

MAS3918: Extremes

Solutions to non-assessed questions

1. When $\xi \neq 0$, we have

$$G(x; \mu, \sigma, \xi) = \exp \left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]_+^{-1/\xi} \right\}.$$

Differentiating w.r.t. x (via the chain rule) gives

$$\begin{aligned} g(x; \mu, \sigma, \xi) &= \exp \left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]_+^{-1/\xi} \right\} \times \frac{1}{\xi} \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]_+^{-1/\xi - 1} \times \frac{\xi}{\sigma} \\ &= \frac{1}{\sigma} \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]_+^{-(1/\xi + 1)} \exp \left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]_+^{-1/\xi} \right\}. \end{aligned}$$

When $\xi = 0$, we have

$$G(x; \mu, \sigma) = \exp \left\{ - \exp \left[- \left(\frac{x - \mu}{\sigma} \right) \right] \right\}.$$

Differentiating w.r.t. x (via the chain rule) gives

$$\begin{aligned} g(x; \mu, \sigma) &= \exp \left\{ - \exp \left[- \left(\frac{x - \mu}{\sigma} \right) \right] \right\} \times \exp \left\{ - \left(\frac{x - \mu}{\sigma} \right) \right\} \times \frac{1}{\sigma} \\ &= \frac{1}{\sigma} \exp \left\{ - \exp \left[- \left(\frac{x - \mu}{\sigma} \right) \right] - \left(\frac{x - \mu}{\sigma} \right) \right\}. \end{aligned}$$

This gives, for $\xi = 0$,

$$\begin{aligned} L(\sigma, \xi; \mathbf{x}) &= \prod_{i=1}^m \frac{1}{\sigma} \exp \left\{ - \exp \left[- \left(\frac{x_i - \mu}{\sigma} \right) \right] - \left(\frac{x_i - \mu}{\sigma} \right) \right\} \\ &= \sigma^{-m} \prod_{i=1}^m \exp \left\{ - \exp \left[- \left(\frac{x_i - \mu}{\sigma} \right) \right] - \left(\frac{x_i - \mu}{\sigma} \right) \right\}, \end{aligned}$$

giving

$$\begin{aligned} \ell(\mu, \sigma; \mathbf{x}) &= -m \log \sigma - \sum_{i=1}^m \left\{ \exp \left[- \left(\frac{x_i - \mu}{\sigma} \right) \right] + \left(\frac{x_i - \mu}{\sigma} \right) \right\} \\ &= -m \log \sigma - \sum_{i=1}^m \exp \left[- \left(\frac{x_i - \mu}{\sigma} \right) \right] - \sum_{i=1}^m \left(\frac{x_i - \mu}{\sigma} \right). \end{aligned}$$

2. This question was set as an assignment question – see assignment solutions.
3. If the X_i are Gumbel distributed, then

$$F_{X_i}(x) = \exp \{ -\exp(-x) \}, \quad i = 1, \dots, n.$$

Now

$$\begin{aligned} \Pr \left(\frac{M_n - b_n}{a_n} \leq z \right) &= \Pr (M_n \leq a_n z + b_n) \\ &= \Pr (\max [X_1, \dots, X_n] \leq a_n z + b_n) \\ &= \Pr (X_1, \dots, X_n \leq a_n z + b_n) \\ &= [\Pr (X_i \leq a_n z + b_n)]^n, \end{aligned}$$

for any $i \in (1, \dots, n)$, since the X_i are IID. Thus, with $a_n = 1$ and $b_n = \log n$, we have

$$\begin{aligned} \Pr \left(\frac{M_n - b_n}{a_n} \leq z \right) &= \left[e^{-e^{-(a_n z + b_n)}} \right]^n \\ &= \left[e^{-e^{-(z + \log n)}} \right]^n \end{aligned}$$

Taking logs, gives

$$\begin{aligned} n \log \left[e^{-e^{-(z + \log n)}} \right] &= n \left[-e^{-(z + \log n)} \right] \\ &= -n e^{-z} e^{\log n^{-1}} \\ &= -n e^{-z} n^{-1} \\ &= e^{-z}. \end{aligned}$$

Exponentiating to 're-balance', gives

$$\Pr \left(\frac{M_n - b_n}{a_n} \leq z \right) = e^{-e^{-z}},$$

which is the distribution function of a Gumbel distribution – i.e. a type I extreme value distribution.

