithm terminates when a complete cycle through the words v generates no new possible values for $A(j, w)$. The algorithm works because any $w \in W$ can be written $w = v_1 v_2 \ldots v_q$, with $\sigma(v_i) = e_{k_i}$.

Is this algorithm practical for hand computation? for electronic computation? The authors have done no experiments, but they suspect that the lists of subsets of A will get out of hand rapidly as the cardinality of A increases. The situation is not entirely satisfactory.

3. The C^*-algebra

Assume conditions (H0)-(H3) hold. Define an abstract untopologized algebra $\mathcal{A}_0$ over $\mathbb{C}$ which depends on A, $(M_j)_{j=1}^r$, D, and δ. The generators of $\mathcal{A}_0$ are $\{s_{u,v}^0; u, v \in \overline{W} \text{ and } t(u) = t(v)\}$. The relations defining $\mathcal{A}_0$ are

$$s_{u,v}^0 s_{v,w}^0 = s_{u,w}^0$$

$$s_{u,v}^0 = \sum_{\substack{w \in W; \sigma(w) = e_j, \\ o(w) = t(u) = t(v)}} s_{uw,vw}^0, \text{ for } 1 \le j \le r$$

$$s_{u,u}^0 s_{v,v}^0 = 0 \text{ for } u, v \in \overline{W}_0, u \ne v.$$

It is trivial to verify that $\mathcal{A}_0$ has an antilinear antiautomorphism defined on the generators by

$$s_{u,v}^0{}^* = s_{v,u}^0.$$

This makes $\mathcal{A}_0$ a $*$-algebra. Let $\mathcal{A}$ be the corresponding enveloping C^*-algebra (c.f. [CK, p.256]) and let $s_{u,v}$ be the image of $s_{u,v}^0$ in $\mathcal{A}$. The generators of $\mathcal{A}$ are therefore

$$\{s_{u,v}; \ u, v \in \overline{W} \text{ and } t(u) = t(v)\}$$

and the defining relations are

(3.1a) $s_{u,v}{}^* \ = \ s_{v,u}$

(3.1b) $s_{u,v} s_{v,w} \ = \ s_{u,w}$

(3.1c) $s_{u,v} \ = \ \displaystyle\sum_{\substack{w \in W; \sigma(w) = e_j, \\ o(w) = t(u) = t(v)}} s_{uw,vw}, \text{ for } 1 \le j \le r$

(3.1d) $s_{u,u} s_{v,v} \ = \ 0 \text{ for } u, v \in \overline{W}_0, u \ne v.$

Remark 3.1. Suppose that $u, v \in \overline{W}$ and $t(u) = t(v)$. Then $s_{u,v}$ is a partial isometry with initial projection $s_{u,v}{}^* s_{u,v} = s_{v,v}$ and final projection $s_{u,v} s_{u,v}{}^* = s_{u,u}$.

Lemma 3.2. *Fix $m \in \mathbb{Z}_+^r$ and let $u, v \in \overline{W}$ with $t(u) = t(v)$. Then*

$$s_{u,v} = \sum_{\substack{w \in W; \sigma(w) = m \\ o(w) = t(u) = t(v)}} s_{uw,vw}.$$

Proof. Using (H1), this follows by induction from (3.1c). $\square$

Lemma 3.3. $s_{u,u} s_{v,v} = 0$ *if $\sigma(u) = \sigma(v)$ and $u \ne v$.*

Proof. The case $\sigma(u) = \sigma(v) = 0$ is exactly the relation (3.1d). Assume that the assertion is true whenever $\sigma(u) = \sigma(v) = m$. Let $\sigma(u') = \sigma(v') = m + e_j$ and let $u = u'|_{[0,m]}$, $v = v'|_{[0,m]}$. By relation (3.1c), we have that $s_{u,u} = \sum_w s_{uw,uw}$ where the sum is over $w \in W_{e_j}$ such that $o(w) = t(u)$. Since $s_{u',u'}$ is one of the terms of the preceding sum we have $s_{u,u} \ge s_{u',u'}$. Similarly $s_{v,v} \ge s_{v',v'}$. If $u \ne v$, this proves that $s_{u',u'} s_{v',v'} = 0$, since by induction $s_{u,u} s_{v,v} = 0$. On the other hand, if $u = v$ then $s_{u',u'}$, $s_{v',v'}$ are distinct terms in the sum $\sum_a s_{uw,uw}$ and are therefore orthogonal. $\square$

Remark 3.4. If $\overline{W}_0 = D$ is finite then it follows from (3.1d) that $\sum_{u \in \overline{W}_0} s_{u,u}$ is an idempotent. From Lemma 3.2 it follows that for any m, $\sum_{u \in \overline{W}_m} s_{u,u} = \sum_{u \in \overline{W}_0} s_{u,u}$. Hence from (3.1b) and Lemma 3.3 it follows that $\sum_{u \in \overline{W}_0} s_{u,u}$ is an identity for $\mathcal{A}$.

The next lemma is an immediate consequence of the definition of an enveloping C^*-algebra.

Lemma 3.5. *Let $\mathcal{H}$ be a Hilbert space and for each $u, v \in \overline{W}$ with $t(u) = t(v)$ let $S_{u,v} \in \mathcal{B}(\mathcal{H})$. If the $S_{u,v}$ satisfy the relations (3.1), then there is a unique *-homomorphism $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\phi(s_{u,v}) = S_{u,v}$.* $\square$

Lemma 3.6. *Any product $s_{u_1,v_1} s_{u_2,v_2}$ can be written as a finite sum of the generators $s_{u,v}$.*

Proof. Choose $m \in \mathbb{Z}^r$ with $m \geq \sigma(v_1)$ and $m \geq \sigma(u_2)$. Use Lemma 3.2 to write s_{u_1,v_1} as a sum of terms s_{u_3,v_3} with $\sigma(v_3) = m$. Likewise, write s_{u_2,v_2} as a sum of terms s_{u_4,v_4} with $\sigma(u_4) = m$. Now in the product $s_{u_1,v_1} s_{u_2,v_2}$ each term has the form

$$
s_{u_3,v_3} s_{u_4,v_4} = \begin{cases} s_{u_3,v_4} & \text{if } v_3 = u_4, \\ s_{u_3,v_3} s_{v_3,v_3} s_{u_4,u_4} s_{u_4,v_4} = 0 & \text{if } v_3 \neq u_4 \end{cases}
$$

by Lemma 3.3. The result follows immediately. $\square$

Corollary 3.7. *The C^*-algebra $\mathcal{A}$ is the closed linear span of the set*

$$
\{s_{u,v}; u, v \in \overline{W} \text{ and } t(u) = t(v)\}.
$$

$\square$

Lemma 3.8. *The algebra $\mathcal{A}$ is nonzero.*

Proof. We must construct a nonzero *-homomorphism from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Consider the set of infinite words

$$
W_\infty = \{w : \mathbb{Z}^r_+ \to A; M_j(w(l + e_j), w(l)) = 1 \text{ whenever } l \geq 0\},
$$

and define the product uv for $u \in \overline{W}$ and $v \in W_\infty$ exactly as in Definition 0.1. Let $\mathcal{H} = l^2(W_\infty)$ and define

$$
(\phi(s_{u,v}))(\delta_w) = \begin{cases} \delta_{uw_1} & \text{if } w = vw_1 \text{ for some } w_1 \in W_\infty, \\ 0 & \text{otherwise.} \end{cases}
$$

It is easy to check that the operators $\{\phi(s_{u,v}); u, v \in \overline{W} \text{ and } t(u) = t(v)\}$ satisfy the relations 3.1 and it follows from Lemma 3.5 that ϕ extends to a *-homomorphism of $\mathcal{A}$. $\square$

Remark 3.9. The Hilbert space $\mathcal{H} = l^2(W_\infty)$ is not separable. However, since the algebra $\mathcal{A}$ is countably generated, there exist nonzero separable, $\mathcal{A}$-stable subspaces of $\mathcal{H}$, and in particular, there exist nontrivial representations of $\mathcal{A}$ on separable Hilbert space.

Lemma 3.10. *If ϕ is a nontrivial representation of $\mathcal{A}$, and if $u \in \overline{W}$ then $\phi(s_{u,u}) \neq 0$. In particular $s_{u,u} \neq 0$.*

Proof. Suppose $\phi(s_{u,u}) = 0$. Choose any $v \in \overline{W}$. Use Lemma 1.6 to find $w \in W$ such that $o(w) = t(u)$ and $t(w) = t(v)$. By Lemma 3.2, we have $0 = \phi(s_{u,u}) = \sum \phi(s_{uw',uw'})$, the sum being taken over all $w' \in W$ such that $\sigma(w') = \sigma(w)$ and $o(w') = t(u)$. Thus $\phi(s_{uw,uw}) = 0$. By Remark 3.1 it follows that that $\phi(s_{uw,v}) = 0$, that $\phi(s_{v,v}) = 0$, and finally that $\phi(s_{v,v'}) = 0$ whenever $t(v) = t(v')$. Hence ϕ is trivial. $\square$

Remark 3.11. When $r = 1$, the algebra $\mathcal{A}$ is a simple Cuntz-Krieger algebra. More precisely, if we write $M = M_1^t$, then the Cuntz-Krieger algebra $\mathcal{O}_M$ is generated by a set of partial isometries $\{S_a; a \in A\}$ satisfying the relations $S_a^* S_a = \sum_b M(a,b) S_b S_b^*$. If $u \in W$, let $S_u = S_{u(0)} S_{u(1)} \ldots S_{t(u)}$ and if $v \in W$ with $t(u) = t(v)$, define $S_{u,v} = S_u S_v^*$ (c.f. [CK, Lemma 2.2]). The map $s_{u,v} \mapsto S_{u,v}$ establishes an isomorphism of $\mathcal{A}$ with $\mathcal{O}_M$. Tensor products of ordinary Cuntz-Krieger algebras

can be identified as higher rank Cuntz-Krieger algebras $\mathcal{A}$. If $\mathcal{A}_1, \mathcal{A}_2$ are simple rank one Cuntz-Krieger algebras, with corresponding matrices M_1, M_2 and alphabets A_1, A_2 then $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the algebra $\mathcal{A}$ arising from the pair of matrices $M_1 \otimes I, I \otimes M_2$ and the alphabet $A_1 \times A_2$. More interesting examples arise from group actions on affine buildings. The details for some $r = 2$ algebras arising in this way are given in Section 7.

4. THE AF SUBALGEBRA

If $m \in \mathbb{Z}_+^r$, let $\mathcal{F}_m$ denote the subalgebra of $\mathcal{A}$ generated by the elements $s_{u,v}$ for $u, v \in \overline{W}_m$.

Lemma 4.1. *There exists an isomorphism* $\mathcal{F}_m \cong \bigoplus_{a \in A} \mathcal{K}(l^2(\{w \in \overline{W}; \sigma(w) = m, t(w) = a\}))$.

