ON THE K-THEORY OF BOUNDARY C^*-ALGEBRAS OF $\widetilde{A}_2$ GROUPS

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Abstract. Let Γ be an $\widetilde{A}_2$ subgroup of $\mathrm{PGL}_3(\mathbb{K})$, where $\mathbb{K}$ is a local field with residue field of order q. The module of coinvariants $C(\mathbb{P}^2_{\mathbb{K}}, \mathbb{Z})_\Gamma$ is shown to be finite, where $\mathbb{P}^2_{\mathbb{K}}$ is the projective plane over $\mathbb{K}$. If the group Γ is of Tits type and if $q \not\equiv 1 \pmod 3$ then the exact value of the order of the class $[\mathbf{1}]_{K_0}$ in the K-theory of the (full) crossed product C^*-algebra $C(\Omega) \rtimes \Gamma$ is determined, where Ω is the Furstenberg boundary of $\mathrm{PGL}_3(\mathbb{K})$. For groups of Tits type, this verifies a conjecture of G. Robertson and T. Steger.

1. Introduction

Let $\mathbb{K}$ be a nonarchimedean local field with residue field k of order q and uniformizer π. Denote by $\mathbb{P}^2_{\mathbb{K}}$ the set of one dimensional subspaces of the vector space $\mathbb{K}^3$, i.e. the set of points in the projective plane over $\mathbb{K}$. Then $\mathbb{P}^2_{\mathbb{K}}$ is a compact totally disconnected space with the topology inherited from $\mathbb{K}$, and there is a continuous action of $G = \mathrm{PGL}_3(\mathbb{K})$ on $\mathbb{P}^2_{\mathbb{K}}$. The group G also acts on its Furstenberg boundary Ω, which is the space of maximal flags $(0) < V_1 < V_2 < \mathbb{K}^3$.

The Bruhat-Tits building Δ of G is a topologically contractible simplicial complex of dimension 2, which is a union of apartments. Each apartment in Δ is a flat subcomplex isomorphic to a tessellation of $\mathbb{R}^2$ by equilateral triangles: that is, an affine Coxeter complex of type $\widetilde{A}_2$. For this reason, Δ is referred to as an affine building of type $\widetilde{A}_2$.

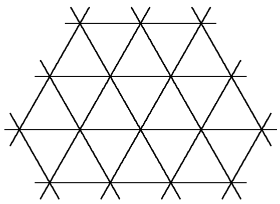

Figure 1. The Coxeter complex of type $\widetilde{A}_2$

An apartment in Δ is a union of six sectors, based at a fixed vertex (Figure 2). These six sectors correspond to six points ω_i in the Furstenberg boundary Ω. They also represent the six edges of a hexagon in the spherical building at infinity Δ^∞. This hexagon (which is a spherical Coxeter complex of type A_2) is an apartment in Δ^∞. For this reason, Δ^∞ is referred to as a spherical building of type A_2. More details are provided in Section 4 below.

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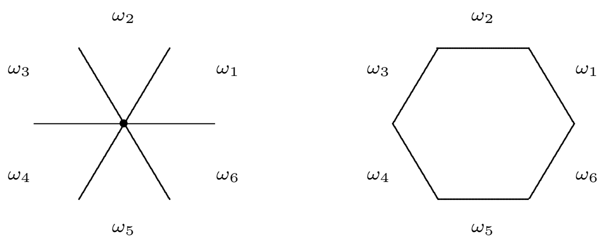

FIGURE 2. The six boundary points of an apartment

Let Γ be a lattice subgroup of G. The abelian group $C(\mathbb{P}^2_{\mathbb{K}}, \mathbb{Z})$ of continuous integer-valued functions on $\mathbb{P}^2_{\mathbb{K}}$ is a Γ-module and the module of coinvariants $C(\mathbb{P}^2_{\mathbb{K}}, \mathbb{Z})_\Gamma = H_0(\Gamma; C(\mathbb{P}^2_{\mathbb{K}}, \mathbb{Z}))$ is a finitely generated group. Now suppose that Γ acts freely and transitively on the vertex set of the Bruhat-Tits building of G, i.e. Γ is an $\widetilde{A}_2$ group [4]. Such a group is a natural analogue of a free group, which acts freely and transitively on the vertex set of a tree (which is a building of type $\widetilde{A}_1$). We prove that $C(\mathbb{P}^2_{\mathbb{K}}, \mathbb{Z})_\Gamma$ is a finite group and that the class $[\mathbf{1}]$ in $C(\mathbb{P}^2_{\mathbb{K}}, \mathbb{Z})_\Gamma$ has order bounded by $q^2 - 1$.

An $\widetilde{A}_2$ group of *Tits type* has a presentation with large automorphism group [4, II, Sections 4,5]. If Γ is of Tits type, and if $q \not\equiv 1 \pmod 3$ it is shown that the class $[\mathbf{1}]$ in $C(\mathbb{P}^2_{\mathbb{K}}, \mathbb{Z})_\Gamma$ has order exactly $q-1$. For groups of Tits type, this proves a conjecture of [8] regarding the order of the class $[\mathbf{1}]_{K_0}$ in the K-theory of boundary C^*-algebras.

Theorem 1.1. *Let Γ be an $\widetilde{A}_2$ group of Tits type and let $\mathfrak{A}_\Gamma$ be the (full) crossed product C^*-algebra $C(\Omega) \rtimes \Gamma$. If $q \not\equiv 1 \pmod 3$ then the order of $[\mathbf{1}]_{K_0}$ in $K_0(\mathfrak{A}_\Gamma)$ is precisely $q-1$.*

It is worth noting that the full crossed product C^*-algebra $C(\Omega) \rtimes \Gamma$ coincides with the reduced crossed product, since the action of Γ on Ω is amenable [6, Section 4.2]. For $q \equiv 1 \pmod 3$, the conjectured order of $[\mathbf{1}]_{K_0}$ is $\frac{q-1}{3}$, and this is verified for $q \le 31$. The element $[\mathbf{1}]_{K_0}$ is significant because the C^*-algebras $\mathfrak{A}_\Gamma$ are classified up to isomorphism by their two K-groups, together with the class $[\mathbf{1}]_{K_0}$ [7, Remark 6.5]. Many of the results are proved in a more general context, where the $\widetilde{A}_2$ group Γ is not necessarily a lattice subgroup of $\mathrm{PGL}_3(\mathbb{K})$.