4. Similar to question 3, we can show that

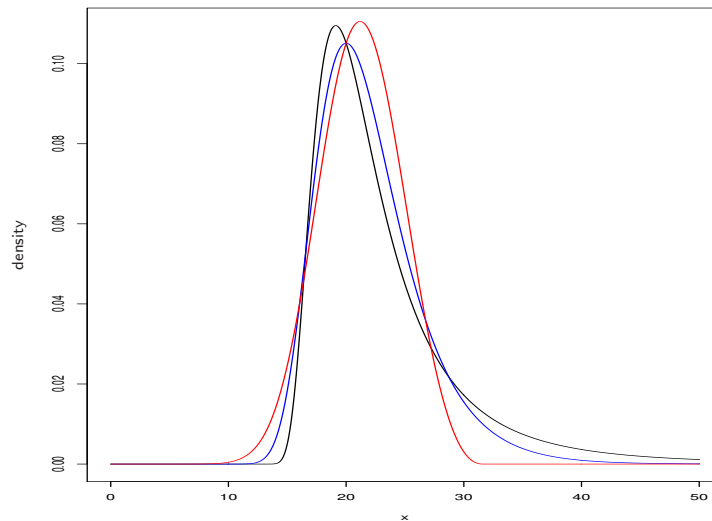
$$\Pr \left(\frac{M_n - b_n}{a_n} \leq z \right) = [\Pr (X_i \leq a_n z + b_n)]^n,$$

for any $i \in (1, \dots, n)$, since the X_i are IID. Recall that, for $X \sim U(0, 1)$, $F_X(x) = x$ and so – with $a_n = 1/n$ and $b_n = 1$ – we have

$$\begin{aligned} \Pr \left(\frac{M_n - b_n}{a_n} \leq z \right) &= \left(\frac{z}{n} + 1 \right)^n \\ &\rightarrow e^z \end{aligned}$$

as $n \rightarrow \infty$. Hence, the limit distribution is of Weibull type, with $\xi = -1$ in the GEV family.

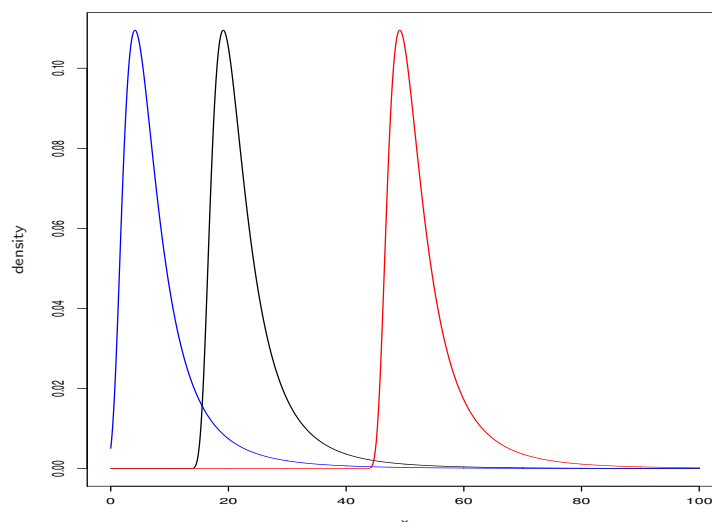
5. This question was set as an assignment question – see assignment solutions.
6. (d) The following plot should be obtained. Here, black represents the case $\xi = 0.3$, blue the case $\xi = 0$ and red the case when ξ is negative (here I have used -0.3 , but any negative value will do).