Proof. Let $u, v \in \overline{W}_m$ with $t(u) = t(v) = a$. Consider the map $E^a_{\delta_u, \delta_v} \mapsto s_{u,v}$, where $E^a_{\delta_u, \delta_v}$ denotes a standard matrix unit in $\mathcal{K}(l^2(\{w \in \overline{W}; \sigma(w) = m, t(w) = a\}))$. This extends to a map which is an isomorphism according to equations (3.1a), (3.1b) and Lemma 3.3. $\square$

The relations (3.1c) show that there is a natural embedding of $\mathcal{F}_m$ into $\mathcal{F}_{m+e_j}$. The C^*-algebras $\{\mathcal{F}_m : m \in \mathbb{Z}_+^r\}$ form a directed system of C^*-algebras in the sense of [KR, p. 864]. By [KR, Proposition 11.4.1] there is an essentially unique C^*-algebra $\mathcal{F}$ in which the union of the algebras $\mathcal{F}_m$ is dense, namely the direct limit of these algebras. We have the following commuting diagram of inclusions.

$$
\begin{array}{ccc}
\mathcal{F}_{m+e_k} & \longrightarrow & \mathcal{F}_{m+e_j+e_k} \\
\uparrow & & \uparrow \\
\mathcal{F}_m & \longrightarrow & \mathcal{F}_{m+e_j}
\end{array}
$$

We may equally well regard $\mathcal{F}$ as the closure of $\bigcup_{j=1}^{\infty} \mathcal{F}_{jp}$, where $p = (1, 1, \ldots, 1)$. In particular $\mathcal{F}$ is an *AF*-algebra.

Proposition 4.2. *Let ϕ be a nonzero homomorphism from $\mathcal{A}$ into some C^*-algebra. Then the restriction of ϕ to $\mathcal{F}$ is an isomorphism.*

Proof. Suppose that ϕ is not an isomorphism on $\mathcal{F}$. Then ϕ is not an isomorphism on some $\mathcal{F}_m$. Since $\bigoplus_{a \in A} \mathcal{K}(l^2(\{w \in \overline{W}; \sigma(w) = m, t(w) = a\})) \cong \mathcal{F}_m$ (Lemma 4.1), it follows from simplicity of the algebra of compact operators that $\phi(s_{u,u}) = 0$ for some u, contradicting Lemma 3.10. $\square$

Define an action α of the r-torus $\mathbb{T}^r$ on $\mathcal{A}$ as follows. If $\sigma(u) - \sigma(v) = m \in \mathbb{Z}^r$ and $t = (t_1, \ldots, t_r) \in \mathbb{T}^r$, let $\alpha_t(s_{u,v}) = t^m s_{u,v}$, where $t^m = t_1^{m_1} t_2^{m_2} \ldots t_r^{m_r}$. The elements $\alpha_t(s_{u,v})$ satisfy the relations (3.1) and generate the C^*-algebra $\mathcal{A}$. By the universal property of $\mathcal{A}$ it follows that α_t extends to an automorphism of $\mathcal{A}$. It is easy to see that that $t \mapsto \alpha_t$ is an action. It is also clear that α_t fixes all elements $s_{u,v}$ with $\sigma(u) = \sigma(v)$ and so fixes $\mathcal{F}$ pointwise. We now show that $\mathcal{F} = \mathcal{A}^\alpha$, the fixed point subalgebra of $\mathcal{A}$. Consider the linear map on $\mathcal{A}$ defined by

$$
(4.1) \qquad\qquad \pi(x) = \int_{\mathbb{T}^r} \alpha_t(x) dt, \qquad \text{for } x \in \mathcal{A}
$$

where dt denotes normalized Haar measure on $\mathbb{T}^r$.

Lemma 4.3. *Let $\pi, \mathcal{F}$ be as above.*

    (1) *The map π is a faithful conditional expectation from $\mathcal{A}$ onto $\mathcal{F}$.*
    (2) *$\mathcal{F} = \mathcal{A}^\alpha$, the fixed point subalgebra of $\mathcal{A}$.*

Proof. Since the action α is continuous, it is easy to see that π is a conditional expectation from $\mathcal{A}$ onto $\mathcal{A}^\alpha$, and that it is faithful. Since α_t fixes $\mathcal{F}$ pointwise, $\mathcal{F} \subset \mathcal{A}^\alpha$. To complete the proof

we show that the range of π is contained in (and hence equal to) $\mathcal{F}$. By continuity of π and Corollary 3.7, it is enough to show that $\pi(s_{u,v}) \in \mathcal{F}$ for all $u, v \in \overline{W}$. This is so because

$$(4.2) \qquad \pi(s_{u,v}) = s_{u,v} \int_{\mathbb{T}^r} t^{\sigma(u)-\sigma(v)} dt = \begin{cases} 0 & \text{if } \sigma(u) \neq \sigma(v), \\ s_{u,v} & \text{if } \sigma(u) = \sigma(v). \end{cases}$$

$\square$

5. Simplicity

We show that the C^*-algebra $\mathcal{A}$ is simple. Consequently any nontrivial C^*-algebra with generators $S_{u,v}$ satisfying relations (3.1) is isomorphic to $\mathcal{A}$. Several preliminary lemmas are necessary. Consequences of (H3), their theme is the existence of words lacking certain periodicities. Recall that for $m \in \mathbb{Z}^r$, $|m| = (|m_1|, \ldots, |m_r|)$.

Lemma 5.1. *Let $m \in \mathbb{Z}^r$ with $m \geq 0$ and let $a \in A$. There exists some $l \geq 0$ and some $w \in W_l$ satisfying*

(1) *If $|p| \leq m$ and $p \neq 0$ then $\tau_p w|_{[0,l] \cap [p,p+l]} \neq w|_{[0,l] \cap [p,p+l]}$,*
(2) *$o(w) = a$.*

Proof. Note that if $w = uw_p v$, and if w_p is not p-periodic, then neither is w. Apply (H3) to obtain for each nonzero p, $|p| \leq m$, a word w_p which is not p-periodic. The final word w is obtained by concatenating these words in some order using "spacers" whose existence is guaranteed by Lemma 1.6. The construction is illustrated in Figure 5, where the spacer $s_{p,q}$ is chosen so that $o(s_{p,q}) = t(w_p)$ and $t(s_{p,q}) = o(w_q)$. $\square$
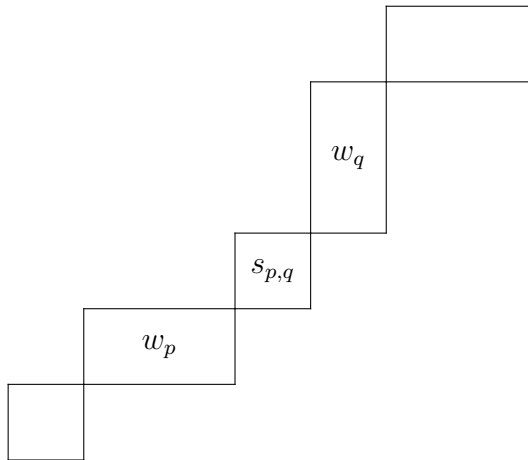

FIGURE 5. Part of the word w; a word $s_{p,q}$ is used to concatenate w_p and w_q.

Lemma 5.2. *One can find $u, u' \in W$ with $\sigma(u) = \sigma(u')$, $o(u) = o(u')$, but $u \neq u'$.*

Proof. Assume the contrary. Considering words of shape e_1, we see that for fixed a, no more than one $b \in A$ satisfies $M_1(b, a) = 1$. By Lemma 1.6, at least one $b \in A$ satisfies $M_1(b, a) = 1$ and at least one $c \in A$ satisfies $M_1(c, a) = 1$. Consequently the directed graph associated to M_1 must be a union of closed cycles. If g is the g.c.d of the cycle lengths, then every $w \in W$ is ge_1-periodic, contradicting (H3). $\square$

Lemma 5.3. *Let $p \in \mathbb{Z}^r$. Let $w_1, w_2 \in W_l$. There exist $l' \geq l$ and $w_1', w_2' \in W_{l'}$ such that*

$$w_1'|_{[0,l]} = w_1, \qquad w_2'|_{[0,l]} = w_2,$$

and

$$\tau_p w_1'|_{[0,l'] \cap [p,p+l']} \neq w_2'|_{[0,l'] \cap [p,p+l']}.$$

Proof. Find two different words u, v with $\sigma(u) = \sigma(v)$ and $o(u) = o(v)$. Choose s so that $p + \sigma(w_1) + \sigma(s) \geq 0$, $o(s) = t(w_1)$ and $t(s) = o(u) = o(v)$. Choose w_2'' so that $w_2''|_{[0,l]} = w_2$ and $\sigma(w_2'') \geq p + \sigma(w_1) + \sigma(s) + \sigma(u)$. Consider $w_2''|_{[p+\sigma(w_1)+\sigma(s), p+\sigma(w_1)+\sigma(s)+\sigma(u)]}$. If this is equal to u, let $w_1'' = w_1 s v$, otherwise, let $w_1'' = w_1 s u$. (This is illustrated in Figure 6.) Finally, let $l' = \sigma(w_1'') \vee \sigma(w_2'')$ and extend w_1'' and w_2'' to words w_1' and w_2' in $W_{l'}$.   □
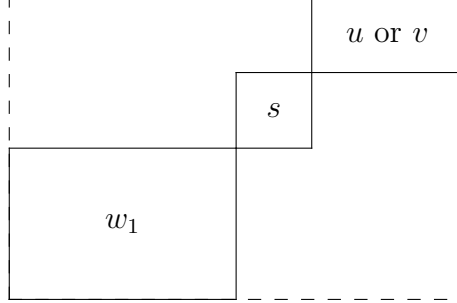

FIGURE 6. The word w_1''.

