

# MAS2906: Computational Probability and Statistics with R

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Semester 1, 2022/23

# Assessments

- Numbas test 1 – feedback video now available in Canvas

- Mini Project

- You should now be well on the way with this
- Deadline: Next Friday, 9th December, 4pm
- Can work in pairs

- Practical next Friday cancelled – continue to work on your project!

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- You should have worked through examples 5.3–5.7 using the video earlier week
  - We'll do a further bonus example today
- We'll then make a start on the sixth and final chapter!

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# Cauchy example (Example 5.7)

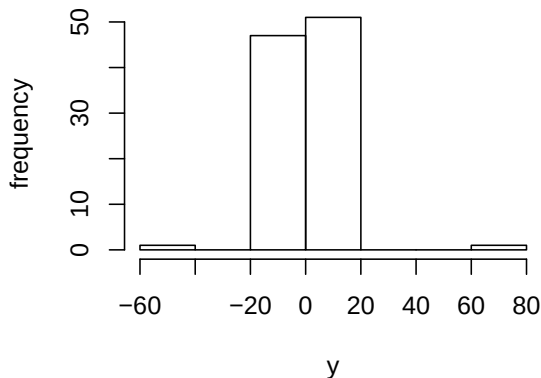
## Example

Suppose  $X \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$  and  $Y = \tan(X)$ .

Use simulation to study the distribution of  $Y$  and explicitly compute the PDF.

# Solution

```
1 y = tan(runif(100, -pi/2, pi/2))  
2 hist(y, xlab = 'y', ylab = 'frequency', main = '')
```



# Solution

This distribution looks odd! Calculate the PDF:

$$h(x) = \tan(x), \quad h^{-1}(y) = \tan^{-1}(y).$$

$$\begin{aligned} f_Y(y) &= \frac{1}{\pi} \frac{d}{dy} (\tan^{-1} y) \quad -\infty < y < \infty, \\ &= \frac{1}{\pi(1+y^2)} \quad -\infty < y < \infty. \end{aligned}$$

This is the **Cauchy distribution**. It has very heavy tails.

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## Bonus example

Hourly wind speed maxima ( $X$  miles per hour), observed on the roof of Claremont Tower as part of the University's *Urban Observatory*, are assumed exponentially distributed with rate 0.1.

Unfortunately, the data recorded are the logged wind speeds.

Use the transformation method to find the PDF of  $Y = \log X$ .

## Bonus example: Solution

Recall that if  $X \sim \text{Exp}(\lambda)$ , then it has PDF

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Now here, we have

$$h(x) = \log x, \quad \text{and} \quad h^{-1}(y) = e^y.$$

Therefore

$$\begin{aligned} f_Y(y) &= f_X(h^{-1}(y)) \times \left| \frac{d(h^{-1}(y))}{dy} \right| \\ &= \lambda \exp\{-\lambda \exp\{y\}\} \times \exp\{y\} \\ &= \lambda \exp\{y - \lambda e^y\}, \end{aligned}$$

with  $\lambda = 0.1$ .

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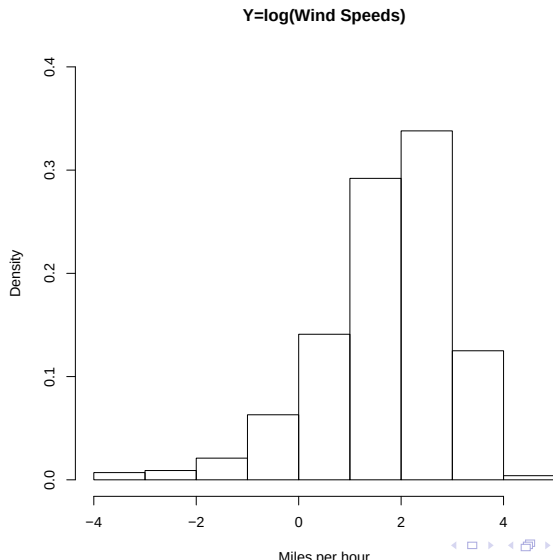
with  $\lambda = 0.1$ .