(e) Comments:

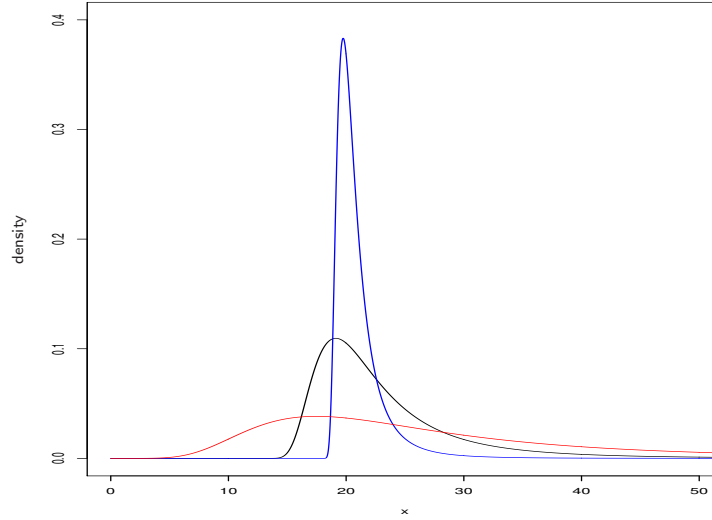
- * When $\xi \geq 0$, we have an unbounded upper tail
- * When $\xi < 0$, we have a bounded upper tail
- * The difference between $\xi = 0$ and $\xi = 0.3$ is that the latter has a heavier tail

(f) Now we keep the original values for σ and ξ , that is $\sigma = 3.5$ and $\xi = 0.3$, but vary μ . I have used $\mu = 5$ (black), $\mu = 20$ (blue) and $\mu = 50$ (red), giving the following plot:



Comment: Densities identical, but for a shift along the x -axis.

- (g) Now we keep the original values for μ and ξ , that is $\mu = 20$ and $\xi = 0.3$, but vary σ . I have used $\sigma = 3.5$ (black), $\sigma = 1$ (blue) and $\sigma = 10$ (red), giving the following plot:



Comment: Densities remain in the same central location, and have the same shape, however, their dispersion changes with σ .

7. (a) (i) Line 1 initialises the parameters at $(\mu, \sigma, \xi) = (\text{mean}, \text{sd}, 0.1)$. The numerical search method used to maximise the likelihood will start at these values of the parameters.
- (ii) We have $\hat{\mu} = 75.81$, $\hat{\sigma} = 34.55$ and $\hat{\xi} = -0.06$. Therefore,

$$\begin{aligned}\hat{z}_{50} &= 75.81 + \frac{34.55}{-0.06} [(-\log(1 - 1/50))^{0.06} - 1] \\ &= 196 \text{ cm} \approx 2 \text{ metres.}\end{aligned}$$

- (iii) The minimum of the negative log-likelihood function at the parameter estimates in (ii) is 264.815, so the maximum of the likelihood function is -264.815.
- (iv) Ordinarily, $I_O = -H$, where H is the Hessian. However, since this function returns the negative log-likelihood, $I_O = H$ and so

$$I_O = \begin{pmatrix} 0.0418 & -0.0124 & 0.6182 \\ -0.0124 & 0.0793 & 0.9519 \\ 0.6182 & 0.9519 & 90.8240 \end{pmatrix}.$$

- (v) We just need to invert I_O ; in this case, we can do this using:

`> solve(A)`

- (b) From V , we can deduce that $\text{s.e.}(\hat{\xi}) = \sqrt{0.015} = 0.1225$. Therefore, a 95% confidence interval for ξ can be obtained as:

$$-0.06 \pm 1.96 \times 0.1225 \longrightarrow (-0.3001, 0.1801).$$

As this confidence interval includes zero, there is evidence to support the use of the simpler Gumbel model here.