Lemma 5.4. *Fix $m \in \mathbb{Z}^r$ with $m \geq 0$. For some $l \geq 0$ there exists a subset $S = \{w_a; \ a \in A\}$ of W_l satisfying the two properties below.*

(1) *For each $a \in A$, $o(w_a) = a$.*
(2) *Let $a, b \in A$. Let $p \neq 0$ be in $\mathbb{Z}^r$ with $|p| \leq m$. Then $w_a|_{[0,l] \cap [p, p+l]} \neq \tau_p w_b|_{[0,l] \cap [p, p+l]}$.*

Proof. The elements of S are chosen as follows. For each $a \in A$, let $w_a \in W$ satisfy the conclusions of Lemma 5.1. By extending the words w_a as necessary we may suppose that $w_a \in W_l$ for some $l \geq 0$. If $a, b \in A$, $a \neq b$ and $p \in \mathbb{Z}^r$ with $|p| \leq m$, we can apply Lemma 5.3 and extend w_a and w_b to w_a' and w_b' where $\tau_p w_a'$ and w_b' do not agree on their common domain. Then one can extend all the other w_c to w_c' of the same shape as w_a' and w_b'. Apply this procedure once for each of the finitely many triples $(a, b, p) \in A \times A \times \mathbb{Z}^r$ with $|p| \leq m$ and $a \neq b$, and the proof is done.   □

Fix $m \in \mathbb{Z}_+^r$. Choose l and S satisfying the conditions of Lemma 5.4. Define

$$(5.1) \qquad Q = \sum_{\substack{w \in \overline{W}; \sigma(w) = m+l \\ w|_{[m, m+l]} \in S}} s_{w,w}.$$

Lemma 5.5. *If $l' \geq l$, then*

$$(5.2) \qquad Q = \sum_{\substack{w \in \overline{W}; \sigma(w) = m+l' \\ w|_{[m, m+l]} \in S}} s_{w,w}.$$

Proof. By (H1) and Lemma 3.2,

$$\sum_{\substack{w \in \overline{W}; \sigma(w) = m+l' \\ w|_{[m, m+l]} \in S}} s_{w,w} = \sum_{\substack{w_1 \in \overline{W}, w_2 \in W \\ \sigma(w_1) = m+l, \sigma(w_2) = l'-l \\ w_1|_{[m, m+l]} \in S, t(w_1) = o(w_2)}} s_{w_1 w_2, w_1 w_2} = Q.$$

□

Lemma 5.6. *Suppose $\sigma(u), \sigma(v) \leq m$ and $t(u) = t(v) = a$. If $\sigma(u) \neq \sigma(v)$, then $Q s_{u,v} Q = 0$.*

Proof. Using Lemma 3.2, write

$$s_{u,v} = \sum_{\substack{v_1; \sigma(v_1) = m+l \\ o(v_1) = a}} s_{u v_1, v v_1}.$$

Apply Lemma 5.5 with $l' = l + \sigma(v)$ so that for each word w in the sum (5.2) $\sigma(w) = m + l + \sigma(v) = \sigma(vv_1)$. By Lemma 3.3, $s_{uv_1,vv_1}s_{w,w} = 0$ unless $vv_1 = w$. Consequently $s_{uv_1,vv_1}Q = 0$ unless $vv_1|_{[m,m+l]} \in S$, which is to say $v_1|_{[m-\sigma(v),m-\sigma(v)+l]} \in S$. Similarly, $Qs_{uv_1,vv_1} = 0$ unless $v_1|_{[m-\sigma(u),m-\sigma(u)+l]} \in S$. But if $w_1 = v_1|_{[m-\sigma(v),m-\sigma(v)+l]} \in S$ and $w_2 = v_1|_{[m-\sigma(u),m-\sigma(u)+l]} \in S$, then w_1 and w_2 would fail condition (2) of Lemma 5.4, with $p = \sigma(u) - \sigma(v)$. $\square$

Remark 5.7. If $x = \sum_i c_i s_{u_i,v_i}$ is a finite linear combination of the generators of $\mathcal{A}$ with $\sigma(u_i), \sigma(v_i) \leq m$ then $QxQ = \sum_{\sigma(u_i)=\sigma(v_i)} c_i Q s_{u_i,v_i} Q = Q\pi(x)Q$, by Lemma 5.6 and equation (4.2).

Lemma 5.8. *The map $x \mapsto QxQ$ is an isometric *-algebra map from $\mathcal{F}_m$ into $\mathcal{A}$.*

Proof. If $\sigma(u) = \sigma(v) = m$ and $t(u) = t(v) = a$, then, with the notation of Lemma 5.4, we have $Qs_{u,v}Q = s_{uw_a,vw_a}$. With the notation of Lemma 4.1, consider the map

$$\bigoplus_{a \in A} \mathcal{K}(l^2(\{w \in \overline{W}; \sigma(w) = m, t(w) = a\})) \to \mathcal{A}$$

given by

$$E^a_{\delta_u,\delta_v} \mapsto s_{uw_a,vw_a},$$

when $t(u) = t(v) = a$. This is easily checked to be an injective *-algebra map, hence an isometry. The map in the statement of the Lemma is the composition of this isometry with that of Lemma 4.1. $\square$

Theorem 5.9. *The C^*-algebra $\mathcal{A}$ is simple.*

Proof. Let ϕ be a nonzero *-homomorphism from $\mathcal{A}$ to some C^*-algebra. It is enough to show that ϕ is an isometry. Let $x = \sum_i c_i s_{u_i,v_i}$ be a finite linear combination of the generators of $\mathcal{A}$. Choose $m \in \mathbb{Z}^r_+$ so that $\sigma(u_i), \sigma(v_i) \leq m$ for all i. Choose l and S as in Lemma 5.4 and let Q be as in equation (5.1). By Remark 5.7, $QxQ = Q\pi(x)Q$. Observe that $\pi(x) \in \mathcal{F}$ and so by Lemma 5.8, $\|Q\pi(x)Q\| = \|\pi(x)\|$. Moreover $Q\pi(x)Q \in \mathcal{F}_{m+l}$, so by Proposition 4.2, $\|\phi(Q\pi(x)Q)\| = \|Q\pi(x)Q\|$. Thus

$$\|\phi(x)\| \geq \|\phi(Q)\phi(x)\phi(Q)\| = \|\phi(QxQ)\| = \|\phi(Q\pi(x)Q)\| = \|Q\pi(x)Q\| = \|\pi(x)\|.$$

The inequality extends by continuity to all $x \in \mathcal{A}$. It follows that ϕ is faithful since if $\phi(y) = 0$ then $0 = \|\phi(y^*y)\| \geq \|\pi(y^*y)\|$. Therefore $y = 0$, by Lemma 4.3. $\square$

Corollary 5.10. *Let $\mathcal{H}$ be a Hilbert space and for each $u, v \in \overline{W}$ with $t(u) = t(v)$ let $S_{u,v} \in \mathcal{B}(\mathcal{H})$ be a nonzero partial isometry. If the $S_{u,v}$ satisfy the relations (3.1) and $\hat{\mathcal{A}}$ denotes the C^*-algebra which they generate, then there is a unique *-isomorphism ϕ from $\mathcal{A}$ onto $\hat{\mathcal{A}}$ such that $\phi(s_{u,v}) = S_{u,v}$.*

Proof. This follows immediately from Lemma 3.5 and Theorem 5.9. $\square$

Proposition 5.11. *The C^*-algebra $\mathcal{A}$ is purely infinite.*

Proof. Note first that by Lemma 3.2, $s_{uw,uw}$ is a subprojection of $s_{u,u}$. By Lemma 1.7, this implies that $s_{u,u}$ is an infinite projection for any u. Any rank one projection in any $\mathcal{F}_l$ is equivalent to $s_{u,u}$ for some $u \in W_l$, and hence is infinite.

We must show that for every nonzero $h \in \mathcal{A}_+$, the C^*-algebra $\overline{h\mathcal{A}h}$ contains an infinite projection. Since π is faithful, we may assume $\|\pi(h^2)\| = 1$. Let $0 < \epsilon < 1$. Approximate h by self-adjoint finite linear combinations of generators. The square of this approximation gives an element $y = \sum_i c_i s_{u_i,v_i}$ with $y \geq 0$ and $\|y - h^2\| \leq \epsilon$. Fix m so that $\sigma(u_i), \sigma(v_i) \leq m$ for all i and then construct Q by Lemma 5.4 and equation (5.1). We have $QyQ = Q\pi(y)Q \in \mathcal{F}_l$ for some l, $QyQ \geq 0$ and $\|QyQ\| = \|Q\pi(y)Q\| = \|\pi(y)\| \geq \|\pi(h^2)\| - \epsilon = 1 - \epsilon$. Since $\mathcal{F}_l$ is a direct sum of (finite or infinite dimensional) algebras of compact operators, there exists a rank one positive

operator $R_1 \in \mathcal{F}_l$ with $\|R_1\| \leq (1-\epsilon)^{-1/2}$ so that $R_1 Q y Q R_1 = P$ is a rank one projection in $\mathcal{F}_l$. Hence P is an infinite projection.

It follows that $\|R_1 Q h^2 Q R_1 - P\| \leq \|R_1^2\| \|Q\|^2 \|y - h^2\| \leq \epsilon/(1-\epsilon)$. By functional calculus, one obtains $R_2 \in \mathcal{A}_+$ so that $R_2 R_1 Q h^2 Q R_1 R_2$ is a projection and $\|R_2 R_1 Q h^2 Q R_1 R_2 - P\| \leq 2\epsilon/(1-\epsilon)$. For small ϵ one can then find an element R_3 in $\mathcal{A}$ so that $R_3 R_2 R_1 Q h^2 Q R_1 R_2 R_3^* = P$.

Let $R = R_3 R_2 R_1 Q$, so that $R h^2 R^* = P$. Consequently, Rh is a partial isometry, whose initial projection $h R^* R h$ is a projection in $h \mathcal{A} h$ and whose final projection is P. Moreover, if V is a partial isometry in $\mathcal{A}$ such that $V^* V = P$ and $V V^* < P$, then $(h R^*) V (Rh)$ is a partial isometry in $h \mathcal{A} h$ with initial projection $h R^* R h$ and final projection strictly less than $h R^* R h$. $\qquad\square$

We can now explain one of the reasons for introducing the set D of decorations. Recall that D is a countable or finite set. Denote by $\mathcal{A}_D$ the algebra $\mathcal{A}$ corresponding to a given D. One special case of interest is when $D = A$ and δ is the identity map. Then the algebra $\mathcal{A}_A$ is a direct generalization of a Cuntz-Krieger algebra [CK]. There is an obvious notion of equivalence for decorations: two decorations $\delta_1 : D_1 \to A$ and $\delta_2 : D_2 \to A$ are equivalent if there is a bijection $\eta : D_1 \to D_2$ such that $\delta_1 = \delta_2 \eta$. Equivalent decorations give rise to isomorphic algebras. Given any set D of decorations we can obtain another set of decorations $D \times \mathbb{N}$, with the decorating map $\delta' : D \times \mathbb{N} \to A$ defined by $\delta'((d,i)) = \delta(d)$.

Lemma 5.12. *There exists an isomorphism of C^*-algebras $\mathcal{A}_{D \times \mathbb{N}} \cong \mathcal{A}_D \otimes \mathcal{K}$.*

Proof. If $u, v \in W$, the isomorphism is given by $s_{((d,i),u),((d',j),v)} \mapsto s_{(d,u),(d',v)} \otimes E_{i,j}$, where the $E_{i,j}$ are matrix units for $\mathcal{B}(l^2(\mathbb{N}))$. The fact that this is an isomorphism follows from Corollary 5.10. $\qquad\square$

This procedure is useful, because it provides a routine method of passing from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{K}$, a technique that is necessary to obtain the results of [CK].

