CENTRALIZERS IN $\widetilde{A}_2$ GROUPS

GUYAN ROBERTSON

Abstract. Let Γ be a torsion free discrete group acting cocompactly on a two dimensional euclidean building Δ. The centralizer of an element of Γ is either a Bieberbach group or is described by a finite graph of finite cyclic groups. Explicit examples are computed, with Δ of type $\widetilde{A}_2$.

1. Introduction

Let Γ be a torsion free discrete group which acts cocompactly on a 2-dimensional euclidean building Δ [Br, Ga, Ro]. For example, Γ may be any torsion free lattice in $\mathrm{PGL}_3(\mathbb{F})$, where $\mathbb{F}$ is a nonarchimedean local field. In that case Γ acts cocompactly on the Bruhat-Tits building Δ of $\mathrm{PGL}_3(\mathbb{F})$. The building Δ is considered as a geodesic metric space [Br, Chapter VI.3] and is a union of apartments which are isometric to the euclidean plane. Define the *translation length* of an element $g \in \Gamma$ to be $|g| = \inf_{x\in\Delta} d(g.x, x)$, and let

$$\mathrm{Min}(g) = \{x \in \Delta : d(g.x, x) = |g|\}.$$

An element $g \in \Gamma$ is *hyperbolic* if $|g| > 0$ and $\mathrm{Min}(g)$ is nonempty. Since Γ is torsion free, all nontrivial elements of Γ are hyperbolic and Γ acts freely on Δ. Each element $g \in \Gamma - \{1\}$ has at least one geodesic line ℓ in Δ (called an *axis* for g) upon which g acts by translation. The axes of g are parallel to each other and their union is the convex set $\mathrm{Min}(g)$ [BH, Theorem II.6.8]. An axis ℓ for g will always be *oriented* in the direction of translation by g, which will be called the positive direction. A nontrivial element $g \in \Gamma$ is said to be *regular* if no axis of g is parallel to a wall of Δ; otherwise g is said to be *irregular*. If g is regular then every axis of g is contained in a unique apartment, as illustrated in Figure 1.

The main result of this article concerns the structure of centralizers.

Theorem 1.1. *With the above hypotheses, let $g \in \Gamma - \{1\}$ and let $\mathrm{Z}_\Gamma(g)$ denote the centralizer of g in Γ.*

Date: January 29, 2011.
2000 *Mathematics Subject Classification.* Primary 20F67, 51E24.
Key words and phrases. Group, Centralizer, Euclidean Building.

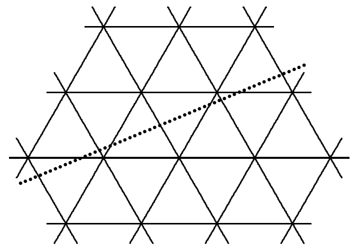


FIGURE 1. Axis of a regular element

(a) *If g has only one axis then* $\mathrm{Z}_\Gamma(g) \cong \mathbb{Z}$.
(b) *If g has more than one axis then one of the following is true.*
- *If g is regular then* $\mathrm{Z}_\Gamma(g)$ *is a Bieberbach group in two dimensions.*
- *If g is irregular then* $\mathrm{Z}_\Gamma(g)/\langle g\rangle$ *is the fundamental group of a finite graph of finite cyclic groups.*

The principal aim of this article is to show how to compute centralizers explicitly. This is done in Section 4 below in the $\widetilde{A}_2$ case. General results on the structure of centralizers in semihyperbolic groups are to be found in [AB, Section 7], [BH, III.4].

In the non-exotic case, where the building is associated to a linear algebraic group of which Γ is a subgroup, the current results can be deduced and interpreted using matrices. All locally finite euclidean buildings of dimension ≥ 3 are associated to linear algebraic groups, and, up to a point, this justifies the restriction to two-dimensional buildings in the present article.

