

MAS3321

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 2 2014/2015

MAS3321

Bayesian Inference

Time allowed: 1 hour 30 minutes

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are TWO questions in Section A and TWO questions in Section B.

Distributions for data and prior distributions are given at the end of the exam paper.

SECTION A

A1. Suppose that the lifetimes X of n patients randomly allocated to receiving a drug treatment follow a $Ga(4, \theta)$ distribution.

- (a) Show that the likelihood function for θ given the lifetimes $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ of a random sample of patients given the treatment is

$$f(\mathbf{x}|\theta) \propto \theta^{4n} e^{-n\bar{x}\theta}, \quad \theta > 0,$$

where $\bar{x}$ is the mean lifetime in the sample.

[4 marks]

- (b) Suppose your prior beliefs about θ were described by a $Ga(g, h)$ distribution. Determine your posterior density for θ . Name this distribution, including its parameters.

[5 marks]

- (c) Is the *Gamma* distribution a conjugate prior distribution? Explain your answer.

[2 marks]

- (d) Explain why, in general, a 95% highest density interval (H.D.I.) for θ is not straightforward to calculate for this posterior distribution.

[3 marks]

- (e) Determine an approximate 95% H.D.I. for θ by approximating the posterior distribution by a normal distribution with the same posterior mean and variance.

[6 marks]

[Total: 20 marks]

A2. The wages X of a random sample of n workers in a city follow a reparameterised log-normal $RLN(\mu, \alpha)$ distribution, where $(\mu, \alpha) \in \mathbb{R}^2$.

(a) Show that the likelihood function is

$$f(\mathbf{x}|\mu, \alpha) \propto e^{n\alpha/2} \exp \left\{ -\frac{e^\alpha}{2} \sum_{i=1}^n (\log x_i - \mu)^2 \right\}.$$

[4 marks]

(b) Show that

$$\begin{aligned} \frac{\partial^2}{\partial \mu^2} \log f(\mathbf{x}|\mu, \alpha) &= -ne^\alpha \\ \frac{\partial^2}{\partial \mu \partial \alpha} \log f(\mathbf{x}|\mu, \alpha) &= e^\alpha \sum_{i=1}^n (\log x_i - \mu) \\ \frac{\partial^2}{\partial \alpha^2} \log f(\mathbf{x}|\mu, \alpha) &= -\frac{e^\alpha}{2} \sum_{i=1}^n (\log x_i - \mu)^2. \end{aligned}$$

[6 marks]

(c) Hence show that the Jeffreys prior is, for $(\mu, \alpha) \in \mathbb{R}^2$

$$\pi(\mu, \alpha) \propto e^{\alpha/2}.$$

Hint: If $X \sim RLN(\mu, \alpha)$ then $E(\log X) = \mu$ and $Var(\log X) = e^{-\alpha}$.

[8 marks]

(d) Are μ and α independent in this prior distribution? Explain your answer.

[2 marks]

[Total: 20 marks]

SECTION B

B3. A random sample $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ is taken from a population which follows a normal $N(\mu, 1/\tau)$ distribution. This sample has mean $\bar{x}$ and variance $s^2 = \sum_{i=1}^n (x_i - \bar{x})^2/n$.

Suppose your prior beliefs about $(\mu, \tau)^T$ follow a normal-gamma $NGa(b, c, g, h)$ distribution.

(a) Show that the likelihood function is

$$f(\mathbf{x}|\mu, \tau) \propto \tau^{n/2} \exp \left[-\frac{n\tau}{2} \{s^2 + (\bar{x} - \mu)^2\} \right].$$

[6 marks]

(b) Show that your posterior distribution for $(\mu, \tau)^T$ is a $NGa(B, C, G, H)$ distribution where

$$B = \frac{bc + n\bar{x}}{c + n}, \quad C = c + n, \quad G = g + \frac{n}{2}, \quad H = h + \frac{cn(\bar{x} - b)^2}{2(c + n)} + \frac{ns^2}{2}.$$

Hint:

$$c(\mu - b)^2 + n(\bar{x} - \mu)^2 + 2h + ns^2 = C(\mu - B)^2 + 2H.$$

[7 marks]

(c) Suppose that Y is the next value randomly sampled from the population.

(i) Writing $Y = \mu + \epsilon$, determine the distributions of $\mu|\tau, \mathbf{x}$ and $\epsilon|\tau$, and hence that of $Y|\tau, \mathbf{x}$.

(ii) Determine the predictive density for Y by integrating out posterior uncertainty about τ . Name this distribution, including its parameters.

Hint 1: you may find your solution more straightforward by working in terms of $k = C/(C + 1)$.