Lemma 5.13. *Let $l : D \to \mathbb{Z}_+^r$ be any map. Define $D' = \{(d,w) \in \overline{W};\ \sigma(w) = l(d)\}$ and define $\delta' : D' \to A$ by $\delta'(\overline{w}) = t(\overline{w})$. Then $\mathcal{A}_{D'} \cong \mathcal{A}_D$.*

Proof. Define $\phi : \mathcal{A}_{D'} \to \mathcal{A}_D$ by $\phi(s_{(\overline{w}_1,u_1),(\overline{w}_2,u_2)}) = s_{\overline{w}_1 u_1, \overline{w}_2 u_2}$, for $\overline{w}_1, \overline{w}_2 \in D'$, $u_1, u_2 \in W$, $o(u_i) = \delta'(\overline{w}_i) = t(\overline{w}_i)$, and $t(u_1) = t(u_2)$. Relations (3.1) (for $\mathcal{A}_{D'}$) are satisfied for $\phi(s_{(\overline{w}_1,u_1),(\overline{w}_2,u_2)})$. By Corollary 5.10 the homomorphism ϕ exists and is injective. Relation (3.1c) for $\mathcal{A}_D$ shows that each generator of $\mathcal{A}_D$ is in the image of ϕ. Hence ϕ is an isomorphism. $\qquad\square$

Corollary 5.14. *For any (D, δ), $\mathcal{A}_D$ is isomorphic to $\mathcal{A}_{D'}$ for some (D', δ') with $\delta' : D' \to A$ surjective.*

Proof. By general hypothesis, D is nonempty and A is finite. Use Lemma 5.13 once to replace D with D'' so that $\#(D'') \geq \#(A)$, and use it again, in conjunction with Lemma 1.6, to construct the pair (D', δ'). $\qquad\square$

Corollary 5.15. *For a fixed alphabet A and fixed transition matrices M_j, the isomorphism class of $\mathcal{A}_D \otimes \mathcal{K}$ is independent of D.*

Proof. By Corollary 5.14, $\mathcal{A}_D \cong \mathcal{A}_{D'}$ for some (D', δ') with $\delta' : D' \to A$ surjective. By Lemma 5.12, $\mathcal{A}_{D' \times \mathbb{N}} \cong \mathcal{A}_{D'} \otimes \mathcal{K}$. Since δ' is surjective, the decorating set $D' \times \mathbb{N}$ is equivalent to the decorating set $A \times \mathbb{N}$: the inverse image of each $a \in A$ is countable. Thus $\mathcal{A}_D \otimes \mathcal{K} \cong \mathcal{A}_{A \times \mathbb{N}}$. $\qquad\square$

Decorating sets other than A and $A \times \mathbb{N}$ arise naturally in the examples associated to affine buildings.

6. Construction of the algebra $\mathcal{A} \otimes \mathcal{K}$ as a crossed product

A vital tool in [CK] was the expression of $\mathcal{O}_A \otimes \mathcal{K}$ as the crossed product of an AF algebra by a $\mathbb{Z}$-action [CK, Theorem 3.8]. The present section is devoted to an analogous result. In view of Lemma 5.12, this is done by establishing an isomorphism from $\mathcal{A}_{A \times \mathbb{N}}$ onto the crossed product of an AF-algebra by a $\mathbb{Z}^r$-action. The AF-algebra will be isomorphic to the algebra $\mathcal{F}$ of Section 4,

relative to the decorating set $A \times \mathbb{N}$, and the action of an element $k \in \mathbb{Z}^r$ will map the subalgebra $\mathcal{F}_m$ onto $\mathcal{F}_{m+k}$ for each $m \geq 0$.

Let a C^*-algebra $\mathcal{A}'$ be defined just as $\mathcal{A}$ is, with $D = A$ and with generators $\{s'_{u,v}; u, v \in W$ and $o(u) = o(v)\}$. The relations are the same as those in (3.1) except that in the sum (3.1c), words are extended from the beginning. The full relations are

$$
\text{(6.1a)} \qquad\qquad {s'_{u,v}}^{*} = s'_{v,u}
$$

$$
\text{(6.1b)} \qquad\qquad s'_{u,v} s'_{v,w} = s'_{u,w}
$$

$$
\text{(6.1c)} \qquad\qquad s'_{u,v} = \sum_{\substack{w \in W; \sigma(w) = e_j, \\ t(w) = o(u) = o(v)}} s'_{wu,wv}
$$

$$
\text{(6.1d)} \qquad\qquad s'_{u,u} s'_{v,v} = 0 \text{ for } u, v \in W_0, u \neq v.
$$

This may be thought of as using words with extension in the negative direction. Alternatively, replacing M_j by the transpose matrix M_j^t for each j in the definition of $\mathcal{A}$ results in an algebra isomorphic to $\mathcal{A}'$. Conditions (H0)–(H3) for the M_j^t follow from the corresponding conditions for the M_j. Consequently all the preceding results are valid for the algebra $\mathcal{A}'$.

By Theorem 5.9, $\mathcal{A}'$ is a simple separable C^*-algebra. Let $\psi : \mathcal{A}' \to \mathcal{B}(\mathcal{H})$ be a nondegenerate representation of $\mathcal{A}'$ on a *separable* Hilbert space $\mathcal{H}$. For $a \in A$, let $\mathcal{H}_a$ be the range of the projection $\psi(s'_{a,a})$. Then $\mathcal{H} = \bigoplus_{a \in A} \mathcal{H}_a$. By Lemmas 3.2 and 3.10, each $\mathcal{H}_a$ is infinite dimensional. Let $\mathcal{C} = \bigoplus_{a \in A} \mathcal{K}(\mathcal{H}_a) \subset \mathcal{K}(\mathcal{H})$.

For each $l \in \mathbb{Z}_+^r$ define a map $\alpha_l : \mathcal{C} \to \mathcal{C}$ by

$$
\text{(6.2)} \qquad\qquad \alpha_l(x) = \sum_{w \in W_l} \psi(s'_{w,o(w)}) x \psi(s'_{o(w),w}).
$$

Note that $\psi(s'_{w,o(w)})$ is a partial isometry with initial space $\mathcal{H}_{o(w)}$ and final space lying inside $\mathcal{H}_{t(w)}$.

Lemma 6.1. *Let α_l be as above.*

    (1) *α_l has image in $\mathcal{C}$.*
    (2) *α_l is a C^*-algebra inclusion.*
    (3) *For $k, l \in \mathbb{Z}_+^r$, $\alpha_k \alpha_l = \alpha_{k+l}$.*

Proof. 1. Clearly $\alpha_l(x) \in \mathcal{K}(\mathcal{H})$. Moreover for fixed $w \in W$ with $t(w) = a$, $\psi(s'_{w,o(w)}) x \psi(s'_{o(w),w}) \in \mathcal{K}(\mathcal{H}_a)$.

2. Fix $w \in W_l$ with $o(w) = a$. Observe that $s'_{w,a}$ is a partial isometry with initial projection $s_{a,a}$. Moreover for two different words $w_1, w_2 \in W_l$, the range projections of $s'_{w_1,o(w_1)}$ and $s'_{w_2,o(w_2)}$ are orthogonal. The result is now clear.

3. If $w_1 \in W_k$ and $w_2 \in W_l$ then according to Lemmas 3.2 and 3.3,

$$
\text{(6.3)} \qquad s'_{w_1,o(w_1)} s'_{w_2,o(w_2)} = \sum_{\substack{w_3 \in W_l \\ t(w_3) = o(w_1)}} s'_{w_3 w_1, w_3} s'_{w_2, o(w_2)} = \begin{cases} 0 & \text{if } t(w_2) \neq o(w_1) \\ s'_{w_2 w_1, o(w_2)} & \text{if } t(w_2) = o(w_1). \end{cases}
$$

Therefore

$$
\alpha_k \alpha_l(x) = \sum_{\substack{w_1 \in W_k \\ w_2 \in W_l}} \psi(s'_{w_1,o(w_1)}) \psi(s'_{w_2,o(w_2)}) x \psi(s'_{o(w_2),w_2}) \psi(s'_{o(w_1),w_1})
$$

$$
= \sum_{\substack{w_1 \in W_k \\ w_2 \in W_l \\ t(w_2) = o(w_1)}} \psi(s'_{w_2 w_1, o(w_2)}) x \psi(s'_{o(w_2), w_2 w_1})
$$

$$
= \alpha_{k+l}(x).
$$

$\square$

For each $m \in \mathbb{Z}^r$ let $\mathcal{C}^{(m)}$ be an isomorphic copy of $\mathcal{C}$, and for each $l \in \mathbb{Z}_+^r$, let

$$\alpha_l^{(m)} : \mathcal{C}^{(m)} \to \mathcal{C}^{(m+l)}$$

be a copy of α_l. Let $\mathcal{E} = \varinjlim \mathcal{C}^{(m)}$ be the direct limit of the category of C^*-algebras with objects $\mathcal{C}^{(m)}$ and morphisms α_l ([KR, Proposition 11.4.1]). Then $\mathcal{E}$ is an AF algebra. (See the discussion preceding Proposition 4.2.)

If $x \in \mathcal{C}$, let $x^{(m)}$ be the corresponding element of $\mathcal{C}^{(m)}$. Then $x^{(m)}$ is identified with $(\alpha_l x)^{(m+l)}$ for all $l \in \mathbb{Z}_+^r$. Define an action ρ of $\mathbb{Z}^r$ on $\mathcal{E}$ by $\rho(l)(x^{(m)}) = x^{(m+l)}$. Since $\mathbb{Z}^r$ is amenable, the full crossed product of $\mathcal{E}$ by this action coincides with the reduced crossed product [Ped, Theorem 7.7.7], and we denote it simply by $\mathcal{E} \rtimes \mathbb{Z}^r$. The defining property of the crossed product says that there is a unitary representation $m \mapsto U^m$ of $\mathbb{Z}^r$ into the multiplier algebra of $\mathcal{E} \rtimes \mathbb{Z}^r$ such that $\rho(l)(x^{(m)}) = U^l x^{(m)} U^{-l}$, that is

$$(6.4) \qquad\qquad U^l x^{(m)} = x^{(m+l)} U^l.$$

Theorem 6.2. *There exists an isomorphism $\phi : \mathcal{A}_{A \times \mathbb{N}} \to \mathcal{E} \rtimes \mathbb{Z}^r$ where, moreover $\phi(\mathcal{F}) = \mathcal{E}$.*

Proof. Let $D = A \times \mathbb{N}$ and $\delta(a, n) = a$. Fix a map $\beta : D \to \mathcal{H}$ so that $\{\beta(d); \delta(d) = a\}$ is an orthonormal basis of $\mathcal{H}_a$. For $\overline{w} = (d, w) \in \overline{W}_m$ define $\beta(\overline{w}) = \psi(s'_{w,o(w)})\beta(d)$, in this way extending β to a map $\beta : \overline{W} \to \mathcal{H}$. Observe that for a fixed $w \in W$ with $o(w) = a$, $\{\beta(d, w); d \in D, \delta(d) = a\}$ is an orthonormal basis for the range of $s'_{w,w}$. Since, moreover the ranges of $\{s'_{w,w}; w \in W_m\}$ are pairwise orthogonal and sum to all of $\mathcal{H}$, we see that $\{\beta(\overline{w}); \overline{w} \in \overline{W}_m\}$ is an orthonormal basis for $\mathcal{H}$.