2. Background

Since $\Gamma\backslash\Delta$ is compact, Δ is uniformly locally finite, in the sense that there exists $q \geq 2$ such that any edge of Δ lies on at most $q+1$ triangles. If Δ is of type $\widetilde{A}_2$ then every edge of Δ lies on exactly $q+1$ triangles. This follows from the facts that the link of a vertex in a building of type $\widetilde{A}_2$ is a generalized 3-gon [Ro, Chapter 3.2], and all vertices in a thick generalized 3-gon have the same valency [Ro, Proposition (3.3)]. The figures below relate to buildings of type $\widetilde{A}_2$, in which apartments are euclidean planes tessellated by equilateral triangles.

The main result is based on the observation that if $g, h \in \mathrm{Aut}(\Delta)$ commute, then h leaves $\mathrm{Min}(g)$ invariant and h maps each axis of g to an axis of g [BH, Theorem II.6.8].

Lemma 2.1. *If an element $g \in \Gamma - \{1\}$ has only one axis then* $\mathrm{Z}_\Gamma(g) \cong \mathbb{Z}$.

Proof. If g has only one axis ℓ then $\mathrm{Min}(g) = \ell$ and so $\mathrm{Z}_\Gamma(g)$ acts freely by translation on ℓ. Therefore $\mathrm{Z}_\Gamma(g) = \mathbb{Z}$. $\square$

This proves Theorem 1.1(a). It is now only necessary to consider elements with more than one axis. The next result proves the first part of Theorem 1.1(b).

Lemma 2.2. *Suppose that $g \in \Gamma - \{1\}$ is a regular element. Then $\mathrm{Z}_\Gamma(g)$ is a Bieberbach group, isomorphic to either $\mathbb{Z}^2$ or the fundamental group of a Klein bottle.*

Proof. Let ℓ be an axis for g. Then ℓ is not parallel to any wall and lies in a unique apartment $\mathcal{A}$ of Δ, which is g-invariant, since ℓ is. Any other axis ℓ' for g is parallel to ℓ and therefore lies in a common apartment with ℓ. However ℓ' also lies in a unique apartment, namely $\mathcal{A}$. Thus $\mathrm{Min}(g) \subseteq \mathcal{A}$. Now since g has more than one axis, g acts by translation (rather than glide reflection) on $\mathcal{A}$, and any geodesic in $\mathcal{A}$ which is parallel to ℓ is an axis for g. Thus $\mathrm{Min}(g) = \mathcal{A}$ and $\mathrm{Z}_\Gamma(g)$ leaves $\mathcal{A}$ invariant.

We claim that $\mathrm{Z}_\Gamma(g)$ acts cocompactly on $\mathcal{A}$. Choose a point ξ on ℓ and a geodesic $p \neq \ell$ in $\mathcal{A}$ passing through ξ. Let $D \subset \Delta$ be a compact fundamental domain for Γ containing ξ, and choose $\xi_n \in p$, $n = 1, 2, \dots$ such that $\mathrm{dist}(\xi_n, \xi) \geq 2n. \mathrm{diam}(D)$. Let ℓ_n be the geodesic in $\mathcal{A}$ passing through ξ_n and parallel to ℓ. Then ℓ_n is an axis for g, since g acts by translation on $\mathcal{A}$ in the direction of ℓ. Choose $g_n \in \Gamma$ such that $g_n\xi_n \in D$. The geodesic $g_n\ell_n$ passes through D and is an axis for $g_n g g_n^{-1}$. The displacement of $g_n\xi_n$ by $g_n g g_n^{-1}$ is equal to the displacement of ξ by g and each $g_n\xi_n$ lies in D. Since Γ is discrete, there are infinitely many pairs $m \neq n$ such that $g_m g g_m^{-1} = g_n g g_n^{-1}$. Choose such a pair. Then $h = g_m^{-1} g_n \in \mathrm{Z}_\Gamma(g)$ and so h leaves $\mathcal{A}$ invariant. By considering h^2 if necessary, we may assume that h is a translation rather than a glide reflection of $\mathcal{A}$ and h moves ξ away from D. Thus the subgroup of $\mathrm{Z}_\Gamma(g)$ generated by g and h acts cocompactly on $\mathcal{A}$. It follows that $\mathrm{Z}_\Gamma(g)$ is a Bieberbach group in two dimensions. $\square$