2. BACKGROUND

2.1. **The Bruhat-Tits building of** $\mathrm{PGL}_3(\mathbb{K})$**.** Given a local field $\mathbb{K}$, with discrete valuation $v : \mathbb{K}^\times \to \mathbb{Z}$, let $\mathcal{O} = \{x \in \mathbb{K} : v(x) \ge 0\}$ and let $\pi \in \mathbb{K}$ satisfy $v(\pi) = 1$. A *lattice* L is a rank-3 $\mathcal{O}$-submodule of $\mathbb{K}^3$. In other words $L = \mathcal{O}v_1 + \mathcal{O}v_2 + \mathcal{O}v_3$, for some basis $\{v_1, v_2, v_3\}$ of $\mathbb{K}^3$. Two lattices L_1 and L_2 are *equivalent* if $L_1 = uL_2$ for some $u \in \mathbb{K}^\times$. The Bruhat-Tits building Δ of $\mathrm{PGL}_3(\mathbb{K})$ is a two dimensional simplicial complex whose vertices are equivalence classes of lattices in $\mathbb{K}^3$. Two lattice classes $[L_0], [L_1]$ are *adjacent* if, for suitable representatives L_0, L_1, we have $L_0 \subset L_1 \subset \pi^{-1}L_0$. A *simplex* is a set of pairwise adjacent lattice classes. The maximal simplices (chambers) are the sets $\{[L_0], [L_1], [L_2]\}$ where $L_0 \subset L_1 \subset L_2 \subset \pi^{-1}L_0$. These inclusions determine a canonical ordering of the vertices in a chamber, up to cyclic permutation.

Each vertex v of Δ has a *type* $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber of Δ has exactly one vertex of each type. If the Haar measure on $\mathbb{K}^3$ is normalized so that $\mathcal{O}^3$ has measure 1 then the type map may be defined by $\tau([L]) = \mathrm{vol}(L) + 3\mathbb{Z}$. The cyclic ordering of the vertices of a chamber coincides with the natural ordering given by the vertex types. The building Δ is of type $\widetilde{A}_2$ and there is a natural action of $\mathrm{PGL}_3(\mathbb{K})$ on Δ induced by the action of $\mathrm{GL}_3(\mathbb{K})$ on the set of lattices.

2.2. $\widetilde{A}_2$ **groups.** More generally, consider any locally finite building Δ of type $\widetilde{A}_2$. Each vertex v of Δ is labeled with a type $\tau(v) \in \mathbb{Z}/3\mathbb{Z}$, and each chamber has exactly one vertex of each type. Each edge e is directed, with initial vertex $o(e)$ of type i and final vertex $t(e)$ of type $i+1$. An automorphism α of Δ is said to be *type rotating* if there exists $i \in \mathbb{Z}/3\mathbb{Z}$ such that $\tau(\alpha(v)) = \tau(v) + i$ for all vertices $v \in \Delta$.

Suppose that Γ is a group of type rotating automorphisms of Δ, which acts freely and transitively on the vertex set of Δ. Such a group Γ is called an $\widetilde{A}_2$ group. The theory of $\widetilde{A}_2$ groups has been developed in [4] and some, but not all, $\widetilde{A}_2$ groups embed as lattice subgroups of $\mathrm{PGL}_3(\mathbb{K})$. Any $\widetilde{A}_2$ group can be constructed as follows [4, I, Section 3]. Let (P, L) be a projective plane of order q. There are q^2+q+1 points (elements of P) and q^2+q+1 lines (elements of L). Let $\lambda : P \to L$ be a bijection. A *triangle presentation* compatible with λ is a set $\mathcal{T}$ of ordered triples (x, y, z) where $x, y, z \in P$, with the following properties.

(i) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for some $z \in P$ if and only if y and $\lambda(x)$ are incident, i.e. $y \in \lambda(x)$.
(ii) $(x, y, z) \in \mathcal{T} \Rightarrow (y, z, x) \in \mathcal{T}$.
(iii) Given $x, y \in P$, then $(x, y, z) \in \mathcal{T}$ for at most one $z \in P$.

In [4] there is displayed a complete list of triangle presentations for $q = 2$ and $q = 3$. Given a triangle presentation $\mathcal{T}$, one can form the group

$$\Gamma = \Gamma_{\mathcal{T}} = \big\langle P \mid xyz = 1 \text{ for } (x, y, z) \in \mathcal{T} \big\rangle. \tag{1}$$

The Cayley graph of Γ with respect to the generating set P is the 1-skeleton of a building Δ of type $\widetilde{A}_2$. Vertices are elements of Γ and a directed edge of the form $e = (a, ax)$ with $a \in \Gamma$ is labeled by a generator $x \in P$. Denote by E_+ the set of directed edges of Δ.

The link of a vertex a of Δ is the incidence graph of the projective plane (P, L), where the lines in L correspond to the inverses in Γ of the generators in P. In other words, $\lambda(x) = x^{-1}$ for $x \in P$.

