IRREDUCIBLE SUBSHIFTS ASSOCIATED WITH $\tilde{A}_2$ BUILDINGS.

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Abstract. Let Γ be a group of type rotating automorphisms of a building $\mathcal{B}$ of type $\widetilde{A}_2$, and suppose that Γ acts freely and transitively on the vertex set of $\mathcal{B}$. The apartments of $\mathcal{B}$ are tiled by triangles, labelled according to Γ-orbits. Associated with these tilings there is a natural subshift of finite type, which is shown to be irreducible. The key element in the proof is a combinatorial result about finite projective planes.

1. Introduction

Let $\mathcal{B}$ be a locally finite thick affine building of type $\widetilde{A}_2$ [Gar]. Such a building $\mathcal{B}$ is a two dimensional simplicial complex which is a union of two dimensional subcomplexes, called *apartments*. The apartments are Coxeter complexes of type $\widetilde{A}_2$, which may be realized as a tilings of the Euclidean plane by equilateral triangles. Buildings of type $\widetilde{A}_2$ are contractible as topological spaces and are natural two dimensional analogues of homogeneous trees. (A homogeneous tree is a building of type $\widetilde{A}_1$.) Each vertex v of $\mathcal{B}$ is labeled with a *type* $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber has exactly one vertex of each type. An automorphism α of $\mathcal{B}$ is said to be *type rotating* if there exists $i \in \mathbb{Z}/3\mathbb{Z}$ such that $\tau(\alpha(v)) = \tau(v) + i$ for all vertices $v \in \mathcal{B}$.

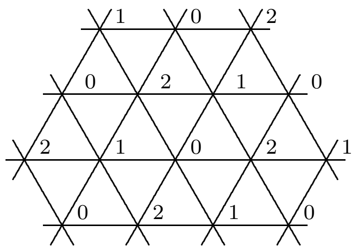

Figure 1. Part of an apartment in an $\tilde{A}_2$ building, showing vertex types.

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If $\mathcal{B}$ is a building of type $\widetilde{A}_2$ then the set S_v of vertices of $\mathcal{B}$ adjacent to any vertex v may be given the structure of a finite projective plane. The projective planes corresponding to different vertices v may be non-isomorphic [RT], but they all have the same order q. If a vertex v of $\mathcal{B}$ has type i then the set P of vertices of type $i+1$ in S_v correspond to the q^2+q+1 points of the projective plane. The set L of vertices of type $i+2$ in S_v correspond to the q^2+q+1 lines of the projective plane. A point $p \in P$ and a line $l \in L$ are incident in the projective plane if and only if there is an edge connecting them in the building. The integer q is called the order of the building and each edge in $\mathcal{B}$ lies on $q+1$ triangles. The reason for this is that every line in the projective plane is incident with $q+1$ points and every point is incident with $q+1$ lines. These facts will be used repeatedly below.

Suppose that $\mathcal{B}$ is a building of type $\widetilde{A}_2$ and that Γ is a group of type rotating automorphisms of $\mathcal{B}$ which acts freely and transitively on the vertex set of $\mathcal{B}$. Such groups Γ are called $\widetilde{A}_2$ groups. In some ways, $\widetilde{A}_2$ groups are rank two analogues of finitely generated free groups, which act in a similar way on buildings of type $\widetilde{A}_1$ (trees). The theory of $\widetilde{A}_2$ groups has been developed in detail in [CMSZ]. The $\widetilde{A}_2$ groups have a detailed combinatorial structure which makes them an ideal place to attack problems involving higher rank groups.

An $\widetilde{A}_2$ group can be described as follows [CMSZ, I,§3]. Let (P, L) be a projective plane of order q. Let $\lambda : P \to L$ be a bijection (a *point–line correspondence*). Let $\mathcal{T}$ be a set of triples (x, y, z) where $x, y, z \in P$, with the following properties.

(i) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for some $z \in P$ if and only if y and $\lambda(x)$ are incident (i.e. $y \in \lambda(x)$).

(ii) $(x, y, z) \in \mathcal{T} \Rightarrow (y, z, x) \in \mathcal{T}$.