(c) We want:

$$\begin{aligned}
\Pr(\text{Snow depth} > 170\text{cm}) &= 1 - G(170; \hat{\mu} = 75.81, \hat{\sigma} = 34.55, \hat{\xi} = -0.06) \\
&= 1 - \exp \left\{ - \left[1 + -0.06 \left(\frac{170 - 75.81}{34.55} \right) \right]_+^{-1/-0.06} \right\} \\
&= 0.049674.
\end{aligned}$$

Thus, $0.049674 \times 200 \approx 10$ years.

- (d) If there is an association between snow depths in successive years, then the standard errors attached to parameter estimates, as reported earlier, will likely be too small as there is less information in the data than the likelihood requires.
8. (a) The points in this plot represent empirical estimates of the $i/(n+1)$ quantiles. The lines represent model-based quantiles (from the GEV in this case), using $\hat{F}^{-1}(i/(n+1))$; the outer lines are the lower and upper bounds of the corresponding 95% confidence intervals.
- (b) This particular return level plot indicates a good fit to the data, with all points lying within the 95% confidence bounds.
- (c) The logairthmic scale used on the x -axis compresses the tail of the fitted distribution, focussing attention on long-period return level estimates.
9. (a) Although the MLEs are not provided, we can easily work these out using the 95% confidence intervals – as they are symmetric, we can find that:

$$\begin{aligned}
\hat{\mu} &= \frac{20.88 + 24.61}{2} = 22.745, \\
\hat{\sigma} &= \frac{3.52 + 7.66}{2} = 5.59 \\
\hat{\xi} &= \frac{0.24 + 0.44}{2} = 0.34.
\end{aligned}$$

For return levels, we need to remember that we have negated the annual minima and performed an analysis (in the usual way, as covered in Chapter 2 of the lecture notes) on the resulting maxima. This will have implications for return level estimates. Although not formally covered in this course (in 24/25), the following distribution function can be shown to be the limiting model for block minima:

$$\tilde{G}(x) = 1 - \exp \left\{ - \left[1 - \xi \left(\frac{x - \tilde{\mu}}{\sigma} \right) \right]^{-1/\xi} \right\} \quad (*)$$

where σ and ξ are identical the scale and shape parameters in the analogous model for block maxima as covered in Chapter 2 of the lecture notes; $\tilde{\mu} = -\mu$, where μ is the GEV location parameter in the analogous model for block maxima. Equating (*) to $1 - 1/r$, as we did for the block maxima case in Chapter 2 of the lecture notes, and solving for $x = z_r$, gives:

$$z_r = \tilde{\mu} + \frac{\sigma}{\xi} [1 - (-\log(1/r))^{-\xi}].$$

Here, we have $\hat{\mu} = -22.745$, $\hat{\sigma} = 5.59$ and $\hat{\xi} = 0.34$; substituting these into the above expression along with $r = 75$, gives an estimate of the 75-year return level as:

$$z_{75} = -22.745 + \frac{5.59}{0.34} [1 - (-\log(1/75))^{-0.34}] = -16.30^\circ\text{C}.$$

- (b) We know that the first element of ∇z_{75} is 1, as this is just the derivative of z_{75} with respect to $\tilde{\mu}$. We can therefore write down that

$$\nabla z_{75} = \begin{pmatrix} 1 \\ 9.795 \\ 145.869 \end{pmatrix}.$$

We also know that for each of our GEV model parameters,

$$\text{standard error} = \frac{95\% \text{ confidence upper bound} - \text{MLE}}{1.96},$$

giving, for each of our GEV parameters,

$$\begin{aligned} \text{var}(\hat{\mu}) &= \left(\frac{24.61 - 22.745}{1.96} \right)^2 = 0.905, \\ \text{var}(\hat{\sigma}) &= \left(\frac{7.66 - 5.59}{1.96} \right)^2 = 1.115 \quad \text{and} \\ \text{var}(\hat{\xi}) &= \left(\frac{0.44 - 0.34}{1.96} \right)^2 = 0.003, \end{aligned}$$

giving

$$V = \begin{pmatrix} 0.905 & 0.5332 & -0.0291 \\ 0.5332 & 1.115 & 0.016 \\ -0.0291 & 0.0186 & 0.003 \end{pmatrix}.$$

