

Bayesian Inference: Assignment 4

Answers to the starred (*) questions should be submitted to Canvas by no later than **4pm, Friday 5th May 2023**. Solutions to non-assessed questions will be made available in class.

Question 1*

The following probability density function (PDF) is often used in geology, for example to model the size of meteorites:

$$f(x|\alpha, \theta) = \frac{\alpha\theta^\alpha}{x^{\theta+1}}, \quad x > \theta$$

where $\alpha > 0$ and $\theta > 0$.

Suppose $x_1, x_2, \dots, x_n$ are a random sample of meteorite sizes; suppose further, from previous studies, we assume that $\alpha = 1$ in the PDF above.

- (a) Determine the Jeffreys prior for θ .
- (b) Determine the posterior distribution for θ , assuming a very large sample.

Question 2*

The salaries X of a random sample of n workers in a city follow a reparameterised log-normal $RLN(\mu, \alpha)$ distribution, where $(\mu, \alpha) \in \mathbb{R}^2$ and

$$f(x|\mu, \alpha) = \frac{e^{\alpha/2}}{x\sqrt{2\pi}} \exp\left\{-\frac{e^\alpha}{2}(\log x - \mu)^2\right\}, \quad x > 0.$$

Find the Jeffreys prior for $(\mu, \alpha)^T$.

Hint: If $X \sim RLN(\mu, \alpha)$ then $E(\log X) = \mu$ and $Var(\log X) = e^{-\alpha}$.

Question 3

A random sample $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ is taken from a log-normal $LN(\mu, \sigma^2)$ distribution.

- (a) Find the Jeffreys prior.
- (b) Is this a proper prior distribution? Explain your answer.

Question 4

Suppose that you have a random sample $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ from a $Ga(\alpha, \lambda)$ distribution.

- (a) Determine the asymptotic posterior distribution for $\underline{\theta} = (\alpha, \lambda)^T$. Hint: you should define the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\lambda}$ in terms of the equations they must satisfy, with these involving the sample mean $\bar{x}$, the geometric mean $\bar{x}_g$ and the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$.
- (b) Suppose a sample of size $n = 100$ gives sample mean $\bar{x} = 3.0$ and geometric mean $\bar{x}_g = 2.5$ leading to maximum likelihood estimates $\hat{\alpha} = 2.8983$ and $\hat{\lambda} = 0.9661$. Determine the asymptotic posterior distribution for $\underline{\theta}$. Also calculate the asymptotic posterior correlation between α and λ . Hint: you will need to use the R functions `digamma` for $\psi(\cdot)$ and `trigamma` for $\psi'(\cdot)$.