(iii) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for at most one $z \in P$.

$\mathcal{T}$ is called a *triangle presentation* compatible with λ. A complete list is given in [CMSZ] of all triangle presentations for $q = 2$ and $q = 3$.

Let $\{a_x : x \in P\}$ be q^2+q+1 distinct letters and form the group

$$\Gamma = \left\langle a_x, x \in P \mid a_x a_y a_z = 1 \text{ for } (x, y, z) \in \mathcal{T} \right\rangle$$

The Cayley graph of Γ with respect to the generators $a_x, x \in P$ is the 1-skeleton of an affine building of type $\widetilde{A}_2$. It is convenient to identify the point $x \in P$ with the generator $a_x \in \Gamma$. If $x \in P$ then the line $\lambda(x)$ corresponds to the inverse a_x^{-1} [CMSZ]. We therefore write x^{-1} for a_x^{-1} and identify x^{-1} with $\lambda(x)$. From now on the notation x and $\lambda(x)$ is used to represent a_x and $a_{\lambda(x)}$ respectively. Note that, with this notation,

$$\mathcal{T} = \{(x, y, z) : x, y, z \in P \text{ and } \ xyz = 1\}.$$

This means that if $x, y \in P$ then $y \in \lambda(x)$ if and only if $xyz = 1$ for some $z \in P$.

The Cayley graph of Γ will be regarded as a directed graph. Vertices are identified with elements of Γ and a directed edge of the form (a, as) with $a \in \Gamma$ is labeled by a generator $s \in P$. Figure 2 illustrates a typical triangle based at a vertex $a \in \mathcal{B}$.

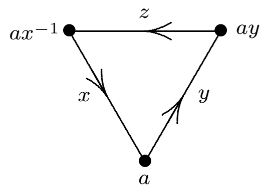


FIGURE 2. A chamber based at a vertex a.

If $q = 2$ there are eight $\widetilde{A}_2$ groups Γ, all of which embed as lattices in the linear group $\mathrm{PGL}(3, \mathbb{F})$ over a local field $\mathbb{F}$. If $q = 3$ there are 89 possible $\widetilde{A}_2$ groups, of which 65 have buildings which are not associated with linear groups [CMSZ].

Example 1.1. The group C.1 of [CMSZ] has presentation

$$\langle x_i, 0 \leq i \leq 6 \mid x_0x_0x_6, x_0x_2x_3, x_1x_2x_6, x_1x_3x_5, x_1x_5x_4, x_2x_4x_5, x_3x_4x_6 \rangle.$$

For this group, $q = 2$, and there are $q^2 + q + 1 = 7$ generators. Thus $P = \{x_0, \dots, x_6\}$ and $L = \{x_0^{-1}, \dots, x_6^{-1}\}$.

Two triangles lie in the same Γ-orbit if and only if they have the same edge labels, where each edge label is a generator of Γ. The combinatorics of the finite projective plane (P, L) shows that there are precisely $(q + 1)(q^2 + q + 1)$ such labellings, which we refer to as $\widetilde{A}_2$ *triangle labellings.* Triangle labellings are in bijective correspondence with the elements of the triangle presentation $\mathcal{T}$. In Figure 3 we illustrate a triangle labelling (one of three) corresponding to the second relation in Example 1.1.

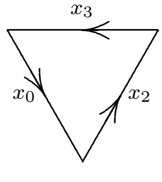


FIGURE 3. A triangle labelling for the group C.1.

The edge labels (or equivalently the tiles) induce a tiling of the apartments in $\mathcal{B}$, as illustrated in Figure 4.

There is a natural $\mathbb{Z}^2$ action on the space of tiled apartments, which gives rise to a so called 2-dimensional subshift of finite type.