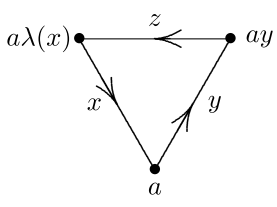

FIGURE 3. A chamber based at a vertex a

3. The group $A_{\mathcal{T}}$

The main results will be proved by defining a homomorphism from an abelian group $\mathcal{A}_{\mathcal{T}}$ onto the module of coinvariants. Suppose that $\mathcal{T}$ is a triangle presentation with associated $\widetilde{A}_2$ group $\Gamma = \Gamma_{\mathcal{T}}$. Define $A_{\mathcal{T}}$ to be the abelian group generated by $P \cup \{\varepsilon\}$ subject to the relations

$$\sum_{y \notin \lambda(x)} y = x, \quad x \in P; \tag{2a}$$

$$x + y + z = \varepsilon, \quad (x,y,z) \in \mathcal{T}; \tag{2b}$$

$$\sum_{x \in P} x = \varepsilon. \tag{2c}$$

It follows immediately from (2a) and (2c) that, for each $x \in P$,

$$\varepsilon = \sum_{y \in P} y = \sum_{y \notin \lambda(x)} y + \sum_{y \in \lambda(x)} y = x + \sum_{y \in \lambda(x)} y.$$

If we define $\widehat{x} = \sum_{y \in \lambda(x)} y$ then we obtain

$$x + \widehat{x} = \varepsilon, \quad x \in P. \tag{3}$$

The group $A_{\mathcal{T}}$ has an alternative presentation with relations (2b), (2c), (3). It turns out that $A_{\mathcal{T}}$ is finite. First of all, observe that the element ε has finite order.

Lemma 3.1. *In the group $A_{\mathcal{T}}$, $(q^2 - 1)\varepsilon = 0$.*

Proof. Define a $\{0,1\}$-matrix M by $M(x,y) = 1 \Leftrightarrow y \notin \lambda(x)$. Then

$$\begin{aligned}
\varepsilon &= \sum_{x \in P} x = \sum_{x \in P} \sum_{y \in P} M(x,y) y \\
&= \sum_{y \in P} (\sum_{x \in P} M(x,y)) y = \sum_{y \in P} q^2 y \\
&= q^2 \varepsilon.
\end{aligned}$$

□

Proposition 3.2. *The group $A_{\mathcal{T}}$ is finite.*

Proof. The $\widetilde{A}_2$ group Γ has Kazhdan's property T [3, 12]. It follows that the abelianization $\Gamma_{ab} = H_1(\Gamma, \mathbb{Z})$ is finite [2, Corollary 1.3.6]. By relation (2b), $A_{\mathcal{T}}/\langle\varepsilon\rangle$ is a quotient of Γ_{ab}. Thus $A_{\mathcal{T}}/\langle\varepsilon\rangle$ is finite. Therefore so also is $A_{\mathcal{T}}$, since $\langle\varepsilon\rangle$ is finite, by Lemma 3.1. □

Question. What is the order of ε in $A_{\mathcal{T}}$?

Lemma 3.3. *The order of ε is at least $\frac{q-1}{(q-1,3)}$.*

Proof. The map $f : P \cup \{\varepsilon\} \to \mathbb{Z}_{q^2-1}$ defined by $f(\varepsilon) = 3(q+1)$ and $f(x) = q+1$ for each $x \in P$, extends to a homomorphism $f : A_{\mathcal{T}} \to \mathbb{Z}_{q^2-1}$. For f preserves the relations (2), since

$$\sum_{y \notin \lambda(x)} f(y) = q^2(q+1) = q+1 = f(x), \quad x \in P,$$

$$f(x) + f(y) + f(z) = 3(q+1) = f(\varepsilon), \quad (x,y,z) \in \mathcal{T},$$

and

$$\sum_{x \in P} f(x) = (q^2+q+1)(q+1) = 3(q+1) = f(\varepsilon).$$

Since $f(\varepsilon) = 3(q+1)$, the order of $f(\varepsilon)$ is

$$\frac{q^2-1}{(q^2-1, 3(q+1))} = \frac{q^2-1}{(q+1)(q-1,3)} = \frac{q-1}{(q-1,3)}.$$

Therefore ε has order at least $\frac{q-1}{(q-1,3)}$. □

Proposition 3.4. *Suppose that there exists a subset $\mathcal{M} \subset \mathcal{T}$ such that each element in P occurs precisely 3 times in the triples belonging to $\mathcal{M}$. Then $(q-1)\varepsilon = 0$.*

Proof. Since $\#P = q^2+q+1$, we have $\#\mathcal{M} = q^2+q+1$. Therefore

$$\begin{aligned} 3\varepsilon = 3\sum_{x \in P} x &= \sum_{(x,y,z) \in \mathcal{M}} (x+y+z) \\ &= \sum_{(x,y,z) \in \mathcal{M}} \varepsilon = (q^2+q+1)\varepsilon. \end{aligned}$$

The result follows, since $q^2\varepsilon = \varepsilon$, by Lemma 3.1. □

Remark 3.5. Such a set $\mathcal{M}$ exists if $\mathcal{T}$ is a triangle presentation of any torsion free $\widetilde{A}_2$ group with $q = 2$. Thus $\varepsilon = 0$ if $q = 2$. To see this, observe that for each $x_0 \in P$ there are 3 elements $y \in P$ such that $y \in \lambda(x_0)$. Thus there are 3 elements in $\mathcal{T}$ of the form (x_0, y, z), where z is uniquely determined by x_0 and y. The 7 possible $x_0 \in P$ give rise to $7.3 = 21$ elements of $\mathcal{T}$. Thus $\#\mathcal{T} = 21$, and each element of P occurs $3+3+3 = 9$ times in the triples belonging to $\mathcal{T}$. Since $\mathcal{T}$ contains no triple of the form (x,x,x), the orbit of an element of $\mathcal{T}$ under cyclic permutations contains 3 elements. Choosing one element of $\mathcal{T}$ from each such orbit, we obtain a set $\mathcal{M}$ containing 7 elements of $\mathcal{T}$, in which each generator appears 3 times.

A weak form of Proposition 3.4 is the following.

