23 Functions: Domains and Ranges

The domain of any given function is the set of 'input values for which the function is defined, and understanding this is the basis for this section of the course. By the end of this section, you should have the following skills:

- An understanding of the definition of a function and domain.
- Find a function and its domain based on the equation of a curve.
- Define the range of a given function.

23.1 Functions

This section defines and gives examples of domains and ranges of functions. These are important properties of a function and we will meet them in subsequent sections. We write a function using the notation \( f(x) \).

The notation means that given a number \( x \) then the function gives another unique number \( f(x) \).

If we write \( y = f(x) \) then we say that \( x \) is the independent variable and \( y \) the dependent variable. Note that a function can be written as \( f(x) = x^3 \) or as \( h(t) = t^3 \) or as \( c(y) = y^3 \). These are all the same function - they all do the same thing, cube a number.

23.1.1 Examples

The following are all functions:

1. \( f(x) = x \), called the identity function.
2. \( t(u) = 2u \).
3. \( f(x) = x^2 \).
4. \( h(z) = 1/z \).

Note that all of these functions are defined for all values of their independent variable apart from the last example as we know that if \( z = 0 \) then \( h(0) \) is not defined. So it is important to say for what values of the independent variable the function exists.
23.2 Domain of a function

The set of values for which a function is defined is called the natural domain (usually shortened to domain).

1. \( f(x) = x^2 \). Defined for all values of \( x \) i.e. the domain is \( \mathbb{R} \).

2. \( f(t) = 2^t \). Defined for all values of \( t \) i.e. the domain is \( \mathbb{R} \).

3. \( f(z) = \sqrt{z} \).
   Defined for \( z \geq 0 \) i.e. the domain is \([0, \infty)\) or \( \mathbb{R}_+ \).

4. \( f(x) = \sqrt{4 + 3x - x^2} \).
   We need \( 4 + 3x - x^2 \geq 0 \) and solving this inequality we see that this is only true for \(-1 \leq x \leq 4\).
   So this function is only defined in this range of values for \( x \) i.e. the domain is \([-1, 4]\).
5. $f(x) = \sin(x)$. Defined for all values so the domain is $\mathbb{R}$.

6. $f(u) = 1/u$. The domain is defined for all $u \neq 0$ and is denoted by $\mathbb{R} - \{0\}$.

7. $f(x) = 1/(x^3 - 3x^2 + 2x)$. 
   Since $x^3 - 3x^2 + 2x = x(x - 1)(x - 2)$ we see that this function is defined for all $x, \ x \neq 0, \ x \neq 1, \ x \neq 2$. This domain is written as $\mathbb{R} - \{0, 1, 2\}$.
8. \( f(x) = \ln(x) \).
   This function is the natural logarithm and is defined for all \( x > 0 \) i.e.
   the domain is \((0, \infty)\).

9. \( f(v) = \sqrt{1/(1 - v)} \).
   Defined for all \( v \) such that \( 1/(1 - v) \geq 0 \) i.e. for all \( v < 1 \) i.e. the
The domain is \((-\infty, 1)\).

Graph of \(\sqrt{1/(1-v)}\).

10. \(f(x) = 1/\sin(x)\).
    Defined for all \(x \neq n\pi, \ n = 0, \pm 1, \pm 2, \ldots\).

Graph of \(1/\sin(x)\).
11. \( f(x) = \ln(\sin(x)) \).
   This function is defined for \( x \) such that \( \sin(x) > 0 \) i.e. for all \( x \) such
   that \( 2n\pi < x < (2n + 1)\pi, \quad n = 0, \pm 1, \pm 2, \ldots \).

23.3 Examples

All of the above examples are functions which give unique values. However, consider the following examples.

Example 1 Find a function \( y = f(x) \) such that \( y^2 = x^2 \).
Graph of $y^2 = x^2$.

**Solution.**

It is clear that $y = \pm x$. However, we see that given $x$, $y$ has two possible values $\pm x$ and so there cannot be such a function. We could insist that the function $y$ satisfies $y \geq 0$. In this case there is a unique function $y = |x|$, the absolute value.
Example 2  Find a function $y = f(x)$ such that $y^2 = x$.