Consider the set of all apartments of $\mathcal{B}$, with each triangle labelled as above. Two matrices M_1, M_2 with entries in $\{0, 1\}$ are defined as follows. If $\alpha, \beta \in \mathcal{T}$, say that $M_1(\alpha, \beta) = 1$ if and only if the triangle

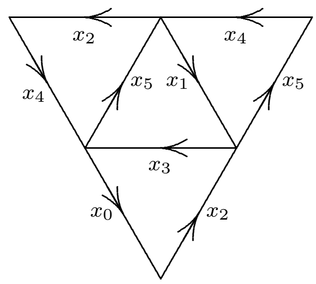


FIGURE 4

labellings $\alpha = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$ lie as shown on the right of Figure 5. A similar definition applies for $M_2(\alpha, \gamma) = 1$, as on the left of Figure 5.

FIGURE 5. Definition of the transition matrices.

The commuting matrices M_1, M_2 are the transition matrices associated with a 2-dimensional subshift, with alphabet $\mathcal{T}$. This subshift is said to be irreducible if for all $\alpha, \beta \in \mathcal{T}$, there exist integers $r, s > 0$ such that the (α, β) component of the matrix $M_1^r M_2^s$ satisfies

$$(M_1^r M_2^s)(\alpha, \beta) > 0.$$

A geometric interpretation of this condition is that any two triangle labellings $\alpha, \beta \in \mathcal{T}$ can be realized so that β lies in some sector with base labelled triangle α, as in Figure 6.

It is important for the simplicity of the C^*-algebras considered in [RS] that this subshift is irreducible. In this article we prove irreducibility by showing that we can actually choose $r > 0$ such that $M_1^r(\alpha, \beta) > 0$. Thus β lies on the wall of a sector as in Figure 7. A similar statement is true for the matrix M_2.

Another way of viewing this is to say that irreducibility is proved for the one dimensional subshift associated with tilings of strips between parallel walls in apartments, as illustrated in Figure 8. This is considerably stronger than irreducibility of the 2-dimensional subshift.

Let Γ be an $\widetilde{A}_2$ group. If Γ has the property that the 2-dimensional subshift described above is irreducible, then the theory developed in [RS, Section 7] applies. This means that one may construct an associated simple C^*-algebra whose structure was analyzed in [RS]. The

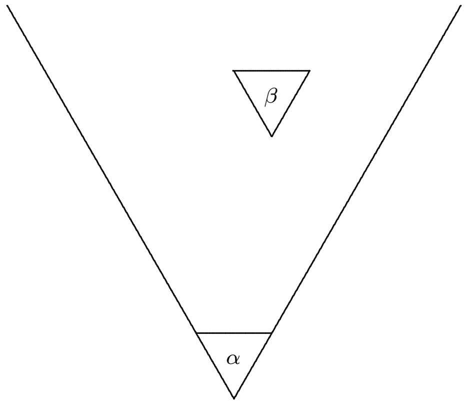


FIGURE 6. The condition for irreducibility.

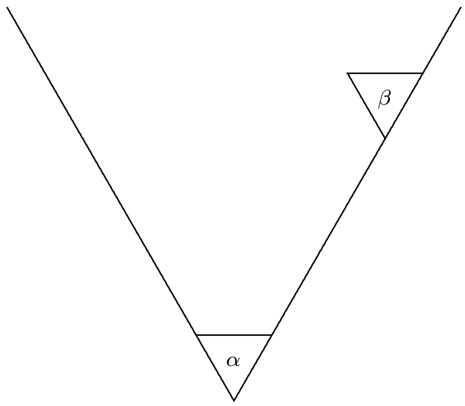


FIGURE 7. Shifting along the wall of a sector.

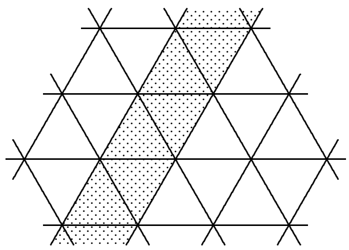
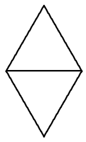


FIGURE 8. A strip in an apartment.

required irreducibility result was proved in [RS, Theorem 7.10] only for the case where Γ is a lattice in $\mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K}$ is a local field of characteristic zero. The argument of [RS, Theorem 7.10] does not apply if $\mathcal{B}$ is the building of $\mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K}$ is a local field of positive characteristic, which is the case for the group C.1 of Example 1.1. Neither does it apply to many examples constructed in [CMSZ], for which $\mathcal{B}$ is not the Bruhat-Tits building of a linear group. The purpose of the present article is to show that irreducibility holds for all $\widetilde{A}_2$ groups. This means that the theory of [RS] now applies to any such group.