# Bonus example: Solution

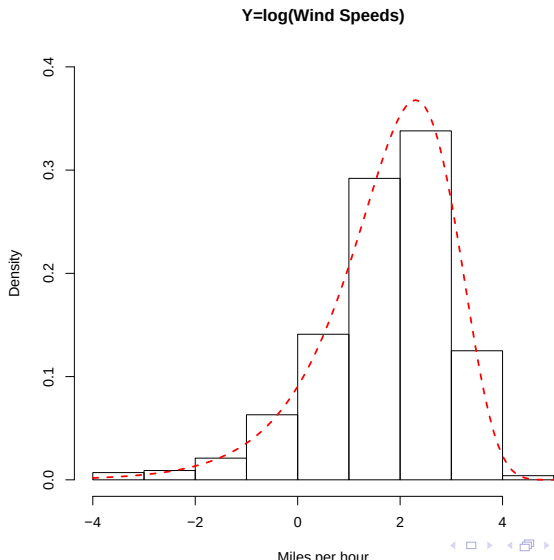
We can check this in R:

```
1 X = rexp(1000, 0.1)
2 Y = log(X)
3 hist(Y, prob=TRUE, main="Y=log(Wind Speeds)", xlab="
  Miles per hour")
4 lambda = 0.1
5 curve(lambda*exp(x-lambda*exp(x)), add=TRUE, col="red",
  lty=2, lwd=2)
```

# Bonus example: Solution



# Bonus example: Solution



# Chapter 6

## Bivariate distributions

## 6.1 Sampling from bivariate distributions

Suppose we have a pair of continuous random variables  $(X, Y)$  with joint PDF  $f_{X,Y}(x, y)$ .

Recall that

$$f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y | X = x)$$

where  $f_X(x)$  is the **marginal** PDF of  $X$  and  $f_{Y|X}(y | X = x)$  is the **conditional** PDF of  $Y$  given  $X = x$ .

One way to simulate realizations of the pair  $(X, Y)$  is:

- 1 Simulate a realization  $x$  of  $X$  from the marginal distribution of  $X$ .
- 2 Simulate a realization  $y$  of  $Y$  from the conditional distribution of  $Y$  given  $X = x$ .

## Example 6.1

### Example

Suppose  $X \sim U(0.1, 0.5)$  and

$$f_{Y|X}(Y | X = x) = \begin{cases} xe^{-xy}, & \text{when } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Write an R function to generate a  $2 \times n$  **matrix** of samples from this distribution, and produce a scatter plot.

## Example 6.1: Solution

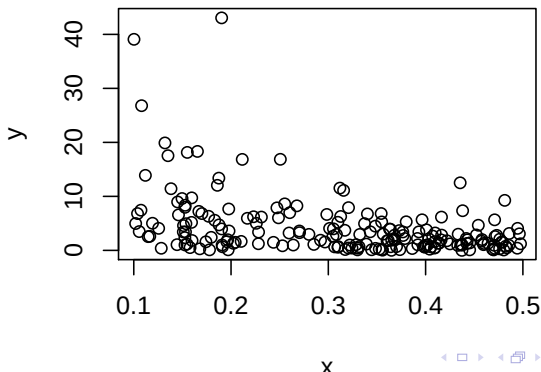
```
1 example6.1 = function(n) {  
2   output = matrix(0, 2, n)  
3   x = runif(n, 0.1, 0.5)  
4   y = rexp(n, rate = x)  
5   output[1, ] = x  
6   output[2, ] = y  
7   return(output)  
8 }
```

**Make some notes about the R function.**

## Example 6.1: Solution

To obtain a scatterplot:

```
1 test = example6.1(200)
2 plot(output[1,], output[2,], xlab = 'X', ylab = 'Y')
```



## Example 6.2: The trivariate case

### Example

Suppose that  $X \sim U(-1, 1)$  and  $Y \sim \text{Exp}(\lambda)$  independently of  $X$  and that

$$f_{Z|X,Y}(Z \mid X = x, Y = y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{1}{2} \left( \frac{z-x}{y} \right)^2}.$$

Write an R function to sample realizations  $(X, Y, Z)$  in the case  $\lambda = 5$  and produce a scatterplot of  $(X, Z)$ .