Solution.

Note that we must have $x \geq 0$. But once again we have two values for $y$ i.e. $y = \pm \sqrt{x}$. So there is no function which gives both values. We get round this by insisting that we take the positive square root and
denote this function by \( y = \sqrt{x} \). The negative square root function is denoted by \( y = -\sqrt{x} \). Once again we have to insist that \( x \geq 0 \) in both functions.

Example 3 Find a function \( y = f(x) \) such that \( \sin(y) = x \).

Solution.

In order to find \( y \) given \( x \) we have to solve the equation \( \sin(y) = x \). Note that we must have \(-1 \leq x \leq 1\). But there are an infinite number of such possible values for \( y \), as given one such solution \( y \) then \( y + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots \) are also solutions. Hence once again we have a problem as there is not a unique value for \( y \). But we can progress if we deliberately choose the unique value for \( y \) which lies between \(-\pi/2\) and \( \pi/2 \). This function we have now defined is called

\[
y = \arcsin(x)
\]

where \(-1 \leq x \leq 1\) and \(-\pi/2 \leq y \leq \pi/2\).
Exercise 1

For each of the following functions find its domain i.e. the set of points where each function is defined. Look at the examples above to see some similar functions.

(a) \( f(x) = x^3 - x \).
(b) \( f(t) = 3^t \).
(c) \( f(z) = \sqrt{z - 1} \).
(d) \( f(z) = \sqrt{z^2 - 1} \).
(e) \( f(x) = \sqrt{6 - x - x^2} \).
(f) \( g(x) = \sin(1/x) \).
(g) \( h(u) = 1/(u - 2) \).
(h) \( f(x) = 1/(x^3 - 3x^2 - 6x - 8) \).
(i) \[ f(x) = \frac{1}{(x^2 + 1)}. \]

(j) \[ f(x) = \ln(x + 3). \]

(k) \[ k(v) = \sqrt{\frac{1}{1 - v^3}}. \]

(l) \[ f(x) = \frac{1}{\cos(x)}. \]

(m) \[ m(x) = \ln(\cos(x)). \]

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**Solutions to exercise 1**

(a) \( f(x) = x^3 - x \) is defined for all values of \( x \) and so the domain is \( \mathbb{R} \).

(b) \( f(t) = 3^t \) is defined for all values of \( t \) and so the domain is \( \mathbb{R} \).

(c) \( f(z) = \sqrt{z - 1} \) is defined for all \( z \) such that \( z - 1 \geq 0 \) \( \Rightarrow \) \( z \geq 1 \) and the domain is the interval \([1, \infty)\).

(d) \( f(z) = \sqrt{z^2 - 1} \) is defined for all \( z \) such that:
\[
z^2 - 1 \geq 0 \Rightarrow z^2 \geq 1 \Rightarrow z \geq 1 \text{ or } z \leq -1\]
and the domain comprises the intervals \([1, \infty)\) and \((-\infty, -1]\).

(e) \( f(x) = \sqrt{6 - x - x^2} \) is defined for all \( x \) such that
\[
6 - x - x^2 \geq 0 \Leftrightarrow x^2 + x - 6 \leq 0 \Rightarrow (x + 3)(x - 2) \leq 0 \Rightarrow -3 \leq x \leq 2.
\]
Hence the domain is the interval \([-3, 2]\).

(f) \( g(x) = \sin(1/x) \) is defined for all \( x \) except for \( x = 0 \). Hence the domain is \( \mathbb{R} - \{0\} \).

(g) \( h(u) = 1/(u - 2) \) is defined for all \( u \) except for \( u = 2 \). The domain is \( \mathbb{R} - \{2\} \).