Corollary 3.6. *Suppose that there exists a subset $\mathcal{M} \subset \mathcal{T}$ such that each of the three coordinate projections from $\mathcal{M}$ onto P is bijective. Then*

$$(q-1)\varepsilon = 0.$$

The rest of this section provides examples where Corollary 3.6 applies.

3.1. **Invariant triangle presentations.** Consider the Desarguesian projective plane $(P,L)=PG(2,q)$. The points and lines are the 1- and 2-dimensional subspaces, respectively, of a 3-dimensional vector space V over $\mathbb{F}_q$, with incidence being inclusion. The field $\mathbb{F}_q$ is a subfield of $\mathbb{F}_{q^3}$ and so $\mathbb{F}_{q^3}$ is a 3-dimensional vector space over $\mathbb{F}_q$. We may therefore identify P with $\mathbb{F}_{q^3}^\times/\mathbb{F}_q^\times$. Now $\mathbb{F}_{q^3}^\times$ is a cyclic group, with generator ζ, say, and $\mathbb{F}_q^\times$ is the subgroup generated by ζ^{1+q+q^2} (the unique subgroup with $q-1$ elements). Thus P is a cyclic group of order $1+q+q^2$ generated by the element $\sigma=\mathbb{F}_q^\times\zeta$. Multiplication by ζ is an $\mathbb{F}_q$-linear transformation and so multiplication by σ is a collineation of (P,L). Thus σ generates a cyclic collineation group of order q^2+q+1 which acts regularly on P (and on L). A group with these properties is called a Singer group, after [10].

Remark 3.7. It is not known whether a projective plane with a Singer group is necessarily Desarguesian. Also, there are no known $\widetilde{A}_2$ groups for which the underlying projective plane (P,L) is not Desarguesian.

Call a triangle presentation S-invariant if it is invariant under a Singer group of collineations of (P,L). Such presentations are studied in [4, I, Section 5]. It is shown in [4, I, Theorem 5.1] that there are exactly $2^{(q+1)/3}$, $2^{q/3}$ or $3(2^{(q-1)/3})$ distinct S-invariant triangle presentations, according as $q\equiv -1$, 0 or 1 mod 3, respectively. These presentations are constructed explicitly in [4] and all of them are also invariant under the Frobenius collineation $x\mapsto x^q$. It is not known whether all the corresponding $\widetilde{A}_2$ groups embed as lattices in $\mathrm{PGL}_3(\mathbb{K})$.

Remark 3.8. If $\lambda: P\to L$ is a bijection and $\mathcal{T}$ is an S-invariant triangle presentation compatible with λ then $y\in\lambda(x)\iff \sigma y\in\lambda(\sigma x)$. Therefore the presentation (2) of the group $\mathcal{A}_\mathcal{T}$ is invariant under the Singer group $<\sigma>$. It is also invariant under the collineation $x\mapsto x^q$.

Corollary 3.9. *If the triangle presentation $\mathcal{T}$ is S-invariant then, in the group $A_\mathcal{T}$, $(q-1)\varepsilon=0$.*

Proof. Fix a triple $(x_0,y_0,z_0)\in\mathcal{T}$. The orbit $\mathcal{M}_0$ of (x_0,y_0,z_0) under the Singer group satisfies the hypotheses of Corollary 3.6. □

Remark 3.10. It follows from Lemma 3.3 that if $\mathcal{T}$ is S-invariant and $q\not\equiv 1 \pmod 3$ then the order of ε is $q-1$. Computations on the family of triangle presentations described below confirm that, up to $q=32$, the order of ε is $\frac{q-1}{(q-1,3)}$. We conjecture that this is true for all S-invariant triangle presentations and for all q.

3.2. **S-invariant triangle presentations of Tits type** [4, I Section 4]**.** The mapping $x\mapsto x^q$ is an automorphism of $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$, i.e. it has fixed field $\mathbb{F}_q$. The trace $\mathrm{Tr}:\mathbb{F}_{q^3}\to\mathbb{F}_q$ is the $\mathbb{F}_q$-linear mapping defined by $\mathrm{Tr}(a)=a+a^q+a^{q^2}$. Now $(x_0,y_0)\mapsto \mathrm{Tr}(x_0y_0)$ is a regular symmetric bilinear form on $\mathbb{F}_{q^3}$, and so the map

$$V\mapsto V^\perp=\{y_0\in\mathbb{F}_{q^3}:\mathrm{Tr}(x_0y_0)=0 \text{ for all } x_0\in V\}$$

is a bijection $P\to L$.

Let $x=\mathbb{F}_q^\times x_0\in P=\mathbb{F}_{q^3}^\times/\mathbb{F}_q^\times$. Write $\mathrm{Tr}(x)=0$ if $\mathrm{Tr}(x_0)=0$ and write $x^\perp$ for $(\mathbb{F}_qx_0)^\perp$. We identify any line, i.e., 2-dimensional subspace, with the set of its 1-dimensional subspaces. Under these identifications, the lines are the subsets $x^\perp = \{y \in P : \mathrm{Tr}(xy) = 0\}$ of the group P. For $x \in P$, let

$$\lambda_0(x) = \left(\frac{1}{x}\right)^\perp = \left\{y \in P : \mathrm{Tr}\left(\frac{y}{x}\right) = 0\right\}.$$

This defines a point-line correspondence $\lambda_0 : P \to L$. The following set of triples is a triangle presentation $\mathcal{T}_0$ compatible with λ_0:

$$\mathcal{T}_0 = \{(x, x\xi, x\xi^{q+1}) \ : \ x, \xi \in P \text{ and } \mathrm{Tr}(\xi) = 0\}. \tag{4}$$

The only nontrivial thing to check is that $\mathcal{T}_0$ is invariant under cyclic permutations. If $(x, x\xi, x\xi^{q+1}) \in \mathcal{T}_0$ then

$$(x\xi, x\xi^{q+1}, x) = (x\xi, (x\xi)\xi^q, (x\xi)(\xi^q)^{q+1}) \in \mathcal{T}_0$$

since $\mathrm{Tr}(\xi_0^q) = \mathrm{Tr}(\xi_0)$ and $\xi_0^{1+q+q^2} \in \mathbb{F}_q^\times$ for each $\xi_0 \in \mathbb{F}_{q^3}^\times$.