For $w \in W$ and $\overline{u} = (d, u) \in \overline{W}$, we have

$$\psi(s'_{w,o(w)})\beta(\overline{u}) = \psi(s'_{w,o(w)})\psi(s'_{u,o(u)})\beta(d) = \begin{cases} 0 & \text{if } o(w) \neq t(u) \\ \psi(s'_{uw,o(u)})\beta(d) & \text{if } o(w) = t(u), \end{cases}$$

by equation (6.3). That is

$$\psi(s'_{w,o(w)})\beta(\overline{u}) = \begin{cases} 0 & \text{if } o(w) \neq t(u) \\ \beta(\overline{u}w) & \text{if } o(w) = t(u). \end{cases}$$

We will now define the map $\phi : \mathcal{A}_D \to \mathcal{E} \rtimes \mathbb{Z}^r$. For $\overline{u}, \overline{v} \in \overline{W}$ with $t(\overline{u}) = t(\overline{v})$ and $\sigma(\overline{u}) = l$ and $\sigma(\overline{v}) = m$ define

$$(6.5) \qquad\qquad \phi(s_{\overline{v},\overline{u}}) = U^{m-l} \left(\beta(\overline{v}) \otimes \overline{\beta(\overline{u})} \right)^{(l)}.$$

We use the notation $\xi \otimes \overline{\eta}$ to denote the rank one operator on a Hilbert space defined by $\zeta \mapsto \langle \zeta, \eta \rangle \xi$, so that when ξ, η have norm one, $\xi \otimes \overline{\eta}$ is a partial isometry with initial projection $\eta \otimes \overline{\eta}$ and final projection $\xi \otimes \overline{\xi}$. If the vectors ξ, η vary through an orthonormal basis for a Hilbert space then the operators $\xi \otimes \overline{\eta}$ form a system of matrix units for the compact operators on that Hilbert space. By equation (6.4) we have

$$(6.6) \qquad\qquad \phi(s_{\overline{v},\overline{u}}) = \left(\beta(\overline{v}) \otimes \overline{\beta(\overline{u})} \right)^{(m)} U^{m-l}.$$

We show that the partial isometries $\phi(s_{\overline{v},\overline{u}})$ satisfy the relations (3.1). Relation (3.1d) is immediate from the definition of $\beta(\overline{w})$, since if $\overline{w} \in \overline{W}_0$ then $\phi(s_{\overline{w},\overline{w}}) = \left(\beta(\overline{w}) \otimes \overline{\beta(\overline{w})} \right)^{(0)}$.

Relation (3.1a) is satisfied since, by (6.6),

$$\phi(s_{\overline{v},\overline{u}})^* = \left(\left(\beta(\overline{v}) \otimes \overline{\beta(\overline{u})} \right)^{(m)} U^{m-l} \right)^* = U^{l-m} \left(\beta(\overline{u}) \otimes \overline{\beta(\overline{v})} \right)^{(m)} = \phi(s_{\overline{u},\overline{v}}).$$

If $t(\overline{w}) = t(\overline{v})$ and $\sigma(\overline{w}) = n$, then

$$\phi(s_{\overline{w},\overline{v}})\phi(s_{\overline{v},\overline{u}}) = U^{n-m}\left(\beta(\overline{w}) \otimes \overline{\beta(\overline{v})}\right)^{(m)} U^{m-l}\left(\beta(\overline{v}) \otimes \overline{\beta(\overline{u})}\right)^{(l)}$$

$$\text{(6.7)} \qquad = U^{n-m}U^{m-l}\left(\beta(\overline{w}) \otimes \overline{\beta(\overline{v})}\right)^{(l)}\left(\beta(\overline{v}) \otimes \overline{\beta(\overline{u})}\right)^{(l)}$$

$$= U^{n-l}\left(\beta(\overline{w}) \otimes \overline{\beta(\overline{u})}\right)^{(l)}$$

$$= \phi(s_{\overline{w},\overline{u}}).$$

Thus (3.1b) is satisfied. Finally (3.1c) is a consequence of the following calculation.

$$\phi(s_{\overline{v},\overline{u}}) = U^{m-l}\left(\beta(\overline{v}) \otimes \overline{\beta(\overline{u})}\right)^{(l)}$$

$$= U^{m-l}\left(\alpha_k(\beta(\overline{v}) \otimes \overline{\beta(\overline{u})})\right)^{(l+k)}$$

$$= U^{m-l}\sum_{w \in W_k}\left(\psi(s'_{w,o(w)})\beta(\overline{v}) \otimes \overline{\psi(s'_{w,o(w)})\beta(\overline{u})}\right)^{(l+k)}$$

$$= U^{m-l}\sum_{\substack{w \in W_k \\ o(w)=t(\overline{u})}}\left(\beta(\overline{v}w) \otimes \overline{\beta(\overline{u}w)}\right)^{(l+k)}$$

$$= \sum_{\substack{w \in W_k \\ o(w)=t(\overline{u})}}\phi(s_{\overline{v}w,\overline{u}w}).$$

All the relations (3.1) are satisfied by the partial isometries $\phi(s_{\overline{v},\overline{u}})$. Therefore by Lemma 3.5, ϕ defines a *-homomorphism. Clearly ϕ is not the zero map; hence it is an isometry, by Theorem 5.9. It only remains to show that ϕ is onto.

Fix $m, n \in \mathbb{Z}^r$ so that $m, m+n \geq 0$. For $\overline{u} \in \overline{W}_m$ and $\overline{v} \in \overline{W}_{m+n}$ with $t(\overline{u}) = t(\overline{v}) = a$, we have

$$\text{(6.8)} \qquad \phi(s_{\overline{v},\overline{u}}) = U^n\left(\beta(\overline{v}) \otimes \overline{\beta(\overline{u})}\right)^{(m)}.$$

As the sets $\{\beta(\overline{v});\ \overline{v} \in \overline{W}_{m+n}, t(\overline{v}) = a\}$ and $\{\beta(\overline{u});\ \overline{u} \in \overline{W}_m, t(\overline{u}) = a\}$ are bases for $\mathcal{H}_a$, the image of ϕ contains a dense subset of $U^n\left(\mathcal{K}(\mathcal{H}_a)\right)^{(m)}$. Therefore the image of ϕ contains $U^n\left(\mathcal{K}(\mathcal{H}_a)\right)^{(m)}$, for each $a \in A$. It therefore contains $U^n\mathcal{C}^{(m)}$.

Also, for any $k \geq 0$,

$$\phi(\mathcal{A}_{A\times\mathbb{N}}) \supseteq U^n\mathcal{C}^{(m)} \supseteq U^n\alpha_k^{(m-k)}\mathcal{C}^{(m-k)} = U^n\mathcal{C}^{(m-k)}.$$

It follows that $\phi(\mathcal{A}_{A\times\mathbb{N}}) = \mathcal{E} \rtimes \mathbb{Z}^r$. It is clear from the definitions that $\phi(\mathcal{F}_m) = \mathcal{C}^{(m)}$ and that $\phi(\mathcal{F}) = \mathcal{E}$. $\qquad \square$

Corollary 6.3. $\mathcal{A} \otimes \mathcal{K} \cong \mathcal{E} \rtimes \mathbb{Z}^r$.

Proof. This follows immediately from Theorem 6.2, Lemma 5.12 and Corollary 5.15. $\qquad \square$

Corollary 6.4. $\mathcal{A}$ *is nuclear.*

Proof. This follows because the class of nuclear C^*-algebras is closed under stable isomorphism and crossed products by amenable groups, and contains the AF algebras. $\qquad \square$

Remark 6.5. Suppose that D is finite, so that $\mathcal{A}$ is unital by Remark 3.4. Then it has been established that the separable unital C^*-algebra $\mathcal{A}$ is simple (Theorem 5.9), nuclear (Corollary 6.4) and purely infinite (Proposition 5.11). Corollary 6.3 also shows that $\mathcal{A}$ belongs to the bootstrap class $\mathcal{N}$, which contains the AF-algebras and is closed under stable isomorphism and crossed products by $\mathbb{Z}$. Thus $\mathcal{A}$ satisfies the Universal Coefficient Theorem [Bl, Theorem 23.1.1]. The work of E. KIRCHBERG and C. PHILLIPS [K1, K2],[Ph] therefore shows that $\mathcal{A}$ is classified by its K-groups.

7. Boundary actions on affine buildings of type $\widetilde{A}_2$

Let $\mathcal{B}$ be a locally finite thick affine building of type $\widetilde{A}_2$. This means that $\mathcal{B}$ is a chamber system consisting of vertices, edges and triangles (*chambers*). An *apartment* is a subcomplex of $\mathcal{B}$ isomorphic to the Euclidean plane tesselated by equilateral triangles. A *sector* (or *Weyl chamber*) is a $\frac{\pi}{3}$-angled sector made up of chambers in some apartment. Two sectors are *equivalent* (or parallel) if their intersection contains a sector. We refer to [Br1, G, Ron] for the theory of buildings. Shorter introductions to the theory are provided by [Br2, Ca, St].

The boundary Ω is defined to be the set of equivalence classes of sectors in $\mathcal{B}$. In $\mathcal{B}$ we fix some vertex O, which we assume to have type 0. For any $\omega \in \Omega$ there is a unique sector $[O, \omega)$ in the class ω having base vertex O [Ron, Theorem 9.6]. The boundary Ω is a totally disconnected compact Hausdorff space with a base for the topology given by sets of the form

$$\Omega(v) = \{\omega \in \Omega : [O, \omega) \ \text{ contains } v\}$$

where v is a vertex of $\mathcal{B}$ [CMS, Section 2]. We note that if $[O, \omega)$ contains v then $[O, \omega)$ contains the parallelogram $\mathrm{conv}(O, v)$.

Let Γ be a group of type rotating automorphisms of $\mathcal{B}$ that acts freely on the vertex set with finitely many orbits. See [CMSZ] for a discussion and examples in the case where Γ acts transitively on the vertex set. There is a natural induced action of Γ on the boundary Ω and we can form the universal crossed product algebra $C(\Omega) \rtimes \Gamma$ [Ped]. The purpose of this section is to identify $C(\Omega) \rtimes \Gamma$ with an algebra of the form $\mathcal{A}$.

Let $\mathfrak{a}$ be a Coxeter complex of type $\widetilde{A}_2$, which we shall use as a model for the apartments of $\mathcal{B}$. Each vertex of $\mathfrak{a}$ has type 0,1, or 2. Fix as the origin in $\mathfrak{a}$ a vertex of type 0. Coordinatize the vertices by $\mathbb{Z}^2$ by choosing a fixed sector in $\mathfrak{a}$ based at the origin and defining the positive coordinate axes to be the corresponding sector panels (*cloisons de quartier*). The coordinate axes are therefore given by two of the three walls of $\mathfrak{a}$ passing through the origin. Let $\mathfrak{t}$ be a model *tile* in $\mathfrak{a}$ and let $\mathfrak{p}_m$ be a model parallelogram in $\mathfrak{a}$ of *shape* $m = (m_1, m_2)$, as illustrated in Figure 7. As per Figure 7, assume that $\mathfrak{t}$ and $\mathfrak{p}_m$ are both based at $(0,0)$. Thus $\mathfrak{t}$ is the model parallelogram of shape $(0,0)$.