Remark 1.2. The subshift studied in [RS] was defined in terms of labelled parallelograms formed by a union of two labelled triangles of the following form.



However, irreducibility of that subshift is an easy consequence of the result presented here.

We now state our main result.

Theorem 1.3. *Given any two $\widetilde{A}_2$ triangle labellings, these labellings can be realized as the initial and final triangles of a sequence of triangles arranged along some wall in $\mathcal{B}$ as follows:*

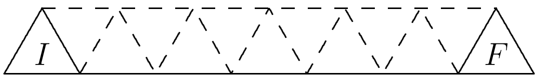
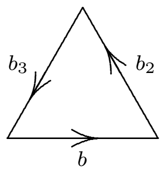


FIGURE 10. Labelled triangles along a wall

The rest of the article is devoted to the proof of Theorem 1.3.

2. Proof of Irreducibility of the 1-dimensional subshift

Fix once and for all the triangle labellings I and F. Consider a triangle labelling of the form below (which we refer to as $\underset{b}{\Delta}$).

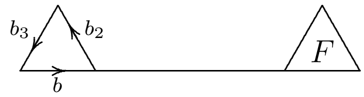


A triangle labelling of the form $\underset{b}{\Delta}$.

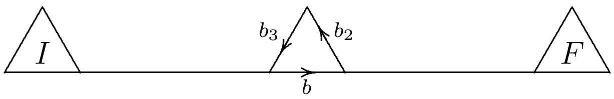
Call such a labelling $\underset{b}{\Delta}$ *reachable* from the *left* if it is the final triangle labelling in some sequence with initial triangle I.

Similarly define *reachable* from the *right*.

Note that for each edge labelling b there are $q+1$ triangles of the form $\underset{b}{\Delta}$. Therefore if we can show that there exists b such that $\underset{b}{\Delta}$ is



reachable from the left for more than $(q+1)/2$ values of the pair (b_2, b_3) and reachable from the right for more that $(q+1)/2$ values of (b_2, b_3), then there exists a labelling (b, b_2, b_3) which is reachable both ways. This will prove Theorem 1.3.



In subsequent arguments, we will need to use a criterion for a triangle labelling of the form $\underset{c}{\Delta}$ to be reachable in one step from a triangle labelling of the form $\underset{b}{\Delta}$, as in Figure 11.

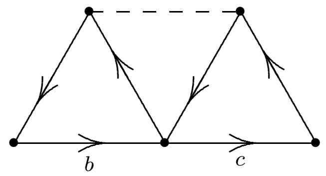


FIGURE 11

Lemma 2.1. *Figure 11 is possible in an apartment of $\mathcal{B}$ if and only if $c \notin \lambda(b)$.*

Proof. Fix a vertex $v \in \mathcal{B}$. Since the 1-skeleton of $\mathcal{B}$ is the Cayley graph of (Γ, P), the vertex v may be considered as an element of Γ. The choice of v is irrelevant, by transitivity of the action of Γ.

As explained in the introduction, the set S_v of vertices adjacent to v has the structure of a finite projective plane. The points of this projective plane are $\{vx\,;\, x \in P\}$ and the lines are $\{v\lambda(x)\,;\, x \in P\}$. Recall that $\lambda(x) = x^{-1}$ in the group Γ. Figure 11 is therefore equivalent to Figure 12.

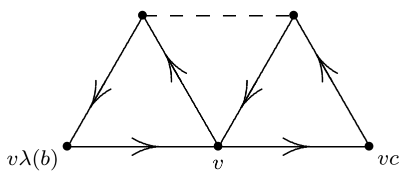


FIGURE 12

If $c \in \lambda(b)$, then there is an edge in $\mathcal{B}$ between $v\lambda(b)$ and vc. Figure 12 is therefore impossible, by contractibility of the building $\mathcal{B}$.