Definition 2.3. A *strip* in Δ is the convex hull of two parallel walls of Δ which contains no other wall of Δ (and so is of minimal width). Recall that a wall in Δ is a geodesic in the 1-skeleton of an apartment which is fixed pointwise by a reflection in the associated euclidean Coxeter group. If $g \in \Gamma - \{1\}$ then an *axial wall* for g is a wall of Δ which is also an axis for g.

Lemma 2.4. *Let $g \in \Gamma - \{1\}$ be an irregular element, with more than one axis. Then any axis ℓ for g is either an axial wall or lies in the interior of a unique strip bounded by two axial walls.*

Proof. Suppose that ℓ is an axis for g which is not a wall of Δ. Let Σ be the unique strip containing ℓ. Since g has more than one axis and $\mathrm{Min}(g)$ is convex, Σ must contain an axis ℓ' for g which is neither a wall nor the median line of Σ. Let ℓ_0 be the unique wall parallel to ℓ' at minimal distance δ to ℓ'. Since g is an isometry, it must map ℓ_0 to a parallel wall $g.\ell_0$, also at distance δ to ℓ'. By uniqueness, $\ell_0 = g.\ell_0$, and ℓ_0 is an axis of g. Finally, the edge ℓ_1 of Σ opposite to ℓ_0 is also an axis of g. □

The preceding argument shows that, if an irregular element $g \in \Gamma$ has only one axis ℓ, then ℓ is either a median line or a wall.

Definition 2.5. Suppose that the element $g \in \Gamma - \{1\}$ is irregular with more than one axis. Define a graph $\mathcal{T}_g$ as follows. A vertex ℓ of $\mathcal{T}_g$ is an axial wall of g. There is an edge $[\ell_1, \ell_2]$ between two such vertices if ℓ_1, ℓ_2 are the boundary walls of a common strip.

Lemma 2.6. *Under the above hypotheses, the graph $\mathcal{T}_g$ is a tree.*

Proof. If ℓ, ℓ' are axial walls of g, then their convex hull $\mathrm{conv}(\ell, \ell')$ is a convex subset of an apartment of Δ, which is the union of n contiguous parallel strips. This defines a path of length n in $\mathcal{T}_g$ from ℓ to ℓ'. The path is unique, since Δ is two dimensional and contractible [Ro, Appendix 4, page 185]. □

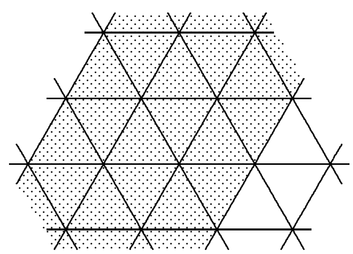


Figure 2. Part of $\mathrm{Min}(g)$ corresponding to path of length 3 in $\mathcal{T}_g$

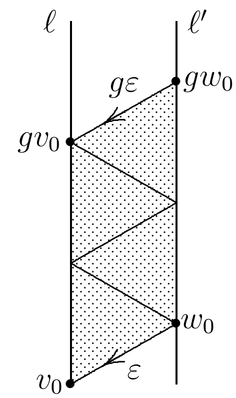
If ℓ is any wall of Δ then the class of all walls parallel to ℓ is a wall ℓ^∞ of the spherical building at infinity Δ^∞. There is an associated tree $T(\ell^\infty)$, whose vertices are the elements of ℓ^∞ [Ro, Chapter 10.2], [We, Chapter 10]. If ℓ is an axis of the element $g \in \Gamma$ then the tree $\mathcal{T}_g$ is a subtree of $T(\ell^\infty)$.