(h) \( f(x) = 1/(x^3 - 3x^2 - 6x - 8) \) is defined for all \( x \) such that \( x^3 - 3x^2 - 6x - 8 \neq 0 \).
So now we look for the values of $x$ such that $x^3 - 3x^2 - 6x - 8 = 0$.

\[
x^3 - 3x^2 - 6x + 8 = 0 \Rightarrow (x - 1)(x^2 - 2x - 8) = 0 \Rightarrow (x - 1)(x - 4)(x + 2) = 0
\]


\[
x = 1 \text{ or } x = 4 \text{ or } x = -2.
\]

The domain is then all $x$ except for $x = -2, \ x = 1, \ x = 4$ and is denoted by $\mathbb{R} - \{-2, 1, 4\}$.

(i) $f(x) = 1/(x^2 + 1)$ is defined for all values of $x$ as $x^2 + 1 > 0$ for all $x$, hence the domain is $\mathbb{R}$.

(j) $f(x) = \ln(x + 3)$ is defined for all $x$ such that $x + 3 > 0 \Rightarrow x > -3$, so the domain is $(-3, \infty)$.

(k) $k(v) = \sqrt{1/(1 - v^3)}$ is defined for all $v$ such that:

\[
\begin{align*}
1/(1 - v^3) &\geq 0 \Leftrightarrow 1 - v^3 > 0 \Leftrightarrow v^3 < 1 \Leftrightarrow v < 1.
\end{align*}
\]

Hence the domain is $(-\infty, 1)$.

(l) $f(x) = 1/\cos(x)$ is defined for all $x$ such that $\cos(x) \neq 0$.
Now $\cos(x) = 0$ when $x = n\pi + \pi/2, \ n = 0, \pm 1, \pm 2$.
Hence the domain is $\mathbb{R} - \{\ldots, -5\pi/2, -3\pi/2, \pi/2, 3\pi/2, 5\pi/2, \ldots\}$.

(m) $m(x) = \ln(\cos(x))$ is defined for all $x$ such that $\cos(x) > 0$.
Looking at the graph of $\cos(x)$ we see that $\cos(x) > 0$ when $-\pi/2 < x < \pi/2$ i.e. $(-\pi/2, \pi/2)$.
Since $\cos(x)$ is periodic of period $2\pi$ we see that the domain comprises all of the intervals of the form $((4n - 1)\pi/2, (4n + 1)\pi/2), \ n = 0, \pm 1, \pm 2 \ldots$.
23.4 Range of a Function

We have already discussed the domain of a function $f(x)$ i.e. the values of $x$ for which $f(x)$ is defined. Next we consider the values $f(x)$ we get as $x$ varies over the domain. This is, not surprisingly, called the range of $f(x)$.

**Example 4** Find the ranges of the following functions.

(a) $f(x) = x$. The domain is $\mathbb{R}$ i.e. all numbers and the range is also $\mathbb{R}$.

(b) $f(x) = x^2$. The domain is once again $\mathbb{R}$, but the range is all positive numbers as $x^2 \geq 0$ i.e. $[0, \infty]$.

(c) $g(x) = \sin(x)$. The domain is $\mathbb{R}$, but the range is given by $[-1, 1]$ as $-1 \leq \sin(x) \leq 1$.

(d) $h(t) = \sqrt{t}$. Remember that this is the positive square root. The domain is $[0, \infty]$ as is the range.

(e) $f(x) = x^2 + 2x + 3$. The domain is $\mathbb{R}$ as $f(x)$ is defined for all values of $x$. We now examine the range i.e. what values do we get from $x^2 + 2x + 3$ as $x$ varies over all the domain i.e. over all values? For quadratics like this it is a good idea to complete the square:

$$f(x) = x^2 + 2x + 3 = (x + 1)^2 + 2.$$  

We see that $f(x)$ has least value 2 when $x = -1$ and that it takes all values $\geq 2$. (Look at the graph). Hence the range is $[2, \infty)$.

(f) $f(x) = 3 - x - x^2$. Once again the domain is $\mathbb{R}$ and we use completing
the square to find the range.

\[ f(x) = -(x^2 + x - 3) = -((x + 1/2)^2 - 1/4 - 3) = -((x + 1/2)^2 - 13/4) = -(x + 1/2)^2 + 13/4. \]

From this we can read off the fact that the maximum value is 13/4 and this occurs at \( x = -1/2 \).
Hence we have that the range is \((-\infty, 13/4]\).