The model tile $\mathfrak{t}$.

A model parallelogram $\mathfrak{p}_m$.
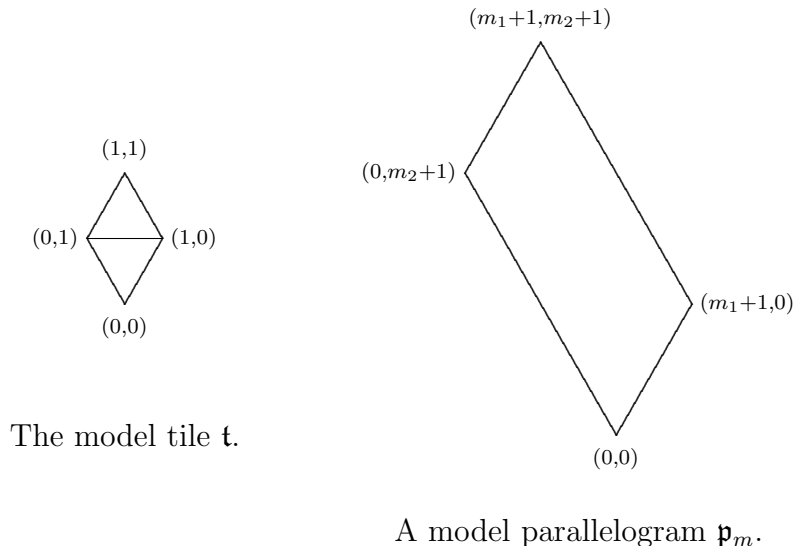
FIGURE 7

Let $\mathfrak{T}$ denote the set of type rotating isometries $i : \mathfrak{t} \to \mathcal{B}$, and let $A = \Gamma \backslash \mathfrak{T}$. We will use the set A as an alphabet to define an algebra $\mathcal{A}$. Let $\mathfrak{P}_m$ denote the set of type rotating isometries $p : \mathfrak{p}_m \to \mathcal{B}$, and let $\mathfrak{W}_m = \Gamma \backslash \mathfrak{P}_m$. Let $\mathfrak{P} = \bigcup_m \mathfrak{P}_m$ and $\mathfrak{W} = \bigcup_m \mathfrak{W}_m$.

If $p \in \mathfrak{P}_m$, then define $t(p) : \mathfrak{t} \to \mathcal{B}$ by $t(p)(l) = p(m + l)$. Then $t(p)$ is a type rotating isometry such that $t(p)(\mathfrak{t})$ lies in $p(\mathfrak{p}_m)$ with $t(p)(1, 1) = p(m_1 + 1, m_2 + 1)$, as illustrated in Figure 8. Thus $t(p) \in \mathfrak{T}$. Similarly $o(p) : \mathfrak{t} \to \mathcal{B}$ is defined by $o(p) = p|_{\mathfrak{t}}$.
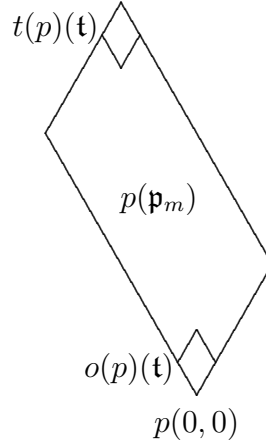
FIGURE 8. The initial and terminal tiles.

The matrices M_1, M_2 with entries in $\{0, 1\}$ are defined as follows. If $a, b \in A$, say that $M_1(b, a) = 1$ if and only if there exists $p \in \mathfrak{P}_{(1,0)}$ such that $a = \Gamma o(p)$ and $b = \Gamma t(p)$. Similarly, if $c \in A$ then $M_2(c, a) = 1$ if and only if there exists $p \in \mathfrak{P}_{(0,1)}$ such that $a = \Gamma o(p)$ and $c = \Gamma t(p)$. The definitions are illustrated in Figure 9, for suitable representative isometries i_a, i_b, i_c in $\mathfrak{T}$.

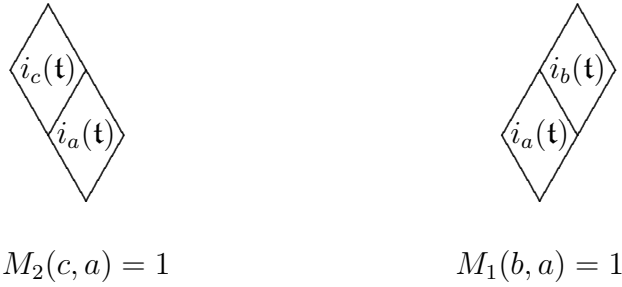$$M_2(c, a) = 1 \qquad\qquad M_1(b, a) = 1$$

FIGURE 9. Definition of the transition matrices.

In order to apply the general results we need to verify that conditions (H0)-(H3) are satisfied. We address this question in the subsection 7.1. Until further notice we simply impose the following

ASSUMPTION: Conditions (H0)-(H3) are satisfied.

We can now define the set of words W_m of shape $m \in \mathbb{Z}_+$ based on the alphabet A and the transition matrices M_1, M_2, as in Section 1. There is also a natural map $\alpha : \mathfrak{P} \to W$, defined as follows. Given $p \in \mathfrak{P}_m$, construct $\alpha(p) = w$ according to the following procedure. For each $n \in \mathbb{Z}_+^2$ with $0 \leq n \leq m$, let $w(n) = \Gamma p_n$ where $p_n \in \mathfrak{T}$ is defined by $p_n(l) = p(n+l)$, for $(0,0) \leq l \leq (1,1)$. Since the translation $l \mapsto n+l$ is a type-rotating isometry of $\mathfrak{a}$, it follows that p_n is type rotating, hence an element of $\mathfrak{T}$. Passing to the quotient by Γ gives a well defined map $\alpha : \mathfrak{W}_m \to W_m$ and hence a map $\alpha : \mathfrak{W} \to W$. The use of α for the different maps should not cause confusion. If $a = \Gamma i \in A = \mathfrak{W}_0$ then $\alpha(a) = a$, so it is clear that $o(\alpha(p)) = \Gamma o(p)$ and $t(\alpha(p)) = \Gamma t(p)$.

Lemma 7.1. *The map α is a bijection from $\mathfrak{W}_m$ to W_m for each $m \in \mathbb{Z}_+^2$.*

Proof. Suppose that $\alpha(p) = \alpha(p')$. Then $\Gamma p_n = \Gamma p'_n$ for $0 \leq n \leq m$. For each n there exists γ_n so that $\gamma_n p_n = p'_n$. Since Γ acts freely on the vertices, each γ_n is uniquely determined. Moreover, since p_n and p_{n+e_j} share a pair of vertices in their image, it must be true that $\gamma_n = \gamma_{n+e_j}$. By induction, the γ_n have a common value, γ, and $\gamma p = p'$. Thus $\Gamma p = \Gamma p'$.

It remains to show that α is surjective. Let $w \in W_m$. Choose a path $(n_0, \ldots, n_k)$ of points in $\mathbb{Z}^2$ so that $n_0 = 0, n_k = m$, and each difference $n_{j+1} - n_j$ is either e_1 or e_2. Choose representative type rotating isometries $i_0, i_1, \ldots, i_k$ from $\mathfrak{t}$ into $\mathcal{B}$ with $\Gamma i_r = w(n_r)$ so that $i_r(\mathfrak{t})$ and $i_{r+1}(\mathfrak{t})$

are adjacent tiles in $\mathcal{B}$, according to the two possibilities in Figure 9. This defines a gallery $\mathfrak{G} = \{i_0(\mathfrak{t}), i_1(\mathfrak{t}), \ldots, i_k(\mathfrak{t})\}$. Let $\mathfrak{G}_0 = \{C_0, C_1, \ldots, C_k\}$ be the corresponding gallery in $\mathfrak{p}_m$ with $C_0 = \mathfrak{t}$. It is clear from Figure 10 that $\mathfrak{G}_0$ is a minimal gallery in $\mathfrak{a}$. (The elements of $\mathfrak{G}$ and $\mathfrak{G}_0$ are tiles rather than chambers, but this is immaterial.) The obvious map $p : \mathfrak{G}_0 \to \mathfrak{G}$ is a strong isometry (preserves generalized distance). Therefore $\mathfrak{G}$ is contained in an apartment and p extends to a strong isometry p from the convex hull $\mathrm{conv}(\mathfrak{G}_0) = \mathfrak{p}_m$ into that apartment. See [Br1, p. 90, Theorem] and [Br2, Appendix B]. Thus $p \in \mathfrak{P}_m$ and since $\alpha(p)$ agrees with w at $n_0, n_1, \ldots, n_k$, (H1) implies that $\alpha(p) = w$. $\qquad\square$
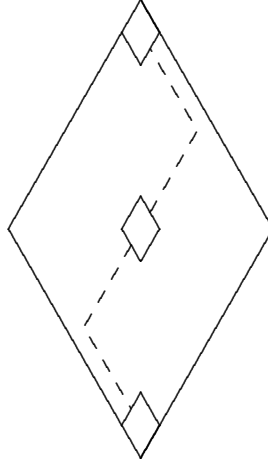

FIGURE 10. A minimal gallery in in $\mathfrak{a}$.

Let $\overline{\mathfrak{W}}_m$ denote the set of type rotating isometries $p : \mathfrak{p}_m \to \mathcal{B}$ such that $p(0,0) = O$ and let $\overline{\mathfrak{W}} = \bigcup_m \overline{\mathfrak{W}}_m$. Let D denote the set of type-rotating isometries $d : \mathfrak{t} \to \mathcal{B}$ such that $d(0,0) = O$. Let $\delta : D \to A$ be given by $\delta(d) = \Gamma d$. The map δ is injective since Γ acts freely on the vertices of $\mathcal{B}$. Moreover δ is surjective if and only if Γ acts transitively on the vertices. Define $\overline{\alpha} : \overline{\mathfrak{W}} \to \overline{W}$ by $\overline{\alpha}(p) = (o(p), \alpha(\Gamma p))$. (Recall that $o(p) = p|_\mathfrak{t}$.)

Lemma 7.2. *The map $\overline{\alpha}$ is a bijection from $\overline{\mathfrak{W}}_m$ onto $\overline{W}_m$ for each $m \in \mathbb{Z}_+^2$.*

Proof. If $\overline{\alpha}(p_1) = \overline{\alpha}(p_2)$ then $o(p_1) = o(p_2)$; moreover $\Gamma p_1 = \Gamma p_2$, by Lemma 7.1. Since Γ acts freely on the vertices, it follows that $p_1 = p_2$. Therefore $\overline{\alpha}$ is injective.