Lemma 2.7. *The group $G = \mathrm{Z}_\Gamma(g)/\langle g \rangle$ acts on the tree $\mathcal{T}_g$.*

Proof. Since each element of $\mathrm{Z}_\Gamma(g)$ maps axes of g to axes of g, the group $\mathrm{Z}_\Gamma(g)$ acts on $\mathcal{T}_g$. Also the cyclic group $\langle g \rangle$ fixes each vertex of $\mathcal{T}_g$, since each axis is g-invariant. □

Remark 2.8. The action of G on $\mathcal{T}_g$ may have inversion. In order to obtain an action without inversion, each geometric edge which is stabilized by a nontrivial element of G is divided into two by inserting a vertex ℓ^* at its midpoint. Then G acts without inversion of the resulting tree $\mathcal{T}_g^*$. The full barycentric subdivision of $\mathcal{T}_g$ would be a good alternative to $\mathcal{T}_g^*$, but we have chosen to minimize the number of additional vertices in order to simplify the graphs for the explicit examples below.

Remark 2.9. Any strip containing an axial wall ℓ is completely determined by ℓ and a single triangle in the strip which contains a fixed edge of ℓ. Since there are at most $q+1$ such triangles, each vertex of the tree $\mathcal{T}_g$ has valency at most $q+1$.



Remark 2.10. Recall that an axis ℓ for g is oriented in the direction of translation by g. Let ℓ, ℓ' be neighbouring axial walls for g and fix an edge $\varepsilon = [v_0, w_0]$ in Δ joining them. Then g translates v_0, w_0 the same distance along the parallel walls ℓ, ℓ'. The edge $g\varepsilon = [gv_0, gw_0]$, is the edge obtained by translating ε along the strip. This observation will be crucial in constructing the explicit examples in Section 4.

Recall that our axes are always *oriented.*

Lemma 2.11. *Two axes ℓ, ℓ' lie in the same $\mathrm{Z}_\Gamma(g)$-orbit if and only if they lie in the same Γ-orbit.*

Proof. Recall that g acts by translation by the same distance on each of the axes. In order to prove the nontrivial implication, let $\gamma \in \Gamma$ with $\gamma\ell = \ell'$. We claim that $\gamma \in \mathrm{Z}_\Gamma(g)$. If $v \in \ell$ then $\gamma v \in \ell'$ and $\gamma[v, gv] = [\gamma v, g\gamma v]$. Therefore $\gamma(gv) = g(\gamma v)$. Since Γ acts freely on Δ, $\gamma g = g\gamma$. □

Lemma 2.12. *Let $g \in \Gamma - \{1\}$. Then the set of $\mathrm{Z}_\Gamma(g)$-orbits of axial walls for g is finite.*

Proof. Since g acts on each axis by translation of distance $|g|$, the axis is determined by a segment of length $|g|$. Since Γ acts cocompactly, every segment of length $|g|$ can be translated by some element $h \in \Gamma$ to some segment belonging to a certain finite set. Each segment in the finite set generates an axial wall for some conjugate of g. So, further translating by one of a finite number of elements $k \in \Gamma$, we obtain one of a finite number of segments generating axial walls for g itself. According to Lemma 2.11, the elements hk lie in $Z_\Gamma(g)$. $\square$

Lemma 2.13. *Let $g \in \Gamma - \{1\}$ and let ℓ be an axis for g. Then the stabilizer $S(\ell)$ of ℓ in $\mathrm{Z}_\Gamma(g)$ is a cyclic group. If $S(\ell)$ is nontrivial then it is generated by an element of minimal positive translation length.*