Note that, given that  $X = x$ ,  $Y = y$  then  $Z$  has a normal distribution:

$$Z \mid X = x, Y = y \sim N(x, y^2).$$

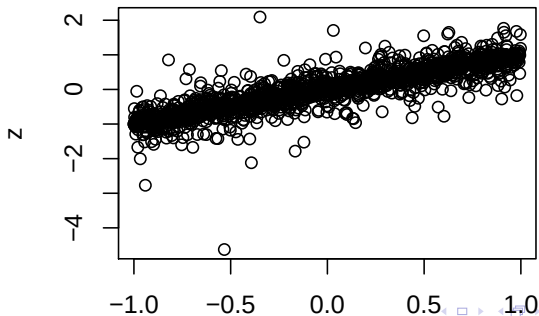
## Example 6.2: Solution

```
1 example6.2 = function(n) {  
2   output = matrix(0, 3, n)  
3   x = runif(n, -1, 1)  
4   y = rexp(n, rate = 5)  
5   z = rnorm(n, mean = x, sd = y)  
6   output[1, ] = x  
7   output[2, ] = y  
8   output[3, ] = z  
9   output  
10 }
```

**Make some notes about the R function.**

## Example 6.2: Solution

```
1 test = example6.2(2000)
2 plot(test[1, ], test[3, ], xlab = 'x', ylab = 'z')
```



## 6.2 The multivariate Normal distribution

You may have met the **bivariate Normal distribution** in **MAS1608**.

The bivariate Normal distribution is a generalization of the Normal distribution to pairs of random variables, or equivalently, to a distribution on vectors in  $\mathbb{R}^2$ .

The multivariate Normal distribution is the analogous distribution to vectors in  $\mathbb{R}^n$ .

## 6.2.1 The bivariate Normal distribution

Suppose  $\mu_x, \mu_y, \sigma_x \geq 0, \sigma_y \geq 0$  and  $-1 \leq \rho \leq 1$  are constants. Define the  $2 \times 2$  matrix  $\Sigma$  by

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}.$$

Then define a joint probability density function by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}Q(x,y)\right)$$

where

$$Q(x,y) = (\underline{x} - \underline{\mu})^T \Sigma^{-1} (\underline{x} - \underline{\mu})$$

and

$$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \underline{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}.$$

## 6.2.1 The bivariate Normal distribution

If random variables  $(X, Y)$  have joint probability density given by  $f_{X,Y}$  above, then we say that  $(X, Y)$  have a **bivariate normal distribution** and write

$$(X, Y)^T \sim N_2(\underline{\mu}, \Sigma).$$

It can be proved that the function  $f_{X,Y}(x, y)$  integrates to 1 and therefore defines a valid joint pdf.

The notes contain expansions of  $Q(x, y)$  and  $\det \Sigma$ .

# Remarks

- 1 The vector  $\underline{\mu} = (\mu_x, \mu_y)^T$  is called the **mean vector** and the matrix  $\Sigma$  is called the **covariance matrix** (or sometimes the variance-covariance matrix).
- 2 Functions of the form  $F(\underline{x}) = \underline{x}^T \Sigma^{-1} \underline{x}$  are called **quadratic forms**. Quadratic forms are functions  $\mathbb{R}^n \rightarrow \mathbb{R}$  which satisfy certain properties. They crop up in several areas of mathematics and statistics.
- 3 The matrix  $\Sigma$  and its inverse  $\Sigma^{-1}$  are **positive definite**. A matrix  $A$  is positive definite if

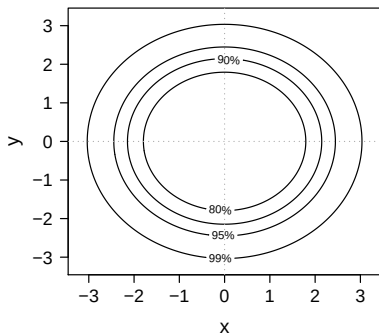
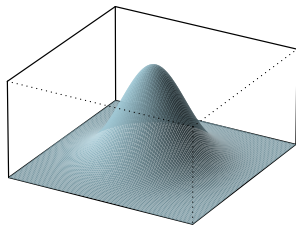
$$\underline{x}^T A \underline{x} \geq 0$$

for all non-zero vectors  $\underline{x}$ .