To see that it is surjective, let $\overline{w} = (d, w) \in \overline{W}_m$, where $w \in W_m$ and $d \in D$. By Lemma 7.1, there exists $p \in \mathfrak{P}_m$ such that $\alpha(\Gamma p) = w$. Then

$$\Gamma d = \delta(d) = o(w) = o(\alpha(\Gamma p)) = \Gamma o(p).$$

Replacing p by γp for suitable $\gamma \in \Gamma$ ensures that $o(p) = d$ and hence $p \in \overline{\mathfrak{W}}_m$ and $\overline{\alpha}(p) = \overline{w}$. $\qquad\square$

If $p \in \overline{\mathfrak{W}}$ then $\mathrm{conv}(O, t(p)(\mathfrak{t})) = \mathrm{conv}(O, p(m_1 + 1, m_2 + 1))$ and we introduce the notation

$$\Omega(p) = \Omega(p(m_1 + 1, m_2 + 1)) = \{\omega \in \Omega; t(p) \subset [O, \omega]\} = \{\omega \in \Omega; p(\mathfrak{p}_m) \subset [O, \omega]\}.$$

Let $i \in \mathfrak{T}$, that is, suppose that $i : \mathfrak{t} \to \mathcal{B}$ is a type rotating isometry. Let

$$\Omega(i) = \{\omega \in \Omega; i(\mathfrak{t}) \subset [i(0,0), \omega]\},$$

those boundary points represented by sectors which originate at $i(0,0)$ and contain $i(\mathfrak{t})$. Clearly $\Omega(\gamma i) = \gamma \Omega(i)$. For $p \in \overline{\mathfrak{W}}_m$ we have $\Omega(p) = \Omega(t(p))$. Indeed, any sector originating at $t(p)(0,0)$ and containing $t(p)(\mathfrak{t})$ extends to a sector originating at O and containing $p(\mathfrak{p}_m)$.

Fix $\overline{w}_1, \overline{w}_2 \in \overline{W}$ with $t(\overline{w}_1) = t(\overline{w}_2) = a \in A$. Let $p_1 = \overline{\alpha}^{-1}(\overline{w}_1)$ and $p_2 = \overline{\alpha}^{-1}(\overline{w}_2)$. Let $\gamma \in \Gamma$ be the unique element such that $\gamma t(p_1) = t(p_2)$. Define a homomorphism $\phi : \mathcal{A} \to C(\Omega) \rtimes \Gamma$ by

$$(7.1) \qquad \phi(s_{\overline{w}_2, \overline{w}_1}) = \gamma \mathbf{1}_{\Omega(p_1)} = \mathbf{1}_{\Omega(p_2)} \gamma.$$

Note that by Lemma 3.5 this does indeed define a *-homomorphism of $\mathcal{A}$ because the operators of the form $\phi(s_{\overline{w}_2, \overline{w}_1})$ are easily seen to satisfy the relations (3.1). We now prove that ϕ is an isomorphism from $\mathcal{A}$ onto $C(\Omega) \rtimes \Gamma$ (Theorem 7.7 below). For this some preliminaries are necessary.

Lemma 7.3. *For any $m \in \mathbb{Z}_+^2$, $\mathbf{1} = \displaystyle\sum_{p \in \overline{\mathfrak{W}}_m} \mathbf{1}_{\Omega(p)}$.*

Proof. This follows from the discussion in [CMS, Section 2]. $\qquad\square$

Lemma 7.4. *The linear span of $\{\mathbf{1}_{\Omega(p)}; p \in \overline{\mathfrak{W}}\}$ is dense in $C(\Omega)$.*

Proof. This follows because the sets $\Omega(p)$ for $p \in \overline{\mathfrak{W}}$ form a basis for the topology of Ω [CMS, Section 2]. $\qquad\square$

Lemma 7.5. *Let $p \in \mathfrak{P}_m$ where $m = (m_1, m_2) \in \mathbb{Z}_+^2$. Let $x = p(0,0)$ and $y = p(m_1 + 1, m_2 + 1)$. Let y' be another vertex of $\mathcal{B}$ whose graph distance to y in the 1-skeleton of $\mathcal{B}$ equals n. Suppose that $n \leq m_1, m_2$. Then $\mathrm{conv}(x, y')$ contains $p(\mathfrak{p}_{(m_1 - n, m_2 - n)})$.*

Proof. Induction reduces us to the case $n = 1$. There is some apartment containing x and the edge from y to y'. In Figure 11 we show one of the six possible positions for y'.
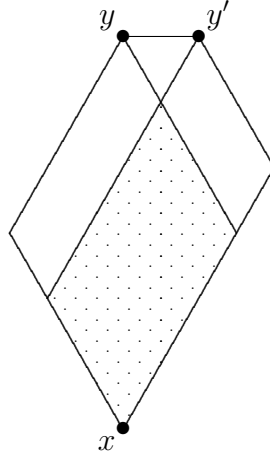

FIGURE 11. Relative positions of x, y and y' in an apartment.

Now $\mathrm{conv}(x, y')$ is the image of some $p' \in \mathfrak{P}_{(m_1 + 1, m_2 - 1)}$. It is evident in this case (and in the other five cases) that $p'(\mathfrak{p}_{(m_1 - 1, m_2 - 1)}) = p(\mathfrak{p}_{(m_1 - 1, m_2 - 1)})$. $\qquad\square$

Corollary 7.6. *Let $p \in \mathfrak{P}_m$ for $m = (m_1, m_2)$. Let $x = p(m_1 + 1, m_2 + 1)$ and $y = p(0,0)$. Let y' be a third vertex of $\mathcal{B}$ at distance n to y. Suppose $n \leq m_1, m_2$. Then $\mathrm{conv}(x, y')$ is the image of some $p' \in \mathfrak{P}$, where $p'(0,0) = y'$ and $t(p') = t(p)$.*

Proof. See Figure 12. According to Lemma 7.5, the convex hull $\mathrm{conv}(x, y')$ contains $t(p)(\mathfrak{t})$. Because $\mathrm{conv}(x, y')$ contains a chamber, it must be the image of a unique $p' \in \mathfrak{P}$ with $p'(0,0) = y'$. Since $t(p)(\mathfrak{t})$ lies in the image of p', namely $\mathrm{conv}(x, y')$, and $t(p)(1,1) = x$, it follows that $t(p') = t(p)$. $\qquad\square$

Theorem 7.7. *The map ϕ is an isomorphism from $\mathcal{A}$ onto $C(\Omega) \rtimes \Gamma$.*
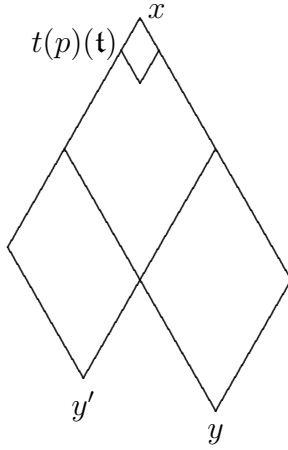
FIGURE 12

Proof. Since $\mathcal{A}$ is simple, ϕ is injective. If $p \in \overline{W}$ then $\mathbf{1}_{\Omega(p)} = \phi(s_{\overline{w},\overline{w}})$, where $\overline{w} = \overline{\alpha}(p)$ and so Lemma 7.4 shows that the range of ϕ contains $C(\Omega)$. It remains to show that it contains Γ. Fix $\gamma \in \Gamma$. Choose $m = (m_1, m_2)$ so that $d(O, \gamma^{-1}O) \leq m_1, m_2$. Now $\gamma = \gamma \mathbf{1} = \sum_{p \in \overline{\mathfrak{W}}_m} \gamma \mathbf{1}_{\Omega(p)}$. For $p \in \overline{\mathfrak{W}}_m$ we claim that $\gamma t(p) = t(p')$ for some $p' \in \overline{\mathfrak{W}}$. Hence $\gamma \mathbf{1}_{\Omega(p)} = \phi\big(s_{\alpha(p'),\alpha(p)}\big)$. This shows the range of ϕ contains Γ and hence is surjective.

To prove the claim above, apply Corollary 7.6 to find $p'' \in \mathfrak{P}$ with $p''(0,0) = \gamma^{-1}(O)$ and $t(p'') = t(p)$. Let $p' = \gamma p''$. Then $p'(0,0) = O$, so $p' \in \overline{\mathfrak{W}}$ and $t(p') = \gamma t(p'') = \gamma t(p)$. $\square$

Remark 7.8. It follows from Theorem 7.7 and Remark 6.5 that $C(\Omega) \rtimes \Gamma$ is simple, nuclear and purely infinite. Simplicity and nuclearity had previously been proved in [RS1] under the additional assumption that Γ acts transitively and in a type rotating manner on the vertices of $\mathcal{B}$. From simplicity it follows that $C(\Omega) \rtimes \Gamma$ is isomorphic to the reduced crossed product $C(\Omega) \rtimes_r \Gamma$. See also [An, QS] for general conditions under which the full and reduced crossed products coincide. Their results apply, for example when Γ is a lattice in a linear group.

7.1. **Conditions** (H0)-(H3) **for affine buildings of type $\widetilde{A}_2$.** Continue the notation and terminology used above, assuming throughout that Γ acts on $\mathcal{B}$ via type rotating automorphisms. We prove that conditions (H0), (H1), and (H3) are satisfied so long as Γ acts freely and with finitely many orbits on the vertices of $\mathcal{B}$. Moreover, if $\mathcal{B}$ is the building of $G = \mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K}$ is a nonarchimedean local field of characteristic zero and Γ is a lattice in $\mathrm{PGL}_3(\mathbb{K})$ we prove that (H2) holds as well. There are several concrete examples in [CMSZ] where all these hypotheses are satisfied.

Proposition 7.9. *Suppose that Γ acts freely and with finitely many orbits on the vertices of $\mathcal{B}$. Then the matrices M_1, M_2 of the previous section satisfy conditions* (H0),(H1), *and* (H3).

Proof. By definition M_1 and M_2 are $\{0,1\}$-matrices. To say that they are nonzero is to say that $\mathfrak{P}_{(1,0)}$ and $\mathfrak{P}_{(0,1)}$ are nonempty, which the are. This proves (H0).

Fix any nonzero $j \in \mathbb{Z}^2$. We will construct $w \in \mathfrak{W}$ which is not j-periodic. Choose $m \in \mathbb{Z}^2$ large enough so that inside $\mathfrak{p}_m$ one can find a minimal gallery of chambers, $(C_0, C_1, \ldots, C_l)$ so that C_l is the j-translate of C_0. Write $\tau : C_0 \to C_l$ for the identification by translation of the two chambers.

Construct an isometry p from this minimal gallery to $\mathcal{B}$ by defining successively $p|_{C_0}$, $p|_{C_1}$, etc. Since the building is thick, one has at least two choices at each step. Once $p|_{C_{l-1}}$ is fixed, no two of the choices for $p|_{C_l}$ can be in the same Γ-orbit, since Γ acts freely on the vertices of $\mathcal{B}$. Therefore, one may choose p so that $p|_{C_0}$ and $p \circ \tau$ are in different Γ-orbits. Now extend p to an isometry $p : \mathfrak{p}_m \to \mathcal{B}$. The element of W associated to p, that is $\alpha(\Gamma p)$, is not j-periodic. This proves (H3).