Proof. Each element z of $S(\ell)$ acts upon the oriented axis ℓ by a translation of signed distance $\rho(z) = \pm|z|$. The map $z \mapsto \rho(z)$ is a homomorphism from $S(\ell)$ into $\mathbb{R}$, which is injective since Γ acts freely on Δ. The set $\{|x| : x \in \Gamma\}$ of translation lengths of elements of Γ is discrete. Therefore the map $z \mapsto \rho(z)$ is an isomorphism from $S(\ell)$ onto a discrete subgroup of $\mathbb{R}$. The result follows. $\square$

3. The graph of groups and proof of the main theorem

The results of the previous section show that $G = \mathrm{Z}_\Gamma(g)/\langle g\rangle$ acts without inversion on the tree $\mathcal{T} = \mathcal{T}_g^*$. Recall the construction of the corresponding graph of groups. Since G acts without inversion, we can choose an orientation on the edges which is invariant under G [Se, I.3.1]. This orientation consists of a partition of the set of edges and a bijective involution $e \mapsto \overline{e}$ which interchanges the two components. Each directed edge e has an initial vertex $o(e)$ and a terminal vertex $t(e)$ such that $o(\overline{e}) = t(e)$. The quotient graph $X = G\backslash\mathcal{T}$ has vertex set V and directed edge set E. There is an induced involution $e \mapsto \overline{e} : E \to E$ and there are maps $o, t : E \to V$ with $o(\overline{e}) = t(e)$.

Choose a maximal tree T in X, and a lifting $\sigma : T \to \mathcal{T}$ to a subtree of $\mathcal{T}$. For each $v \in V$, let G_v be the stabilizer of $\sigma(v)$ in G. If e is an edge of T, define $G_e = G_{\overline{e}}$ to be the stabilizer of $\sigma(e)$ in G and define $\varphi_e : G_e \to G_{t(e)}$ to be the inclusion map. For each $e \in E_+ - T$, define $\sigma(e)$ to be the directed edge of $\mathcal{T}$ lying over e and with $o(\sigma(e)) = \sigma(o(e))$, and define G_e to be the stabilizer of $\sigma(e)$. We take $\varphi_{\overline{e}}$ to be the inclusion map, but as $\sigma(\overline{e}) \neq \overline{\sigma(e)}$, we choose $\gamma_e \in G$ with $\gamma_e(\sigma(\overline{e})) = \overline{\sigma(e)}$. Then φ_e is defined as conjugation by γ_e.

Now (X, G) is a graph of groups in the sense of [Se, I.5]. By Bass-Serre theory [Se, I.5.4], G is isomorphic to the fundamental group $\pi_1(X, G)$, which is generated by the groups G_v and the elements γ_e subject to the following relations, for $e \in E$.

- $\gamma_{\overline{e}} = \gamma_e^{-1}$;
- $\gamma_e = 1, \quad e \in T$;
- $\gamma_e \varphi_e(x) \gamma_e^{-1} = \varphi_{\overline{e}}(x), \quad x \in G_e$.

The next result, which is an immediate consequence of the definitions, is useful in computing the fundamental group in the examples of the next section.

Lemma 3.1. *If φ_e is surjective then $\gamma_e G_{t(e)} \gamma_e^{-1} \subseteq G_{o(e)}$. In particular, if $e \in T$, then $G_{t(e)} \subseteq G_{o(e)}$.*

Proof of Theorem 1.1. This follows from the results of the previous section. The graph X has finitely many vertices by Lemma 2.12, and finitely many edges by Remark 2.9. The stabilizer in G of any vertex of $\mathcal{T}$ is a finite cyclic group by Lemma 2.13. The same is true of edge stabilizers, since they are subgroups of vertex stabilizers. □