(g) \( h(t) = 1/t \). The domain is \(-\{0\}\) as we do not allow \( t = 0 \). Now as \( t \) varies we can get any value from \( 1/t \) we like as long as it is non-zero.
Let \( h \neq 0 \) be any value then we have to find \( t \) such that \( 1/t = h \). But clearly \( t = 1/h \) will do.
So the range is \( \mathbb{R} \setminus \{0\} \).

(h) \( f(x) = 1/(x - 1) \). The domain is \( \mathbb{R} \setminus \{1\} \) as we cannot let \( x = 1 \) (why?). So as \( x \) takes values in the domain what values do we get for \( f(x) = 1/(x - 1) \)? Let \( f \) be a value in the range. There has to be an \( x \neq 1 \) such that:

\[ f = 1/(x - 1) \Rightarrow \]

\[ x - 1 = 1/f \Rightarrow \]

\[ x = 1/f + 1. \]

So as long as \( f \neq 0 \) we can find an \( x \). Hence the range is all \( f \neq 0 \) i.e.
\[ \mathbb{R} \setminus \{0\} . \]

(i) \( f(x) = x^2/(1 + x^2) \). The domain is \( \mathbb{R} \) as although we are dividing by \( x^2 + 1 \) it is never 0 as \( x^2 + 1 \geq 1 \) for all \( x \).
As for the range, we note that

(a) \( \frac{x^2}{x^2 + 1} \geq 0. \)
(b) \( x^2 < x^2 + 1 \Rightarrow \frac{x^2}{x^2 + 1} < 1. \)

Hence we have

\[ 0 \leq f(x) < 1. \]

In fact the range is all \( f \) such that \( 0 \leq f < 1 \) and we can check this as follows.

We show that any value \( f \neq 1 \) which lies between 0 and 1 is in the range.

Let \( f \) be any such value, we want to find \( x \) such that \( f = f(x) \) i.e. such that

\[
\begin{align*}
    f &= \frac{x^2}{x^2 + 1} \quad \Rightarrow \\
    f(x^2 + 1) &= x^2 \quad \Rightarrow \\
    x^2(f - 1) &= -f \quad \Rightarrow \\
    x^2 &= f/(1 - f) \quad \Rightarrow \\
    x &= \pm \sqrt{f/(1 - f)}.
\end{align*}
\]

So there is a value of \( x \), in fact there are two, such that \( f = f(x) \).
Note that \( f/(1 - f) \) is \( \geq 0 \) as \( 0 \leq f < 1 \) so we can take the square root.
Graph of \( y = x^2/(1 + x^2) \).

Hence all values between 0 and 1, but not including 1 are in the range i.e. \([0, 1)\).

(j) \( h(t) = t^4 + 2t^2 \). Domain is all \( t \) i.e. \( \mathbb{R} \). In order to obtain the range note that \( h(t) = t^4 + 2t^2 = (t^2 + 1)^2 - 1 \geq 0 \).
We now show that the range is all \( h \) such that \( h \geq 0 \).
Let \( h \geq 0 \) then

\[
(t^2 + 1)^2 - 1 = h \Rightarrow \\
t^2 + 1 = \sqrt{1 + h} \Rightarrow \\
t^2 = \sqrt{1 + h} - 1.
\]

As \( h \geq 0 \) we see that \( \sqrt{1 + h} - 1 \geq 0 \) and we can take the square root so that \( t = \pm \sqrt{1 + h} - 1 \) is such that \( h(t) = h \). Hence the range is all \( h \) such that \( h \geq 0 \).