- 4 It follows that when  $\mu_x = \mu_y = 0$ ,  $Q(x, y)$  is a positive definite quadratic form.

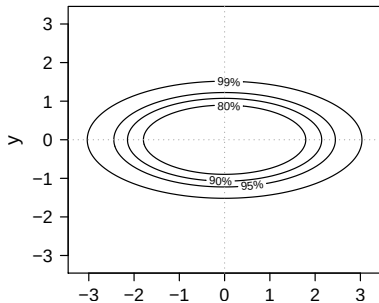
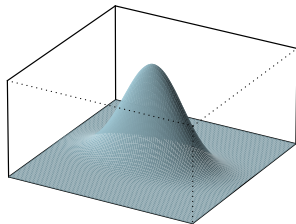
# Pictures

$$\sigma_x = \sigma_y, \rho = 0$$



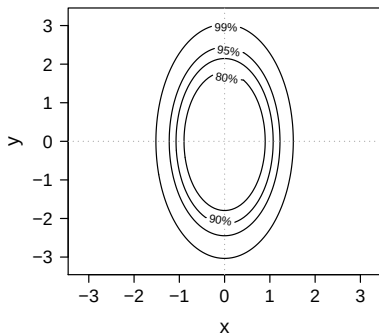
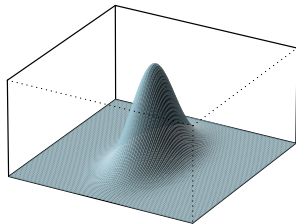
# Pictures

$$\sigma_x = 2\sigma_y, \rho = 0$$



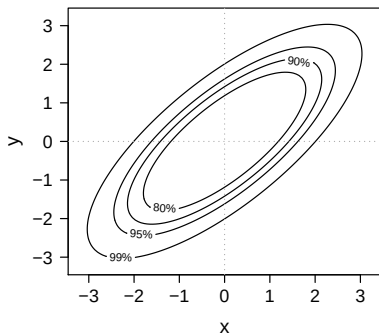
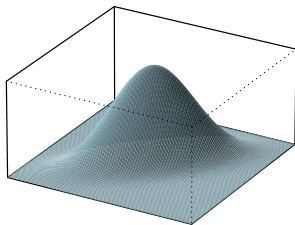
# Pictures

$$2\sigma_x = \sigma_y, \rho = 0$$



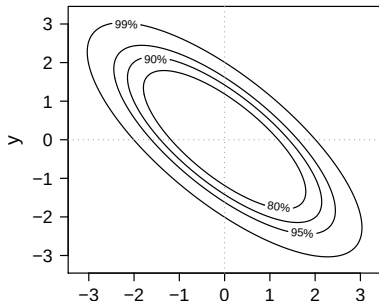
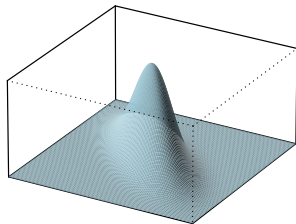
# Pictures

$$\sigma_x = \sigma_y, \rho = 0.75$$



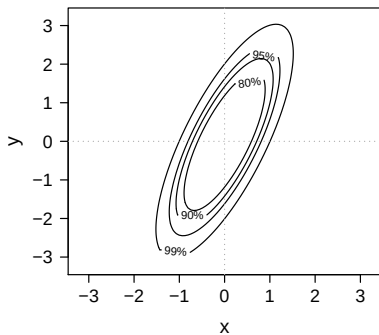
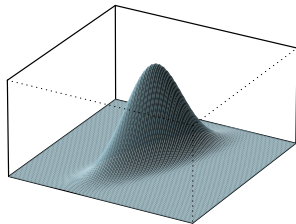
# Pictures