4. Examples in an $\widetilde{A}_2$ group

A class of groups which act regularly on the vertices of a building of type $\widetilde{A}_2$ has been studied in [CMSZ]. Consider the group C.1 of [CMSZ]. This group was originally defined in [Mu]: see also [KO, Theorem 2.2.1], [GM]. It is a torsion free lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$, with seven generators $x_0, x_1, \ldots, x_6$ and seven relators

$$x_0x_0x_6 \quad x_0x_2x_3 \quad x_1x_2x_6 \quad x_1x_3x_5 \quad x_1x_5x_4 \quad x_2x_4x_5 \quad x_3x_4x_6$$

The 1-skeleton of the Bruhat-Tits building Δ of $\mathrm{PGL}_3(\mathbb{Q}_2)$ is the Cayley graph of Γ relative to this set of generators. Vertices in Δ are identified with elements of Γ and the directed edge $[a, ax]$, where $a \in \Gamma$, is labelled by the generator x. To facilitate computations, it is convenient to tabulate, for each i, the values of j for which there is a relation of the form $x_ix_jx_k = 1$.

i	0	1	2	3	4	5	6
j	0, 2, 6	2, 3, 5	3, 4, 6	0, 4, 5	1, 5, 6	1, 2, 4	0, 1, 3

4.1. **Example: the tree $\mathcal{T}_g^*$ for $g = x_0x_5$.** The element g has an axis ℓ passing through $1, g, g^2, \ldots$, and g acts on ℓ by translation of distance $|g| = 2$. We seek all possible axial walls for g which are neighbours of ℓ. By Remark 2.10, any strip containing ℓ and another axial wall must have periodic labelling of period 2. The entire strip is completely determined by ℓ and a single triangle containing a fixed edge of ℓ. Therefore there are at most three such strips. In fact, there are exactly three, corresponding to three edges in $\mathcal{T}_g$ joining ℓ to vertices ℓ_1, ℓ_2, ℓ_3. The three cases are illustrated on the left of Figure 3, where the edge label j stands for the generator x_j. The stabilizer in $\mathrm{Z}_\Gamma(g)$ of each strip is the subgroup $\langle g\rangle$, and this acts upon the strip by translation. In each case, a fundamental domain for this action is shaded.

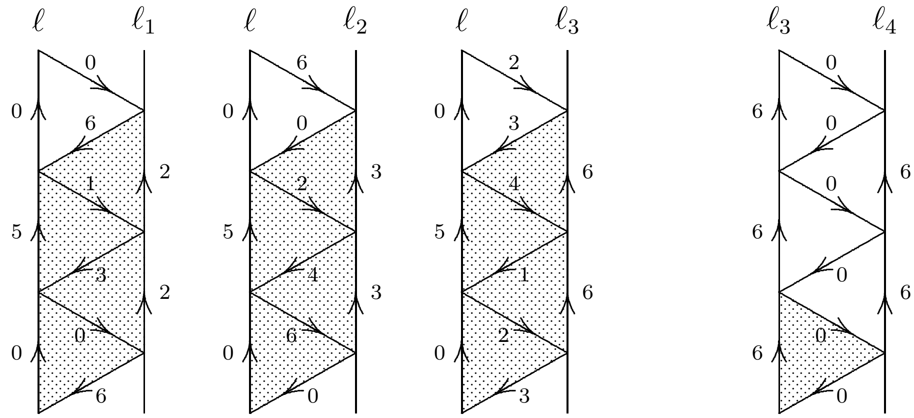


Figure 3

Continuing to extend the tree, it turns out that there is only one further edge emanating from ℓ_3 to a vertex ℓ_4 as illustrated on the right of Figure 3. As indicated by their edge labels, ℓ_3, ℓ_4 are in the same Γ-orbit, hence in the same $Z_\Gamma(g)$-orbit, by Lemma 2.11. The element $x_3^{-1}x_0x_3$ acts with inversion on the edge $[\ell_3,\ell_4]$ of $\mathcal{T}_g$, since it acts by glide reflection on the corresponding strip, in which a fundamental domain is shaded. We therefore divide $[\ell_3,\ell_4]$ in two by inserting a vertex ℓ_4^* at its midpoint, corresponding to the median line of the strip. Since $x_3^{-1}x_0x_3$ acts by translation on this median line, it stabilizes ℓ_4^*. In fact $S(\ell_4^*)=\langle x_3^{-1}x_0x_3\rangle\cong\mathbb{Z}$.