On the other hand, if $c \notin \lambda(b)$ then $v\lambda(b)$ and vc are not adjacent in S_v. Now $v\lambda(b)$ and vc lie in a hexagon H whose vertices belong to S_v.

This is because the projective plane S_v has the structure of a spherical building, whose apartments are hexagons. The vertices of the hexagon H are alternately points and lines of the projective plane S_v. The only way in which the line $v\lambda(b)$ and the point vc can fail to be adjacent in the hexagon H is if they are opposite vertices of the hexagon, as shown in Figure 13.

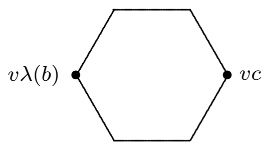


FIGURE 13

This means that Figure 12 is possible in $\mathcal{B}$, where each labelled triangle has one edge on the hexagon H.

□

Lemma 2.2. *If $b \in P$ then the numbers*

$$\mathcal{L}(b) = \#\{(b_2, b_3) : (b, b_2, b_3) \text{ is reachable from the left}\},$$
$$\mathcal{R}(b) = \#\{(b_2, b_3) : (b, b_2, b_3) \text{ is reachable from the right}\}$$

are independent of b.

Proof. It is clearly enough to prove the assertion for $\mathcal{L}(b)$. Given $b' \in P$, we must show that $\mathcal{L}(b) = \mathcal{L}(b')$. Now the diagram in Figure 14 can be completed by choosing c such that $c \notin \lambda(b)$ and $b' \notin \lambda(c)$. This is possible, since there exist $q+1$ elements $c \in \lambda(b)$, there exist $q+1$ elements c such that $b' \in \lambda(c)$, and $2(q+1) < q^2+q+1 = \#(P)$.

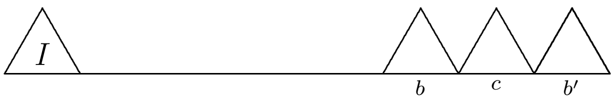


FIGURE 14

Choose and fix such an element $c \in P$. Then each labelling of $\underset{b}{\Delta}$ uniquely determines the labelling of $\underset{b'}{\Delta}$, and vice versa. That is, for fixed b, c, b', the number of labellings of $\underset{b}{\Delta}$ is the same as the number of labellings of $\underset{b'}{\Delta}$. It follows that $\mathcal{L}(b) \le \mathcal{L}(b')$. By symmetry, $\mathcal{L}(b) = \mathcal{L}(b')$. □

It follows from Lemma 2.2 that, in order to prove Theorem 1.3, it is enough to find an elements $b_1, b_2 \in P$ such that

(1a) $$\mathcal{L}(b_1) > (q+1)/2\,,$$

(1b) $$\mathcal{R}(b_2) > (q+1)/2.$$

It is clearly enough to verify (1a).

Given the initial triangle labelling I, denote by D the set of all $d \in P$ for which Figure 15 is possible. Thus D contains precisely q elements. For each $d \in D$ let S_d denote the set of $f \in P$ such that Figure 15 is possible. Therefore $\#(S_d) = q$.

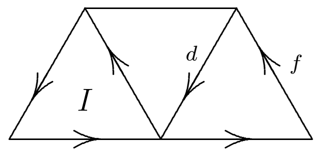


FIGURE 15

Lemma 2.3. *If $d_1, d_2 \in D$ and $d_1 \neq d_2$, then $S_{d_1} \cap S_{d_2}$ contains at most one element.*

Proof. If $f \in S_{d_1} \cap S_{d_2}$ then d_1, $d_2 \in \lambda(f)$. The two points d_1, d_2 in the projective plane determine the line $\lambda(f)$ uniquely. That is, f is uniquely determined. □

Let $S = \bigcup_{d \in D} S_d$. Then S is the set of elements $f \in P$ such that a diagram like Figure 15 is possible, for the given initial triangle I. There are $q(q-1)/2$ sets of the form $S_{d_i} \cap S_{d_j}$, each of which contains at most one element. It follows from the exclusion-inclusion principle that

$$\#(S) \geq q.q - \frac{q(q-1)}{2} = \frac{q^2+q}{2}. \tag{2}$$

This gives a lower bound on the number of possible edge labels f in Figure 15. Now let $f \in S$ be such an edge label. Then $f \in S_d$ for some $d \in P$. Consider diagrams of the form illustrated in Figure 16.