The group $\Gamma_\mathcal{T}$ associated with the presentation (4) is called a group of Tits type [4, I Section 4]. The presentation (4) is clearly invariant under multiplication by any $x \in P$. That is, it is S-invariant. It is also easy to see that (4) is invariant under the Frobenius collineation $x \mapsto x^q$.

Remark 3.11. Consider the related triangle presentation

$$\begin{aligned} \mathcal{T}_0' &= \left\{\left(\frac{1}{z}, \frac{1}{y}, \frac{1}{x}\right) \ : \ (x, y, z) \in \mathcal{T}_0\right\} \\ &= \{(x, x\xi, x\xi^{q^2+1}) \ : \ x, \xi \in P \text{ and } \mathrm{Tr}(\xi) = 0\} \end{aligned} \tag{5}$$

which is compatible with λ_0 and S-invariant. For $q = 2, 3$, the two presentations $\mathcal{T}_0, \mathcal{T}_0'$ are the only S-invariant triangle presentations [4, II,p.165,Remark].

The group $\Gamma_{\mathcal{T}_0'}$ is isomorphic to $\Gamma_{\mathcal{T}_0}$, via the map $x \mapsto \left(\frac{1}{x}\right)$, but the groups $A_{\mathcal{T}_0'}$, $A_{\mathcal{T}_0}$ are not isomorphic. In fact, writing $q = p^r$, p prime, the MAGMA computer algebra package calculates the following expressions for the groups $A_{\mathcal{T}_0}$, $A_{\mathcal{T}_0'}$ for $2 \le q \le 32$.

	$q \not\equiv 1 \pmod 3$	$q \equiv 1 \pmod 3$
$A_{\mathcal{T}_0}$	$\mathbb{Z}_{q-1}$	$\mathbb{Z}_{q-1} \oplus \mathbb{Z}_3$
$A_{\mathcal{T}_0'}$	$(\mathbb{Z}_p)^{3r} \oplus \mathbb{Z}_{q-1}$	$(\mathbb{Z}_p)^{3r} \oplus \mathbb{Z}_{q-1} \oplus \mathbb{Z}_3$

In all cases, the order of ε is precisely $\frac{q-1}{(q-1,3)}$.

Example 3.12. We verify the stated value of ε in the case $q = 4$. The set P is identified with $\mathbb{F}_{64}^\times/\mathbb{F}_4^\times$. Write $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$, where $\alpha^2+\alpha+1 = 0$ and $\mathbb{F}_{64} = \mathbb{F}_2(\alpha)(\delta)$, where $\delta^3 + \delta^2 + \alpha\delta + \alpha + 1 = 0$. Then $\delta^{21} = \alpha + 1$ and $\delta^{42} = \alpha$ are the roots of x^2+x+1. A short calculation shows that $(\delta^7)^4 = (\alpha+1)\delta^7$, so that the fixed points of the Frobenius collineation $x \mapsto x^4$ on $P = \mathbb{F}_{64}^\times/\mathbb{F}_4^\times$ are $\dot{1}, \omega, \omega^2$, where $\dot{1} = \mathbb{F}_4^\times$ and $\omega = \delta^7\mathbb{F}_4^\times$. It is easy to check that $\{\xi \in P : \mathrm{Tr}(\xi) = 0\} = \{\omega, \omega^2, s, s^4, s^{16}\}$, where $s = \delta^3\mathbb{F}_4^\times$. It follows from relation (3) in the group $A_{\mathcal{T}_0}$, with $x = \dot{1}$, that

$$\begin{aligned} \varepsilon &= \dot{1} + (\omega + \omega^2 + s + s^4 + s^{16}) \\ &= (\dot{1} + \omega + \omega^2) + (s + s^4 + s^{16}) \\ &= \varepsilon + (s + s^4 + s^{16}). \end{aligned}$$

In the last line, the fact that $\dot{1}+\omega+\omega^2=\varepsilon$ follows from (2b), since $(\dot{1},\omega,\omega^2)\in\mathcal{T}_0$ by (4). Therefore

$$\begin{aligned}\varepsilon+s^3&=\varepsilon+(s^3+s^4+s)+s^{16}\\&=\varepsilon+(s^3+s^4+s^8)+s^2\\&=\varepsilon+\varepsilon+s^2,\end{aligned}$$

using the facts that $s^8=\delta^{24}\mathbb{F}_4^\times=s$ and $(s^3,s^4,s^8)\in\mathcal{T}_0$. Hence $s^3=\varepsilon+s^2$. By invariance of the presentation $\mathcal{T}_0$ (Remark 3.8), we deduce that $s-\dot{1}=\varepsilon$ and therefore that $\varepsilon=s^{16}-\dot{1}=s^2-\dot{1}$. Thus $s^2=s$ and, by invariance, $s=\dot{1}$, which gives $\varepsilon=0$. It is also easy to see that $A_{\mathcal{T}_0}=\mathbb{Z}_3\oplus\mathbb{Z}_3$, with generators δ and δ^2.

3.3. **More examples.** Consider any triangle presentation $\mathcal{T}$. If ϕ is an order-3 collineation such that $\mathcal{T}$ is fixed by the map $x\mapsto\phi(x)$, then one obtains a new triangle presentation

$$\mathcal{T}^\phi=\{(x,\phi(y),\phi^2(z)):(x,y,z)\in\mathcal{T}\}\tag{6}$$

relative to the point–line correspondence $\phi\circ\lambda:P\to L$. The corresponding group

$$\Gamma_{\mathcal{T}^\phi}=\langle P\mid x\phi(y)\phi^2(z)=1\text{ for }(x,y,z)\in\mathcal{T}\rangle\tag{7}$$

is *not* in general isomorphic to $\Gamma_\mathcal{T}$ [4, II].