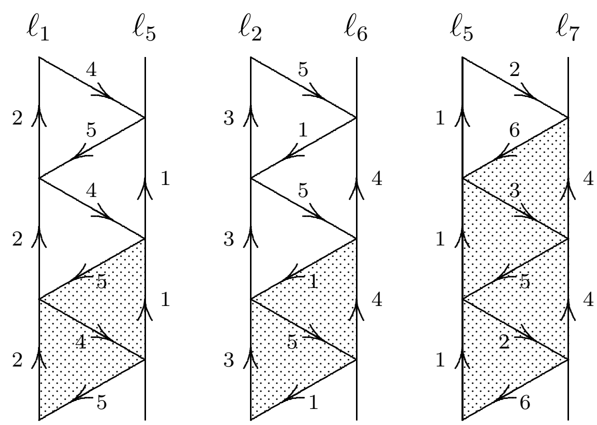


Figure 4

Continue to build the tree $\mathcal{T}_g^*$, adding vertices ℓ_5, ℓ_6, ℓ_7 (as in Figure 4), noting that ℓ_6, ℓ_7 are in the same $Z_\Gamma(g)$-orbit, by Lemma 2.11.

The action of $G = Z_\Gamma(g)/\langle g\rangle$ on $\mathcal{T}_g^*$ is described by the quotient graph of groups in Figure 5. The vertices of the quotient graph $G\backslash\mathcal{T}_g^*$ are labelled according to their inverse images in $\mathcal{T}_g^*$. For example the stabilizer of the vertex ℓ_1 is generated by an element h conjugate to x_2: in fact $h = x_6^{-1}x_2x_6$. The same applies to all vertices in the same Γ orbit as ℓ_1; that is all axes of g with the periodic edge labelling $(\dots 2, 2, 2 \dots)$. Thus the vertex $G.\ell_1$ of $G\backslash\mathcal{T}_g^*$ has label (2). Similarly, the label [0] is attached to the vertex $G.\ell_4^*$. A square bracket is used to indicate that ℓ_4^* is a median line rather than a wall. The vertex and edge groups are shown in Figure 5, except where they are trivial. For example, attached to the vertex (2) is the group $S(\ell_1)/\langle g\rangle = \langle x_6^{-1}x_2x_6\rangle/\langle g\rangle \cong \mathbb{Z}/2\mathbb{Z}$.

Computing the fundamental group of this graph of groups as described in Section 3, using Lemma 3.1, we see that

$$Z_\Gamma(g)/\langle g\rangle \cong \mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^{*2} * (\mathbb{Z}/4\mathbb{Z}).$$

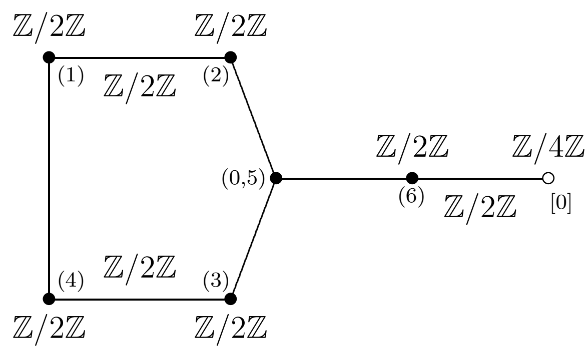


FIGURE 5. Graph of groups with fundamental group $\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z})^{*2} * (\mathbb{Z}/4\mathbb{Z})$