Thus, performing the matrix multiplication gives

$$\text{var}(\hat{z}_{75}) = 223.1055,$$

and so $\text{s.e.}(\hat{z}_{75}) = \sqrt{223.1055} = 14.937$. Therefore, to complete the table, we would have

$$-16.30 \pm 1.96 \times 14.937 \longrightarrow (-45.577, 12.977)^\circ\text{C}.$$

- (c) The results seem very different – note that the shape parameter 95% confidence interval is wholly positive for Grand Prairie, and wholly negative for Dawson Creek. The positive shape parameter for Grand Prairie indicates an unbund upper tail, which might explain why the estimate of the 75-year return level here is considerably higher than that for Dawson Creek. An unbounded tail for temperature extremes seems unlikely!
- (d) The recording error might be the sign of the estimated shape parameter.

10. When $\xi \neq 0$, we have

$$H(y; \sigma, \xi) = 1 - \left(1 + \frac{\xi y}{\sigma} \right)_+^{-1/\xi}.$$

Differentiating w.r.t. y (via the chain rule) gives

$$\begin{aligned} h(y; \sigma, \xi) &= \frac{1}{\xi} \left(1 + \frac{\xi y}{\sigma} \right)_+^{-1/\xi-1} \times \frac{\xi}{\sigma} \\ &= \frac{1}{\sigma} \left(1 + \frac{\xi y}{\sigma} \right)_+^{-(1/\xi+1)} \end{aligned}$$

Now, forming the likelihood, we get

$$\begin{aligned} L(\sigma, \xi; \mathbf{y}) &= \prod_{i=1}^k \frac{1}{\sigma} \left(1 + \frac{\xi y_i}{\sigma} \right)_+^{-(1/\xi+1)} \\ &= \sigma^{-k} \prod_{i=1}^k \left(1 + \frac{\xi y_i}{\sigma} \right)_+^{-(1/\xi+1)}. \end{aligned}$$

Taking logs gives

$$\ell(\sigma, \xi; \mathbf{y}) = -k \log \sigma - (1/\xi + 1) \sum_{i=1}^k \log \left(1 + \frac{\xi y_i}{\sigma} \right)_+,$$

as required. When $\xi = 0$, we have

$$H(y; \sigma) = 1 - \exp(-y/\sigma).$$

Differentiating w.r.t. y gives

$$h(y; \sigma) = \frac{1}{\sigma} \exp(-y/\sigma).$$

Forming the likelihood gives

$$L(\sigma; \mathbf{y}) = \sigma^{-k} \prod_{i=1}^k \exp(-y_i/\sigma);$$

finally, taking logs gives

$$\ell(\sigma; \mathbf{y}) = -k \log \sigma - \sigma^{-1} \sum_{i=1}^k y_i,$$

as required.

11. This question was set as an assignment question – see assignment solutions.
12. Recall from the lecture notes that we require

$$\frac{1 - F(u + y)}{1 - F(u)}.$$

For our unit Fréchet margins, this gives

$$\frac{1 - e^{-(u+y)^{-1}}}{1 - e^{-u^{-1}}}.$$

Now the above expression $\rightarrow 1$ as $u \rightarrow \infty$ (for fixed y). Some students might include mathematical arguments here; some might include simple numerical results/R code which shows this. Either are fine.

As this represents the survival function of a threshold excess y , we therefore want

$$\left[1 + \frac{\xi y}{\sigma} \right]^{-1/\xi} \rightarrow 1$$

for some σ and ξ . If $\xi = 1$, we get

$$\frac{1}{1 + \frac{y}{\sigma}},$$

which $\rightarrow 1$ as $\sigma \rightarrow \infty$. As we also require this result for $u \rightarrow \infty$, setting $\sigma = u$ will give the required limit, that is, the limiting distribution for threshold exceedances is $GPD(u, 1)$.

13. This question was set as an assignment question – see assignment solutions.