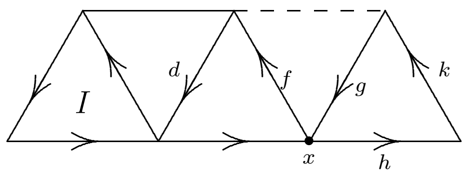


FIGURE 16

In the projective plane of nearest neighbours of x label the points p_f, p_g and lines l_f, l_h as in Figure 17. (By duality, the words 'point' and 'line' could be interchanged here. The specified choice makes the wording of a later argument easier.)

Then (g, h, k) is reachable from f, (i. e. the diagram is possible), if and only if $l_h \neq l_f$, $p_g \neq p_f$ and $p_g \in l_h \cap l_f$. That is $p_g = (l_f - \{p_f\}) \cap l_h$, where $l_h \neq l_f$.

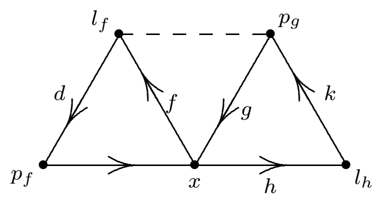


FIGURE 17

For $h \in P$ the set of possible g is in bijective correspondence with the set

$$\begin{aligned} T_h &= \bigcup_{f\in S-\{h\}} \{(l_f - \{p_f\}) \cap l_h\} \\ &= l_h \cap \bigcup_{f\in S-\{h\}} l_f - \{p_f\}\,. \end{aligned}$$

If we can show that $\#(T_h) > \frac{q+1}{2}$ for some h, then (1a) is satisfied with $b_1 = h$.

The proof of Theorem 1.3 therefore reduces to the following combinatorial result about projective planes. Recall from (2) that $\#(S) \geq \frac{q^2+q}{2}$.

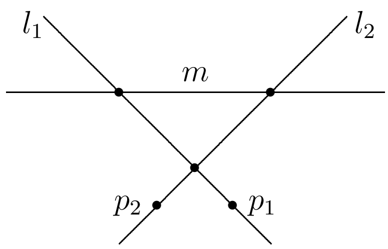
Lemma 2.4. *In a projective plane of order q, let $\left\{l_j : 1 \leq j \leq \frac{q^2+q}{2}\right\}$ be a family of distinct lines. For each j, let p_j be a point on l_j and let $l'_j = l_j - \{p_j\}$. Then there exists a line m such that*

$$\#\left(m \cap \bigcup \{l'_j : l_j \neq m\}\right) > \frac{q+1}{2}\,.$$

Proof. This divides into 3 separate cases, which are dealt with in increasing order of difficulty.

Case 1: $q = 2$. Here $\frac{q^2+q}{2} = 3$, and so there are three distinct lines l_1, l_2, l_3, each containing three points. Each set l'_j therefore contains exactly two points.

Choose a line m which meets a point of $l'_1 - l_2$ and a point of $l'_2 - l_1$. Then $m \cap (l'_1 \cup l'_2)$ contains $2 > \frac{3}{2}$ elements.



Case 2: $q \geq 4$.
Each line contains $q+1$ points, so $\#(l'_j) = q$. Two distinct lines meet in exactly one point. Hence

$$\#(l'_1 \cup l'_2 \cup l'_3) \geq 3q - 3\,. \tag{3}$$

Assume that the conclusion of the Lemma is false. Then we *claim* that for $3 \le k \le \frac{q^2+q}{2}$,

$$\#(l'_1 \cup l'_2 \cup \cdots \cup l'_k) \ge (3q-3) + (k-3)\lceil \frac{q-1}{2} \rceil\,, \tag{4}$$

where $\lceil t \rceil$ denotes the ceiling of t, the least integer not less than t.