4.2. **Example: the graph of groups for** $g = x_0x_1x_4$**.** In this case the corresponding graph of groups is illustrated in Figure 6. This may be verified as before, and the notation used for the vertices is the same. For example, the vertex labelled $[2, 3, 5]$ is the image of a vertex of $\mathcal{T}_g^*$ which is stabilized by an element conjugate to $x_2x_3x_5$. Computing the fundamental group of this graph of groups gives

$$Z_\Gamma(g)/\langle g\rangle \cong \mathbb{Z}^{*2} * (\mathbb{Z}/2\mathbb{Z})^{*5}.$$

Other groups for which such computations are possible are those which act regularly on the vertices of a building of type $\widetilde{A}_1 \times \widetilde{A}_1$, that is a product of trees. Explicit presentations of such groups are given in [KR]. The groups studied in [RR] have the property that the centralizer of any non-trivial element is either $\mathbb{Z}$ or $\mathbb{Z}^2$ (with both possibilities occurring).

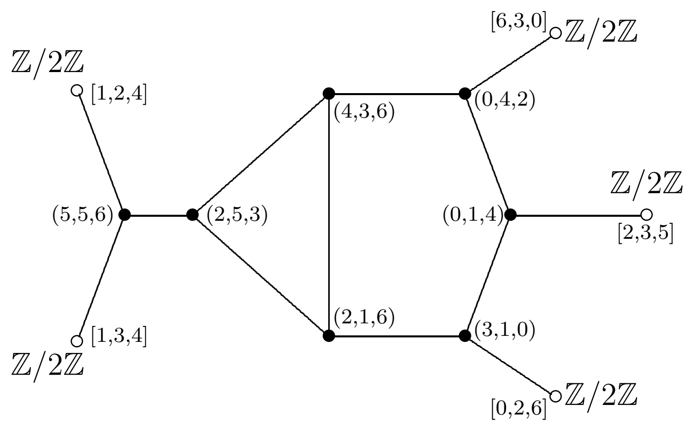


FIGURE 6. Graph of groups with fundamental group $\mathbb{Z}^{*2} * (\mathbb{Z}/2\mathbb{Z})^{*5}$

REFERENCES

[AB] J. M. Alonso and M. R. Bridson, Semihyperbolic groups. *Proc. London Math. Soc.* **70** (1995), 56–114.

[BH] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.

[Br] K. Brown, *Buildings*, Springer-Verlag, New York, 1989.

[CMSZ] D. I. Cartwright, A. M. Mantero, T. Steger and A. Zappa, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, I and II, *Geom. Ded.* **47** (1993), 143–166 and 167–223.

[Ga] P. Garrett, *Buildings and Classical Groups*, Chapman & Hall, London, 1997.

[GM] Y. Glasner and S. Mozes, Automata and square complexes, *Geom. Dedicata* **111** (2005), 43–64.

[KO] F. Kato and H. Ochiai, Arithmetic structure of CMSZ fake projective planes, *J. Algebra* **305** (2006), 1166–1185.

[KR] J. Kimberley and G. Robertson, Groups acting on products of trees, tiling systems and analytic K-theory, *New York J. Math.* **8** (2002), 111–131.

[Mu] D. Mumford, An algebraic surface with K ample, $(K^2) = 9$, $p_g = q = 0$, *Amer. J. Math.* **101** (1979), 233–244.

[Ro] M. Ronan, *Lectures on Buildings*, Perspectives in Mathematics, Vol. 7, Academic Press, New York, 1989.

[RR] D. Rattaggi and G. Robertson, Abelian subgroup structure of square complex groups and arithmetic of quaternions, *J. Algebra*, **286** (2005), 57–68.

[Se] J.-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980.

[We] R. M. Weiss, *The Structure of Affine Buildings*, Princeton, New Jersey, 2009.

SCHOOL OF MATHEMATICS AND STATISTICS, NEWCASTLE UNIVERSITY, NEWCASTLE UPON TYNE NE1 7RU, ENGLAND, UK.

E-mail address: `a.g.robertson@ncl.ac.uk`