Condition (H1c) is vacuous for $r = 2$. Consider the configuration of Figure 13. Given the tiles a, b, and c, there is exactly one tile d which completes the picture. Since a, b, and c make up a minimal gallery, this follows by the same argument used in proving Lemma 7.1. Translating this fact to matrix terms, we have that if $(M_2 M_1)(c, a) > 0$ then $(M_1 M_2)(c, a) = 1$. Likewise, if $(M_1 M_2)(c, a) > 0$, then $(M_2 M_1)(c, a) = 1$. Conditions (H1a) and (H1b) follow, and by Lemma 1.4, so does condition (H1). $\qquad\square$
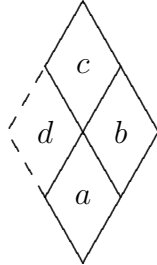

FIGURE 13. Uniqueness of extension.

It is not much harder to prove (H1) directly, bypassing conditions (H1a)-(H1c). It remains to prove condition (H2). The next result and its corollary prove a strong version of condition (H2).

Theorem 7.10. *Let $\mathcal{B}$ be the building of $G = \mathrm{SL}_3(\mathbb{K})$, where $\mathbb{K}$ is a local field of characteristic zero. Let Γ be a lattice in $\mathrm{SL}_3(\mathbb{K})$ which acts freely on the vertices of $\mathcal{B}$ with finitely many orbits. Let the alphabet $A = \Gamma\backslash\mathfrak{T}$ and the transition matrices M_1, M_2 be defined as at the beginning of Section 7. Then for each $i = 1, 2$, the directed graph with vertices $a \in A$ and directed edges (a, b) whenever $M_i(b, a) = 1$ is irreducible.*

Proof. We use an idea due to S. Mozes [M2, Proposition 3]. Fix a model half-infinite strip $\mathfrak{s}$ of tiles in $\mathfrak{a}$ based at $(0, 0)$ and let $\mathfrak{s}_k = \mathfrak{p}_{(k,0)}$ be the initial segment consisting of $k + 1$ tiles.

FIGURE 14. The strip $\mathfrak{s}$.

Let $\mathfrak{S}_0$ [respectively $\mathfrak{S}_k$, $k \geq 1$] be the set of type-preserving isometric embeddings s of $\mathfrak{s}$ [respectively $\mathfrak{s}_k$] into the building $\mathcal{B}$. Thus $\mathfrak{S}_k$ is the subset of type-preserving maps in $\mathfrak{P}_{(k,0)}$ and if $s \in \mathfrak{S}_k$, then $o(s)$ and $t(s)$ are defined as before. Also, if $s \in \mathfrak{S}$ define $o(s) = s|_{\mathfrak{t}}$ and let $s_k = t(s|_{\mathfrak{s}_k})$ for $k \geq 0$.

It is desired to prove that given $a, b \in A$, there exists $k \in \mathbb{Z}_+$ and $s \in \mathfrak{S}_k$ such that $a = \Gamma o(s)$ and $b = \Gamma t(s)$. The group G acts transitively on the set of apartments of $\mathcal{B}$ [St, Section 5]. Moreover the stabilizer of an apartment acts transitively on the set of sectors of the apartment. It follows that G acts transitively on $\mathfrak{S}_0$. Therefore $\mathfrak{S}_0 = G/H_0$ for some subgroup H_0. In fact

$$H_0 = \{\begin{pmatrix} a_1 & b_2 & b_3 \\ 0 & a_2 & c \\ 0 & d & a_3 \end{pmatrix} \in G; \ |a_j| = 1, |b_j|, |c| \leq 1, \text{ and } |d| < 1\}.$$

Say that two elements of $\mathfrak{S}_0$ are equivalent if, beyond a certain point dependent on the two elements, they agree on all tiles of $\mathfrak{s}$. Let $\mathfrak{S}$ be the space of equivalence classes. Since G acts transitively on $\mathfrak{S}_0$, a fortiori it acts transitively on $\mathfrak{S}$. Thus $\mathfrak{S} = G/H$ for some H. In fact

$$H = \{\begin{pmatrix} a_1 & b_2 & b_3 \\ 0 & a_2 & c \\ 0 & d & a_3 \end{pmatrix} \in G; \ |a_j| = 1, |c| \leq 1, \text{ and } |d| < 1\}.$$

The only relevant facts about H are that it is closed and noncompact. On $\mathfrak{S}_0$ and $\mathfrak{S}$ we put the topologies and measures obtained from the isomorphisms with G/H_0 and G/H.

The Howe–Moore Theorem [Z, Theorems 10.1.4 and 2.2.6] shows that Γ acts *ergodically* on $\mathfrak{S}$. Suppose that there exist $a, b \in A$ which cannot occur as $a = \Gamma o(s)$ and $b = \Gamma t(s)$ with $s \in \mathfrak{S}_k$, for any k. Let $\mathfrak{S}_0' = \{s \in \mathfrak{S}_0; \Gamma o(s) = a\}$ and let $\mathfrak{S}'$ be the projection of $\mathfrak{S}_0'$ to $\mathfrak{S}$. The sets $\mathfrak{S}_0'$ and $\mathfrak{S}'$ are clearly Γ-invariant. The set $\mathfrak{S}'$ is open, since $\mathfrak{S}_0'$ is open. Therefore $\mathfrak{S}'$ is not of null measure. By the ergodicity of the Γ-action $\mathfrak{S}'$ has full measure. Also of full measure will be the inverse image of $\mathfrak{S}'$ in $\mathfrak{S}_0$, which consists of all those $s' \in \mathfrak{S}_0$ which are equivalent to some $s \in \mathfrak{S}_0$ with $\Gamma o(s) = a$. A fortiori

$$\{s \in \mathfrak{S}_0; \text{there exists } K \geq 0 \text{ such that } b \neq \Gamma s_k \text{ for all } k \geq K\}$$

is of full measure in $\mathfrak{S}_0$. Now, in $\Gamma \backslash \mathfrak{S}_0$ the set

$$\{\Gamma s \in \Gamma \backslash \mathfrak{S}_0; \text{there exists } K \geq 0 \text{ such that } b \neq \Gamma s_k \text{ for all } k \geq K\}$$

will also be of full measure.

On $\Gamma \backslash \mathfrak{S}_0$ use the measure obtained in the usual way from the unique (up to positive constant) positive G-invariant measure on $\mathfrak{S}_0$. The following condition defines the new measure, $d\dot{s}$, in terms of the old measure, ds:

$$\int_{\Gamma \backslash \mathfrak{S}_0} \sum_{\gamma \in \Gamma} F(\gamma(s)) \, d\dot{s} = \int_{\mathfrak{S}_0} F(s) \, ds$$

for any $F \in C_c(\mathfrak{S}_0)$. Since $\Gamma \backslash G$ is of finite total measure, it follows that $\Gamma \backslash \mathfrak{S}_0$ is too. Assume that this total measure is one. One can easily verify that relative to $d\dot{s}$ the distribution of Γs_k is independent of k. In fact, $|\{\Gamma s \in \Gamma \backslash \mathfrak{S}_0; \Gamma s_k = b\}| = \frac{1}{\#A}$ for all $k \geq 0$.

Let $0 < \epsilon < \frac{1}{\#A}$. The monotone convergence theorem implies that there exists K such that

$$|\{\Gamma s \in \Gamma \backslash \mathfrak{S}_0; b \neq \Gamma s_k \text{ for all } k \geq K\}| > 1 - \epsilon.$$

But this means that

$$|\{\Gamma s \in \Gamma \backslash \mathfrak{S}_0; \Gamma s_K = b\}| \leq \epsilon.$$

This contradicts $|\{\Gamma s \in \Gamma \backslash \mathfrak{S}_0; \Gamma s_K = b\}| = \frac{1}{\#A}$, and so proves the result. $\square$

Corollary 7.11. *Let $\mathcal{B}$ be the building of $G = \mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K}$ is a local field of characteristic zero. Let Γ be a lattice in $\mathrm{PGL}_3(\mathbb{K})$ which acts freely on the vertices of $\mathcal{B}$. Then the conclusions of Theorem 7.10 hold.*

Proof. The image of $SL_3(\mathbb{K})$ in $PGL_3(\mathbb{K})$ has finite index. Let Γ' be the pullback to $SL_3(\mathbb{K})$ of Γ. Then $\Gamma' \backslash SL_3(\mathbb{K})$ also has finite volume and the proof of Theorem 7.10 applies. Moreover, the Γ-orbits of tiles of $\mathcal{B}$ are made up of unions of Γ'-orbits. So if we wish to construct $s \in \mathfrak{S}_k$ having first and last tiles in certain Γ-orbits, we just pick Γ'-orbits contained in the two Γ-orbits and thereafter work with Γ'. $\square$

Remark 7.12. We needed to use an indirect argument in the previous Corollary because the Howe–Moore theorem does not apply in its simplest form to $PGL_3(\mathbb{K})$.

Remark 7.13. In work which will appear elsewhere, it will be shown how to extend the methods of the proof of the Howe–Moore theorem so as to prove the necessary ergodicity in greater generality. It is enough to suppose that Γ acts freely and with finitely many orbits on the vertices of a thick building of type $\widetilde{A}_2$. Since ergodicity implies (H2), and since (H0), (H1), and (H3) always hold, Theorem 7.7 is likewise true in this generality.

Not only does this allow one to work with the buildings associated to $\mathrm{PGL}_3(\mathbb{K})$ when $\mathbb{K}$ has positive characteristic, but it also makes available those buildings of type $\widetilde{A}_2$ which are associated to no linear group. Note finally that direct combinatorial proofs of (H2) can be constructed for the $\widetilde{A}_2$ groups listed in [CMSZ].

7.2. Examples of type $\widetilde{A}_1 \times \widetilde{A}_1$.

Analogous results hold for groups acting on buildings of type $\widetilde{A}_1 \times \widetilde{A}_1$. Consider by way of illustration a specific example studied in [M1] and generalized in [BM]. In [M1, Section 3], there is constructed a certain lattice subgroup Γ of $G = PGL_2(\mathbb{Q}_p) \times PGL_2(\mathbb{Q}_q)$, where $p, q \equiv 1 \pmod 4$ are two distinct primes. The building Δ of G is a product of two homogeneous trees T_1, T_2, so that the chambers of Δ are squares and the apartments are copies of the euclidean plane tesselated by squares. If $a \in \Gamma$ then there are automorphisms a_1, a_2 of T_1, T_2, respectively such that $a(u, v) = (a_1 u, a_2 v)$ for each vertex (u, v) of Δ. However, even though each $a \in \Gamma$ is a direct product of automorphisms, the group Γ is not a direct product of groups Γ_1 and Γ_2.

The group Γ acts freely and transitively on the vertices of Δ. The preceding results all extend to this situation. The tiles are now squares instead of parallelograms. The boundary of Δ is defined as before, using $\frac{\pi}{2}$-angled sectors. The condition (H1) is a consequence of [M1, Theorem 3.2] and the irreducibility condition (H2) follows from [M2, Proposition 3].

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