We prove the claim by induction. If $k = 3$ then it is true, by (3). Assume that (4) holds for a given value of k. Since we are assuming that the conclusion of the Lemma fails,

$$\#(l'_{k+1} \cap (l'_1 \cup \cdots \cup l'_k)) \le \#(l_{k+1} \cap (l'_1 \cup \cdots \cup l'_k)) \le \frac{q+1}{2}\,.$$

Hence,

$$\#(l'_{k+1} - (l'_1 \cup \cdots \cup l'_k)) \ge q - \frac{q+1}{2} = \frac{q-1}{2}\,.$$

Therefore

$$\begin{aligned}\#(l'_1 \cup l'_2 \cup \cdots \cup l'_k \cup l'_{k+1}) &\ge (3q-3) + (k-3)\lceil \tfrac{q-1}{2} \rceil + \lceil \tfrac{q-1}{2} \rceil \\ &= (3q-3) + ((k+1)-3)\lceil \tfrac{q-1}{2} \rceil\,.\end{aligned}$$

Thus we have established (4).

In particular, since (4) holds for $k = (q^2+q)/2$, and there are q^2+q+1 points in the projective plane, we have

$$q^2 + q + 1 \ge (3q-3) + \left(\frac{q^2+q}{2} - 3\right) \lceil \frac{q-1}{2} \rceil\,. \tag{5}$$

Now (5) has been derived from the assumption that the conclusion of the Lemma was false. Therefore all that is required now is to show that (5) is false. Now (5) fails when $q = 4$, since in that case

$$q^2 + q + 1 = 21 \not\ge 23 = (3q-3) + \left(\frac{q^2+q}{2} - 3\right) \lceil \frac{q-1}{2} \rceil\,.$$

On the other hand, if $q \ge 5$, write $q = r + 5$, $r \ge 0$. Then

$$\begin{aligned}4\left(3q - 3 + \left(\frac{q^2+q}{2} - 3\right)\left(\frac{q-1}{2}\right) - (q^2+q+1)\right) &= q^3 - 4q^2 + q - 10 \\ &= r^3 + 11r^2 + 36r + 20 \\ &\ge 20\,.\end{aligned}$$

Therefore (5) also fails when $q \ge 5$. This proves Case 2.

Case 3: $q = 3$. This requires separate treatment. Here $\frac{q^2+q}{2} = 6$, $\frac{q+1}{2} = 2$.

Given distinct lines $l_1, l_2 \ldots, l_6$ we delete a point from each to obtain sets $l'_1, \ldots, l'_6$. We must find a line m such that

$$\#\left(m \cap \bigcup \{l'_j : l_j \ne m\}\right) > 2\,. \tag{6}$$

It is known that there is a unique projective plane of order 3, namely the Desarguesian plane arising from a 3-dimensional vector space over $\mathbb{F}_3$ [Bl, Theorem 2.3.1].

There are thirteen points and thirteen lines in the projective plane. Label the points $0, 1, 2, \ldots, 12$ and label the lines $(0), (1), (2), \ldots, (12)$, as indicated in the table below [Bl, Section 1.4]. For example, line (8) contains the points $5, 6, 8, 1$.

(12)	(11)	(10)	(9)	(8)	(7)	(6)	(5)	(4)	(3)	(2)	(1)	(0)
1	2	3	4	5	6	7	8	9	10	11	12	0
2	3	4	5	6	7	8	9	10	11	12	0	1
4	5	6	7	8	9	10	11	12	0	1	2	3
$\underline{10}$	$\underline{11}$	12	0	1	2	3	4	5	6	7	8	9

By permuting the lines $l_1, l_2 \ldots, l_6$, if necessary, we may suppose that $l_1 \cap l_2$ is not equal to either of the excluded points p_1 or p_2.