(k) \( f(x) = \sqrt{x^2 + 2x - 3} \). This is only defined if \( x^2 + 2x - 3 \geq 0 \).
But \( x^2 + 2x - 3 = (x + 3)(x - 1) \) hence

\[
x^2 + 2x - 3 \geq 0 \Rightarrow x \leq -3 \text{ or } x \geq 1.
\]
So the domain is \( x \leq -3 \text{ or } x \geq 1 \) i.e. \( (-\infty, -3] \text{ or } [1, \infty) \).
If we complete the square for the quadratic \( x^2 + 2x - 3 \) we have \( x^2 + 2x - 3 = (x + 1)^2 - 4 \geq -4 \) and we see that \( x^2 + 2x - 3 \) can take all values \( \geq -4 \).
Hence in particular it can take all values \( \geq 0 \) and so \( f(x) = \sqrt{x^2 + 2x - 3} \) can take all values \( \geq 0 \). So the range is \([0, \infty)\) for \( f(x) \).
Exercise 2

Find the ranges of the following functions:

1. \( f(x) = 3 - 2x. \)
2. \( f(x) = 3x^2 - 2. \)
3. \( g(x) = \sin(x) + \cos(x). \)
4. \( h(t) = \sqrt{t^2 + 1}. \)
5. \( f(x) = x^2 + 2x - 15. \)
6. \( f(x) = 5 - 3x - 2x^2. \)
7. \( h(t) = 1/(2t + 7). \)
8. \( f(x) = (x - 1)/(x + 1). \)
9. \( f(x) = (1 + x^2)/(2 + 3x^2). \)
10. \( h(t) = 2t^4 + 3t^2 - 5 \geq -5. \)

Solutions to exercise 2
(a) \( f(x) = 3 - 2x \). The domain is \( \mathbb{R} \).

To find the range, let \( f \) be a value in the range then

\[
3 - 2x = f \Rightarrow x = (3 - f)/2.
\]

This shows than no matter what value \( f \) we choose we can find \( x \) such that \( f(x) = f \), hence the range is also \( \mathbb{R} \).

(b) \( f(x) = 3x^2 - 2 \). The domain is once again \( \mathbb{R} \), but the range is all \( f \geq -2 \) as given \( f \geq -2 \) then \( f = 3x^2 - 2 \Rightarrow x = \sqrt{f + 2}/3 \) gives \( f(x) = f \) i.e. the range is \([-2, \infty)\).

(c) \( g(x) = \sin(x) + \cos(x) \). The domain is \( \mathbb{R} \), but the range is given by the following: \( g(x) = \sin(x) + \cos(x) = \sqrt{2} \sin(x + \pi/4) \).

Hence the range is \([-\sqrt{2}, \sqrt{2}]\) as \(-1 \leq \sin(x + \pi/4) \leq 1\).

(d) \( h(t) = \sqrt{t^2 + 1} \). Remember that this is the positive square root.

The domain is \( \mathbb{R} \) as \( t^2 + 1 > 0 \) for all \( t \).

To find the range, let \( h = h(t) \geq 0 \).

\[
\begin{align*}
h &= \sqrt{t^2 + 1} \Rightarrow \quad \quad \quad h^2 &= t^2 + 1 \Rightarrow \quad \quad \quad t &= \sqrt{h^2 - 1}.
\end{align*}
\]

So we see that if \( h \geq 1 \) then we can find \( t \) such that \( h(t) = h \) and so the range is \([1, \infty)\).

(e) \( f(x) = x^2 + 2x - 15 \). The domain is \( \mathbb{R} \) as \( f(x) \) is defined for all values of \( x \). We now examine the range i.e. what values do we get from \( x^2 + 2x - 15 \) as \( x \) varies over all the domain i.e. over all values.

For quadratics like this it is a good idea to complete the square:

\[
f(x) = x^2 + 2x - 15 = (x + 1)^2 - 16.
\]

We see that \( f(x) \) has least value \(-16\) when \( x = -1 \) and that it takes all values \( \geq -16 \).

Hence the range is \([-16, \infty)\).
(f) \( f(x) = 5 - 3x - 2x^2 \). Once again the domain is \( \mathbb{R} \) and we use completing the square to find the range.

\[
\begin{align*}
f(x) &= -(2x^2 + 3x - 5) \\
&= -(2(x + 3/4)^2 - 9/8 - 5) \\
&= -(2(x + 3/4)^2 - 49/8) \\
&= -2(x + 3/4)^2 + 49/8.
\end{align*}
\]

From this we can read off the fact that the maximum value is 49/8 and this occurs at \( x = -3/4 \).
Hence we have that the range is \(( -\infty, 49/8 \)\).