One possible choice for ϕ is the collineation $x\mapsto x^q$. Applied to the triangle presentations (4), ϕ and ϕ^2 give two more triangle presentations which are *not* in general S-invariant. If $q\equiv 1\pmod 3$ then another possible choice of ϕ is $x\mapsto\omega x$, where $\omega=\sigma^{(q^2+q+1)/3}$.

Corollary 3.13. *If the triangle presentation $\mathcal{T}$ is S-invariant and if ϕ is an order-3 collineation of (P,L) such that $\mathcal{T}$ is fixed by the map $x\mapsto\phi(x)$ then $(q-1)\varepsilon=0$ in $A_{\mathcal{T}^\phi}$.*

Proof. Let $\mathcal{M}=\{(x,\phi(y),\phi^2(z)):(x,y,z)\in\mathcal{M}_0\}$, where $\mathcal{M}_0$ is defined in the proof of Corollary 3.9. This satisfies the hypotheses of Corollary 3.6. □

4. The boundary action

Associated with the building Δ is the building at infinity Δ^∞. This is a spherical building of type A_2 [11, Theorem 8.24]. In the geometrical realization of Δ^∞, a point $\xi\in\Delta^\infty$ is an equivalence class of rays (subsets of Δ isometric to $[0,\infty)$), where two rays are equivalent if they are *parallel*, i.e. at finite Hausdorff distance from each other [1, 11.8]. For each vertex x of Δ and each $\xi\in\Delta^\infty$, there is a unique ray $[x,\xi)$ with initial vertex x in the parallelism class of ξ. A *sector* is a $\frac{\pi}{3}$-angled sector made up of chambers in some apartment. Each sector in Δ determines a 1-simplex (chamber) of Δ^∞ whose points are equivalence classes of rays in the sector. Two sectors determine the same chamber of Δ^∞ if and only if they contain a common subsector.

If $[x,\xi)$ is a sector wall then ξ is a *vertex* of Δ^∞. If the initial edge of this sector wall is $[x,y]$, and if $\tau(y)=\tau(x)+i$ then the vertex ξ is said to be of type $i-1$. This definition is independent of the base vertex x, since two parallel rays lie in a common apartment. Denote by $\mathcal{P}$ the set of vertices $\xi\in\Delta^\infty$ of type 0 and denote by $\mathcal{L}$ the set of vertices $\eta\in\Delta^\infty$ of type 1. Then $(\mathcal{P},\mathcal{L})$ is a projective plane. A point $\xi\in\mathcal{P}$ and a line $\eta\in\mathcal{L}$ are incident if and only if they are the two vertices of a common chamber in Δ^∞. This is equivalent to saying that there is a sector with base vertex x whose walls are the rays $[x,\xi)$ and $[x,\eta)$.

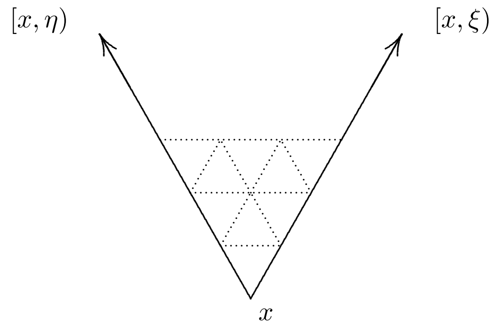

FIGURE 4. Sector walls

If $e \in E_+$, let $\Omega(e)$ denote the set of points $\xi \in \mathcal{P}$ which have representative rays with initial edge e. That is,

$$\Omega(e) = \{\xi \in \mathcal{P} : e \subset [o(e),\xi)\}\,.$$

The sets $\Omega(e)$, $e \in E_+$, form a basis of clopen sets for a totally disconnected compact topology on $\mathcal{P}$. The topological space $\mathcal{P}$ is called the *minimal boundary* of Δ.

Remark 4.1. If Δ is the Bruhat-Tits building of $\mathrm{PGL}_3(\mathbb{K})$ then there is a natural identification of $\mathcal{P}$ and $\mathbb{P}^2_{\mathbb{K}}$ as topological G-spaces. If $\lambda \in \mathbb{P}^2_{\mathbb{K}}$ and L_0 is a lattice, define a sequence of lattices inductively by $L_{i+1} = L_i + (\lambda \cap \pi^{-1}L_i)$. Then $L_i \subset L_{i+1} \subset \pi^{-1}L_i$ and $\tau([L_{i+1}]) = \tau([L_i]) + 1$, since L_i is maximal in L_{i+1}. The sequence of vertices $[L_0],[L_1],[L_2],\dots$ defines a ray whose parallelism class ξ_λ is an element of $\mathcal{P}$ and the map $\lambda \mapsto \xi_\lambda$ is a bijection from $\mathbb{P}^2_{\mathbb{K}}$ onto $\mathcal{P}$. If $e = ([L_0],[L_1])$, where $L_0 \subset L_1 \subset \pi^{-1}L_0$ and $\tau([L_1]) = \tau([L_0]) + 1$, then $\Omega(e)$ may be identified with the set of lines $\lambda \in \mathbb{P}^2_{\mathbb{K}}$ such that $L_1 = L_0 + (\lambda \cap \pi^{-1}L_0)$.