To check this assertion, suppose that it does not already hold for the given choice of l_1, l_2. Since each point is incident at most four of the lines $l_1, l_2 \ldots, l_6$, we may assume that $l_1 \cap l_2 = p_1$ but that $l_1 \cap l_5 \neq p_1$ and $l_1 \cap l_6 \neq p_1$. If $l_1 \cap l_5 \neq p_5$ or $l_1 \cap l_6 \neq p_6$ we are done. On the other hand, if $l_1 \cap l_5 = p_5$, $l_1 \cap l_6 = p_6$ and $p_5 \neq p_6$ then $l_5 \cap l_6$ is not equal to either p_5 or p_6. It remains to consider the case $l_1 \cap l_5 = p_5$, $l_1 \cap l_6 = p_6$ with $p_5 = p_6$. In that case, $l_2 \cap l_5 \neq p_5$, $l_2 \cap l_6 \neq p_6$ (since $l_1 \cap l_2 = p_1$) and either $l_2 \cap l_5 \neq p_2$ or $l_2 \cap l_6 \neq p_2$.

Having verified this assertion, we can assume that $l_1 \cap l_2$, p_1 and p_2 are three noncollinear points. Now the automorphism group $\mathrm{PGL}_3(\mathbb{F}_3)$ acts transitively on triples of noncollinear points. Map these three points to the points $2, 10, 11$ respectively We may therefore suppose that l_1, l_2 are lines $(12), (11)$ respectively with excluded points $10, 11$ (underlined in the table). Thus

$$l_1' = \{1, 2, 4\} \qquad l_2' = \{2, 3, 5\} .$$

Now for $j = 3, 4, 5, 6$, the set l_j' contains a point not in l_1 or l_2, namely one of the points 0,6,7,8,9,12. Let $j \in \{3, 4, 5, 6\}$.

(a) If $0 \in l_j'$ then

$$(0) \cap (l_1' \cup l_2' \cup l_j') = \{1, 3, 0\} ,$$

and

$$(9) \cap (l_1' \cup l_2' \cup l_j') = \{4, 5, 0\} .$$

One can choose as line m to satisfy inequality (6) whichever of (0), (9) is not equal to l_j. Both choices of m may be possible.

(b) If $6 \in l_j'$ then

$$(8) \cap (l_1' \cup l_2' \cup l_j') = \{1, 5, 6\} ,$$

and

$$(10) \cap (l_1' \cup l_2' \cup l_j') = \{4, 3, 6\} .$$

One can choose as line m whichever of (8), (10) is not equal to l_j.

(c) If $7 \in l_j'$ then

$$(9) \cap (l_1' \cup l_2' \cup l_j') = \{4, 5, 7\} ,$$

so if $l_j \neq (9)$ we can choose $m = (9)$.
If $8 \in l_j'$ then

$$(8) \cap (l_1' \cup l_2' \cup l_j') = \{1, 5, 8\} ,$$

so if $l_j \neq (8)$ we can choose $m = (8)$.
If $9 \in l_j'$ then

$$(0) \cap (l_1' \cup l_2' \cup l_j') = \{1, 3, 9\} ,$$

so if $l_j \neq (0)$ we can choose $m = (0)$.
If $12 \in l_j'$ then

$$(10) \cap (l_1' \cup l_2' \cup l_j') = \{4, 3, 12\} ,$$

so if $l_j \neq (10)$ we can choose $m = (10)$.

(d) By choosing $j = 3, 4, 5, 6$ in parts (a), (b) and (c) above that, we see that we can choose m to satisfy inequality (6) except in one case. Up to a permutation of the set $\{3, 4, 5, 6\}$, this is the case where

$$l_3,\ l_4,\ l_5,\ l_6 = (9),\ (8),\ (0),\ (10)$$

respectively with

$$7 \in (9)',\ 8 \in (8)',\ 9 \in (0)',\ 12 \in (10)' .$$

We work with the three lines $l_1 = (12)$, $l_3 = (9)$, $l_6 = (10)$. There are two possibilities to consider:
If $6 \in (10)'$ then $(7) \cap (l_1' \cup l_3' \cup l_6') = \{2, 7, 6\}$; so take $m = (7)$.
If $6 \notin (10)'$ then $(10)' = \{3, 4, 12\}$, $(2) \cap (l_1' \cup l_3' \cup l_6') = \{1, 7, 12\}$; so take $m = (2)$.

□

Remark 2.5. Careful examination of the proof of Theorem 1.3 shows that six steps are enough to get from initial to final triangle, exactly as indicated in Figure 10.

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