$$\sigma_x = \sigma_y, \rho = -0.75$$



# Pictures

$$2\sigma_x = \sigma_y, \rho = 0.75$$



# Comments

- 1  $Q(x, y) \geq 0$  with equality only when  $\underline{x} = \underline{\mu}$ . It follows that the density function has its mode at  $\underline{x} = \underline{\mu}$ .
- 2 Changing the values of  $\mu_x, \mu_y$  does not change the shape of the plots, but corresponds to a translation of the  $xy$ -plane i.e. changing  $\mu_x, \mu_y$  just shifts the contours / surface to a new mode position.
- 3 The contours of equal density are **circular** when  $\sigma_x = \sigma_y$  and  $\rho = 0$  and **elliptical** when  $\sigma_x \neq \sigma_y$  or  $\rho \neq 0$ .
- 4  $\sigma_x$  and  $\sigma_y$  control the extent to which the distribution is **dispersed**.
- 5 The parameter  $\rho$  is the **correlation** of  $X, Y$  i.e.  $\text{Cor}(X, Y) = \rho$ . Thus for non-zero  $\rho$ , the contours are at an angle to the axes.

## 6.2.1 Marginals and conditionals

Suppose  $(X, Y)^T \sim N_2(\underline{\mu}, \Sigma)$ . Then:-

- 1 The marginal distributions are normal:

$$X \sim N(\mu_x, \sigma_x^2) \quad \text{and}$$

$$Y \sim N(\mu_y, \sigma_y^2).$$

- 2 The conditional distributions are normal:

$$X|Y = y \sim N(\mu_x + \rho \frac{\sigma_x}{\sigma_y}(y - \mu_y), \sigma_x^2(1 - \rho^2)) \quad \text{and}$$

$$Y|X = x \sim N(\mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2)).$$

- 3 When  $\rho = 0$ ,  $X$  and  $Y$  are independent.
- 4 Linear combinations of  $X$  and  $Y$  are also normally distributed:

$$aX + bY \sim N(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho\sigma_x\sigma_y)$$

where  $a, b$  are constants.

## 6.2.2 Simulating bivariate normal random variables

R has built-in functions to simulate from the bivariate normal distribution, which we will see soon.

However, we can use the simulation strategy outlined in Section 6.1, as we have results for the **marginals** and **conditionals**.

## Example 6.3

### Example

Suppose  $(X, Y)^T \sim N_2(\underline{\mu}, \Sigma)$  where  $\mu_x = 2$ ,  $\mu_y = 3$ ,  $\sigma_x = 1$ ,  $\sigma_y = 1$  and  $\rho = 0.5$ .

Simulate a sample of size 500 from this distribution and draw a scatter plot.

Use simulation to find  $\Pr(X^2 + Y^2 < 9)$ .

## Example 6.3: Solution

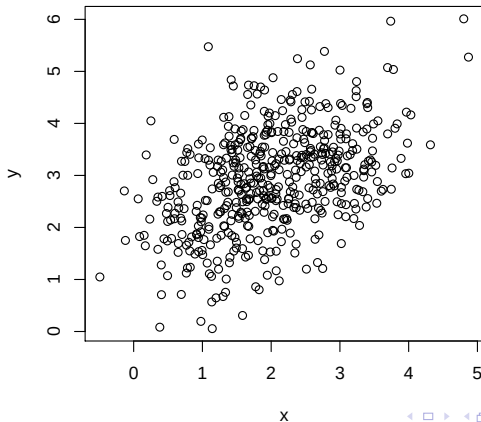
The marginal distribution of  $X$  is  $X \sim N(2, 1^2)$ .

Using the formula for the conditionals, we have:

$$\begin{aligned} Y|X = x &\sim N\left(\mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2)\right) \\ &\sim N(3 + 0.5(x - 2), 0.75). \end{aligned}$$

## Example 6.3: Solution

```
1 npts = 500
2 x = rnorm(npts, mean=2, sd = 1)
3 y = rnorm(npts, mean=3+0.5*(x-2), sd=sqrt(0.75))
```



## Example 6.3: Solution

To find  $\Pr(X^2 + Y^2 < 9)$  approximately, count the number of points in the region:

```
1 npts = 10000
2 x = rnorm(npts, mean=2, sd = 1)
3 y = rnorm(npts, mean=3+0.5*(x-2), sd=sqrt(0.75))
4 f = x^2+y^2
5 sum(f<9)/npts
```

Answer  $\simeq 0.2776$ .