If v is a fixed vertex of Δ, then $\mathcal{P}$ may be expressed as a disjoint union

$$\mathcal{P} = \bigsqcup_{o(e)=v} \Omega(e)\,. \tag{8}$$

Also, if $e \in E_+$, then $\Omega(e)$ can be expressed as a disjoint union

$$\Omega(e) = \bigsqcup_{\substack{o(e')=t(e)\\ \Omega(e')\subset\Omega(e)}} \Omega(e')\,. \tag{9}$$

If Γ is an $\widetilde{A}_2$ group acting on Δ, then Γ acts on $\mathcal{P}$, and the abelian group $C(\mathcal{P},\mathbb{Z})$ has the structure of a Γ-module, with $(g\cdot f)(\xi) = f(g^{-1}\xi)$, $g \in \Gamma$, $\xi \in \mathcal{P}$. The module of Γ-coinvariants, $C(\mathcal{P},\mathbb{Z})_\Gamma$, is the quotient of $C(\mathcal{P},\mathbb{Z})$ by the submodule generated by $\{g\cdot f - f : g \in \Gamma, f \in C(\mathcal{P},\mathbb{Z})\}$. If $f \in C(\mathcal{P},\mathbb{Z})$ then $[f]$ denotes its class in $C(\mathcal{P},\mathbb{Z})_\Gamma$. Also, $\mathbf{1}$ denotes the constant function defined by $\mathbf{1}(\xi) = 1$ for $\xi \in \mathcal{P}$.

If $e \in E_+$, let χ_e be the characteristic function of $\Omega(e)$. For each $g \in \Gamma$, the functions χ_e and $g\cdot\chi_e = \chi_{ge}$ project to the same element in $C(\mathcal{P},\mathbb{Z})_\Gamma$. Therefore, for any edge $e = (a,ax)$ with $a \in \Gamma$ and $x \in P$, it makes sense to denote by $[x]$ the class of χ_e in $C(\mathcal{P},\mathbb{Z})_\Gamma$.

Suppose that $e, e' \in E_+$ with $o(e') = t(e) = v$, so that $t(e') = yv$ and $o(e) = x^{-1}v$ for (unique) $y, x \in P$. Then $\Omega(e') \subset \Omega(e)$ if and only if $y \notin \lambda(x)$. This is because $\Omega(e') \subset \Omega(e)$ if and only if the edges e and e' lie as shown in Figure 5, in some apartment.

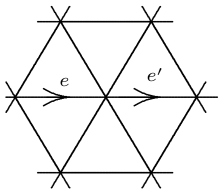

FIGURE 5

Equations (9) and (8) imply the following relations in $C(\mathcal{P}, \mathbb{Z})_\Gamma$.

$$\sum_{y \notin \lambda(x)} [y] = [x], \quad x \in P; \tag{10a}$$

$$\sum_{x \in P} [x] = [\mathbf{1}]. \tag{10b}$$

These should be compared with relations (2a),(2c), respectively. Now we seek an analogue of (2b). The following fact is the key.

Lemma 4.2. [9, Lemma 9.4] *Given any chamber c and any sector S in Δ, there exists a sector $S_1 \subset S$ such that S_1 and c lie in a common apartment.*

If c is a chamber of Δ and if $\xi \in \mathcal{P}$, then ξ has a representative ray that lies relative to c in one of the three positions in Figure 6, in some apartment containing them both. This is because, by Lemma 4.2, we can choose a ray $[x, \xi)$ such that c and $[x, \xi)$ lie in a common apartment. Now choose an appropriate ray parallel to $[x, \xi)$.

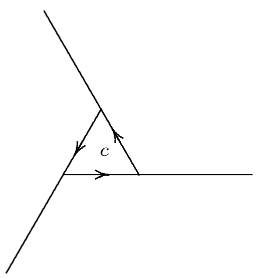

FIGURE 6. Relative positions of a chamber and representative rays

The next lemma follows immediately. Equation (11) is the desired analogue of (2b).

Lemma 4.3. *If $e_0, e_1, e_2 \in E_+$ are the edges of a chamber $c \in \Delta$ then*

$$\Omega(e_0) \sqcup \Omega(e_1) \sqcup \Omega(e_2) = \mathcal{P}.$$

Consequently, if $\mathcal{T}$ is a triangle presentation and $(x, y, z) \in \mathcal{T}$ then

$$[x] + [y] + [z] = [\mathbf{1}]. \tag{11}$$

Now we specify generators for $C(\mathcal{P},\mathbb{Z})_\Gamma$.

Lemma 4.4. *The group $C(\mathcal{P},\mathbb{Z})_\Gamma$ is finitely generated, with generating set $\{[x] : x \in \mathcal{P}\}$.*

Proof. Every clopen set V in $\mathcal{P}$ may be expressed as a finite disjoint union of sets of the form $\Omega(e)$, $e \in E_+$. Any function $f \in C(\mathcal{P},\mathbb{Z})$ is bounded, by compactness of $\mathcal{P}$, and so takes finitely many values $n_i \in \mathbb{Z}$. Therefore f may be expressed as a finite sum $f = \sum_j n_j \chi_{e_j}$, with $e_j \in E_+$. The result follows. □

Proposition 4.5. *There is a homomorphism θ from $A_{\mathcal{T}}$ onto $C(\mathcal{P},\mathbb{Z})_\Gamma$ defined by $\theta(x) = [x]$, for $x \in P$ and $\theta(\varepsilon) = [\mathbf{1}]$.*

Proof. Equations (10a),(10b),(11) show that θ preserves the relations (2a),(2c),(2b), respectively. Therefore θ extends to a homomorphism. Surjectivity is a consequence of Lemma 4.4. □

5. The main results

This section collects the main consequences.

Corollary 5.1. *Let Γ be an $\widetilde{A}_2$ group acting on an $\widetilde{A}_2$ building Δ of order q with minimal boundary $\mathcal{P}$. Then $C(\mathcal{P},\mathbb{Z})_\Gamma$ is a finite group and the class $[\mathbf{1}]$ in $C(\mathcal{P},\mathbb{Z})_\Gamma$ has order bounded by q^2-1. If Γ has an S-invariant triangle presentation then $(q-1)[\mathbf{1}] = 0$.*

Proof. This follows immediately from Proposition 3.2, Proposition 4.5, Lemma 3.1 and Corollary 3.9. □

The final statement of Corollary 5.1 applies, in particular, to the groups of Tits type (Section 3.2).