(g) \( h(t) = 1/(2t + 7) \). The domain is \( \mathbb{R} - \{7/2\} \) as we do not allow \( t = -7/2 \). To find the range, let \( h \) be a number in the range then we have to find \( t \) such that \( 1/(2t - 7) = h \).
On solving this for \( t \) we get
\[
t = (7h + 1)/(2h).
\]
Note that \( h \neq 0 \) as otherwise there does not exist such a \( t \). So the range is all \( h \neq 0 = \mathbb{R} - \{0\} \).

(h) \( f(x) = (x - 1)/(x + 1) \). The domain is \( \mathbb{R} - \{-1\} \) as we cannot let \( x = -1 \) (why?).
To find the range, let \( f \) be a value in the range. There has to be an \( x \neq -1 \) such that:

\[
\begin{align*}
f &= (x - 1)/(x + 1) \\
f &= 1 - 2/(x + 1) \\
x &= 2/(1 - f) - 1.
\end{align*}
\]
So as long as \( f \neq 1 \) we can find an \( x \). Hence the range is all \( f \neq 1 \) i.e.
\[
\mathbb{R} - \{1\}.
\]

(i) \( f(x)) = (1 + x^2)/(2 + 3x^2) \). The domain is \( \mathbb{R} \) as although we are dividing by \( 2 + 3x^2 \) it is never 0 as \( 2 + 3x^2 \geq 2 \) for all \( x \).
As for the range, Let $f > 0$ be any such value, we want to find $x$ such that $f = f(x)$ i.e. such that

\[
\begin{align*}
    f &= (1 + x^2)/(2 + 3x^2) \\
    f(2 + 3x^2) &= 1 + x^2 \
    x^2(3f - 1) &= 1 - 2f \\
    x^2 &= (1 - 2f)/(3f - 1) \Rightarrow x = \pm \sqrt{(1 - 2f)/(3f - 1)}.
\end{align*}
\]

As $f > 0$ then $(1 - 2f)/(3f - 1) \geq 0$ only if $1/3 < f \leq 1/2$ and we can find $x$ such that $f(x) = f$.

Hence the range is $(1/3, 1/2]$.

![Graph of $y = (1 + x^2)/(2 + 3x^2)$.

Hence all values between $1/3$ and $1/2$, but not including $1/3$ are in the range, see the graph.

(j) $h(t) = 2t^4 + 3t^2 - 5 \geq -5$. Domain is all $t$ i.e. $\mathbb{R}$. In order to obtain the range note that $h(t) = 2t^4 + 3t^2 - 5 = 2(t^2 + 3/4)^2 - 49/8$.

We now show that the range is all $h$ such that $h \geq -5$.

Let $h \geq -5$ be any value in the range then

\[
\begin{align*}
    2(t^2 + 3/4)^2 - 49/8 &= h \Rightarrow \\
    t^2 + 3/4 &= \sqrt{49 + 8h}/4 \Rightarrow \\
    t^2 &= \sqrt{49 + 8h}/4 - 3/4.
\end{align*}
\]
As \( h \geq -5 \) is any value we see that \( \sqrt{49 + 8h/4 - 3/4} \geq 0 \) and we can take the square root so that \( t = \pm \sqrt{49 + 8h/4 - 3/4} \) is such that \( h(t) = h \). Hence the range is \([-5, \infty)\).

### 23.5 Videos

**Square Root Domain 1**

This video shows how to find the domain of the linear function \( f(x) = \sqrt{x + 3} \).

**Square Root Domain 2**

This video shows how to find the domain of the function \( f(x) = \sqrt{x^2 - 4x + 3} \).

**Range of a Quadratic 1**

This video finds the range of the function \( g(z) = z^2 - 5z + 4 \).

**Range of a Quadratic 2**

This video finds the range of the function \( h(y) = 1 + 2y - y^2 \).

**Range of the Reciprocal of a Linear Function**

This video finds the range of the function \( k(x) = 1/(x - 1) \).