## 6.2.3 The multivariate normal distribution

The multivariate normal distribution is defined on vectors in  $\mathbb{R}^n$ .

Suppose that  $\underline{X}$  is a random vector with  $n$  entries, i.e.

$$\underline{X} = (X_1, \dots, X_n)^T.$$

Then

$$\underline{X} \sim N_n(\underline{\mu}, \Sigma)$$

if  $X_1, \dots, X_n$  have joint PDF given by

$$f_{\underline{X}}(\underline{x}) = \frac{1}{2\pi\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2}Q(\underline{x})\right)$$

where

$$Q(\underline{x}) = (\underline{x} - \underline{\mu})^T \Sigma^{-1} (\underline{x} - \underline{\mu}).$$

This definition makes sense for any column vector  $\underline{\mu} \in \mathbb{R}^n$  and any **positive definite**  $n \times n$  matrix  $\Sigma$ .

# Remarks

- 1 The vector  $\underline{\mu}$  is the **mean** of the distribution and  $\Sigma$  is called the **covariance** matrix.
- 2 All the marginal distributions of  $\underline{X}$  are normal. (We do not specify their parameters here, however).
- 3 Similarly, all the conditional distributions of  $\underline{X}$  are normal. (Again, we do not specify the parameters of these distributions here).

## Example 6.4

### Example

Suppose  $\underline{X} \sim N_3(0, \Sigma)$  where

$$\Sigma = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Use R to sample from this distribution. Create a scatterplot of  $(X, Y)$  samples and compute the correlation.

## Example 6.4: Solution

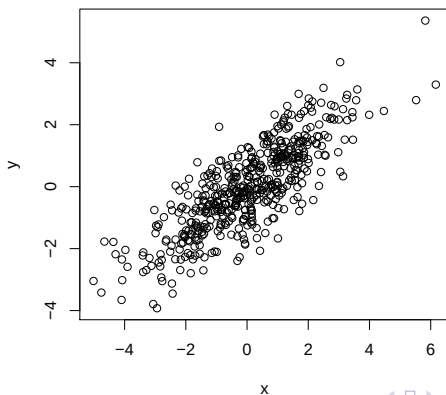
In R, you can use the `mvrnorm` function to sample:

```
1 library(MASS)
2 npts = 500
3 mu = c(0, 0, 0)
4 sigma = matrix(data = c(3,2,1,2,2,1,1,1,3), nrow = 3,
5               byrow = TRUE)
6 x = mvrnorm(npts, mu, sigma)
```

## Example 6.4: Solution

Now  $x$  has dimension  $npts \times 3$ , so column 1 is  $X$  and column 2 is  $Y$ :

```
1 plot(x[:,1], x[:,2], xlabel = 'x', ylabel = 'y')
```



## Example 6.4: Solution

The sample correlation is:

```
1 cor(x[,1], x[,2])  
2 [1] 0.812629
```

## Bonus example

Suppose

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2 \left[ \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} \right].$$

The random variable  $Z$  is defined by  $Z = X + 3Y$ . What is the distribution of  $Z$ ?

## Bonus example

We have  $Z = X + 3Y$ . Using result 4 on page 51, we have

$$E[Z] = 1 \times \mu_x + 3 \times \mu_y = 1 \times 4 + 3 \times 1 = 7.$$

Now from the variance-covariance matrix, we have  $\rho\sigma_x\sigma_y = 2$ .  
Thus

$$\begin{aligned}\text{Var}(Z) &= 1^2 \times \sigma_x^2 + 3^2 \times \sigma_y^2 + 2 \times 1 \times 3 \times (\rho\sigma_x\sigma_y) \\ &= 1 \times 8 + 9 \times 5 + 2 \times 1 \times 3 \times 2 \\ &= 65.\end{aligned}$$

Therefore  $Z \sim N(7, 65)$ .

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