5.1. **K-theory.** The *Furstenberg boundary* Ω of Δ is the set of chambers of Δ^∞, endowed with a compact totally disconnected topology in which basic open sets have the form

$$\Omega(v) = \{\omega \in \Omega : [x,\omega) \ \text{ contains } v\}$$

where v is a vertex of Δ and $[x,\omega)$ is the representative sector for ω with base vertex x [3, Section 2]. If Δ is the Bruhat-Tits building of $G = \mathrm{PGL}_3(\mathbb{K})$ then Ω is isomorphic as a topological G-space to the space of maximal flags $(0) < V_1 < V_2 < \mathbb{K}^3$. The mapping which sends each sector to its wall of type 0 induces a natural surjection $\Omega \to \mathcal{P}$, under which $\mathcal{P}$ has the quotient topology. Since this surjection is equivariant, there is an induced epimorphism $C(\mathcal{P},\mathbb{Z})_\Gamma \to C(\Omega,\mathbb{Z})_\Gamma$.

The topological action of an $\widetilde{A}_2$ group Γ on the maximal boundary is encoded in the full crossed product C^*-algebra $\mathfrak{A}_\Gamma = C(\Omega) \rtimes \Gamma$, which is studied in [6, 7, 8]. The natural embedding $C(\Omega) \to \mathfrak{A}_\Gamma$ induces a homomorphism

$$\psi : C(\Omega,\mathbb{Z})_\Gamma \to K_0(\mathfrak{A}_\Gamma) \tag{12}$$

and $\psi([\mathbf{1}]) = [\mathbf{1}]_{K_0}$, the class of $\mathbf{1}$ in the K_0-group of $\mathfrak{A}_\Gamma$. The article [5] estimates the order of $[\mathbf{1}]_{K_0}$ for various boundary C^*-algebras, and contains an extensive bibliography.

In [8], T. Steger and the second author performed extensive computations which determined the order of $[\mathbf{1}]_{K_0}$ for many $\widetilde{A}_2$ groups with $q \le 13$. The computations were done for all the $\widetilde{A}_2$ groups in the cases $q = 2, 3$ and for several representative groups for each of the other values of $q \le 13$. If $q = 2$ there are precisely eight $\widetilde{A}_2$ groups Γ, all of which embed as lattices in a linear group $\mathrm{PGL}(3,\mathbb{K})$ where $\mathbb{K} = \mathbb{F}_2((X))$ or $\mathbb{K} = \mathbb{Q}_2$. If $q = 3$ there are 89 possible $\widetilde{A}_2$ groups, of which 65 do not embed naturally in linear groups. The experimental evidence suggested that for boundary crossed product algebras associated with $\widetilde{A}_2$ groups it is always true that $[\mathbf{1}]_{K_0}$ has order $q-1$ for $q \not\equiv 1 \pmod 3$ and has order $(q-1)/3$ for $q \equiv 1 \pmod 3$. It is striking that the order of $[\mathbf{1}]$ appears to depend only on the parameter q. It is shown in [8] that the order of $[\mathbf{1}]_{K_0}$ is bounded above by q^2-1 and below by $\frac{q-1}{(q-1,3)}$.

Corollary 5.2. *Let $\mathcal{T}$ be an S-invariant triangle presentation and $\Gamma = \Gamma_{\mathcal{T}}$. Then the class $[\mathbf{1}]_{K_0}$ in $K_0(\mathfrak{A}_\Gamma)$ has order bounded by $q-1$. If $q \not\equiv 1 \pmod 3$ then the order of $[\mathbf{1}]_{K_0}$ is precisely $q-1$.*

Proof. This follows directly from Corollary 5.1, since $\psi([\mathbf{1}]) = [\mathbf{1}]_{K_0}$. □

Theorem 1.1 is an immediate consequence of this Corollary.

Remark 5.3. It also follows from the computations in Remark 3.11 that the order of $[\mathbf{1}]_{K_0}$ is $\frac{q-1}{3}$, for all groups of Tits type with $q \equiv 1 \pmod 3$ and $q \le 31$.

Remark 5.4. For each $\widetilde{A}_2$ group Γ, the algebra $\mathfrak{A}_\Gamma$ has the structure of a higher rank Cuntz-Krieger algebra [7, theorem 7.7]. These algebras are classified up to isomorphism by their two K-groups, together with the class $[\mathbf{1}]_{K_0}$, [7, Remark 6.5]. It was proved in [8, Theorem 2.1] that

$$K_0(\mathfrak{A}_\Gamma) = K_1(\mathfrak{A}_\Gamma) = \mathbb{Z}^{2r} \oplus T, \tag{13}$$

where $r \ge 0$ and T is a finite group. The computations done in [8] give rise to some striking observations. For example, there are precisely three torsion-free $\widetilde{A}_2$ subgroups of $\mathrm{PGL}_3(\mathbb{Q}_2)$ and these three groups are distinguished from each other by $K_0(\mathfrak{A}_\Gamma)$.

There is also strong evidence that, for any torsion free $\widetilde{A}_2$ group Γ, the integer r in (13) is equal to the rank of $H_2(\Gamma,\mathbb{Z})$. A typical example is provided by the group $\Gamma = \Gamma_{\mathcal{T}_0}$, associated with the triangle presentation of Tits type defined in Section 3.2, with $q = 13$. In that case,

$$K_0(\mathcal{A}_\Gamma) = \mathbb{Z}^{1342} \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})^6 \oplus (\mathbb{Z}/13\mathbb{Z})^6.$$

As expected, the class $[\mathbf{1}]_{K_0}$ has order $4 = \frac{q-1}{3}$ and $H_2(\Gamma,\mathbb{Z}) = \mathbb{Z}^{671}$.

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