Hint 2:

$$\int_0^\infty \theta^{a-1} e^{-b\theta} d\theta = \frac{\Gamma(a)}{b^a}.$$

[11 marks]

[Total: 24 marks]

B4. Suppose you have a random sample $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ of wind directions from the circular von Mises $vM(\mu, \lambda)$ distribution, where μ is the mean direction ($0 \leq \mu < 2\pi$) and λ is a concentration parameter ($\lambda > 0$). Your prior distribution has $\mu \sim U(0, 2\pi)$ and $\lambda \sim Ga(g, h)$, independently, for known values g and h .

(a) Show that the likelihood function is

$$f(\mathbf{x}|\mu, \lambda) \propto I_0(\lambda)^{-n} \exp \left\{ \lambda \sum_{i=1}^n \cos(x_i - \mu) \right\}.$$

[3 marks]

(b) Show that your posterior density is, for $0 \leq \mu \leq 2\pi$, $\lambda > 0$

$$\pi(\mu, \lambda|\mathbf{x}) \propto \lambda^{g-1} I_0(\lambda)^{-n} \exp \left\{ \lambda \left[\sum_{i=1}^n \cos(x_i - \mu) - h \right] \right\}.$$

[4 marks]

(c) Determine the posterior conditional densities for $\mu|\lambda$ and $\lambda|\mu$ up to a multiplicative constant.

[6 marks]

You decide to develop an MCMC algorithm with separate steps for each parameter to simulate realisations from the posterior distribution.

(d) You decide to use a normal random walk proposal with variance Σ_μ for μ , that is, $q(\mu^*|\mu) = N(\mu, \Sigma_\mu)$. The acceptance probability for the proposed value μ^* can be written as $\min(1, A_\mu)$. Determine A_μ .

[4 marks]

(e) You decide to use a normal random walk proposal with variance Σ_λ for λ , that is, $q(\lambda^*|\lambda) = N(\lambda, \Sigma_\lambda)$. The acceptance probability for the proposed value λ^* can be written as $\min(1, A_\lambda)$. Determine A_λ .

[4 marks]

(f) Explain how you would initialise your algorithm.

[2 marks]

(g) Write down the steps in your MCMC algorithm for simulating realisations from the posterior distribution.

[8 marks]

- (h) In what circumstances might the acceptance probabilities in (d) and (e) be low and how would you remedy this problem?

[5 marks]

[Total: 36 marks]

Distributions for data

Gamma distribution: If $X \sim Ga(\alpha, \lambda)$ then it has density

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha > 0, \lambda > 0.$$

Also, $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$.

Reparameterised log-normal distribution: If $X \sim RLN(\mu, \alpha)$ then it has density

$$f(x|\mu, \alpha) = \frac{e^{\alpha/2}}{x\sqrt{2\pi}} \exp\left\{-\frac{e^\alpha}{2}(\log x - \mu)^2\right\}, \quad x > 0$$

where $-\infty < \mu < \infty$ and $-\infty < \alpha < \infty$. Also, $E(X) = \exp\{\mu + e^{-\alpha}/2\}$, $Var(X) = \{\exp(e^{-\alpha}) - 1\} \exp(2\mu + e^{-\alpha})$. Further $E(\log X) = \mu$ and $Var(\log X) = e^{-\alpha}$.

Normal distribution: If $X \sim N(\mu, 1/\tau)$ then it has density

$$f(x|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x - \mu)^2\right\}, \quad -\infty < x < \infty$$

where $-\infty < \mu < \infty$ and $\tau > 0$. Also, $E(X) = \mu$ and $Var(X) = 1/\tau$. The distribution has the following quantiles

x	1.2816	1.6449	1.9600	2.3263	2.5758
$\Pr(X < x)$	0.9	0.95	0.975	0.99	0.995

von Mises distribution: If $X \sim vM(\mu, \lambda)$ then it has probability density

$$f(x|\mu, \lambda) = \frac{1}{2\pi I_0(\lambda)} \exp\{\lambda \cos(x - \mu)\}, \quad 0 \leq x < 2\pi$$

where $0 \leq \mu < 2\pi$ and $\lambda > 0$. Also $I_0(\cdot)$ is the modified Bessel function of order zero.

Prior distributions

Gamma distribution: If $\theta \sim Ga(g, h)$ then it has density

$$\pi(\theta) = \frac{h^g \theta^{g-1} e^{-h\theta}}{\Gamma(g)}, \quad \theta > 0, \quad g > 0, h > 0.$$

Also, $E(\theta) = g/h$ and $Var(\theta) = g/h^2$.

Generalised t distribution: If $\mu \sim t_a(b, c)$ then it has density

$$\pi(\mu) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{ac\pi} \Gamma\left(\frac{a}{2}\right)} \left\{ 1 + \frac{(\mu - b)^2}{ac} \right\}^{-\frac{a+1}{2}}, \quad -\infty < \mu < \infty$$

where $-\infty < b < \infty$ and $a, c > 0$. Also, $E(\mu) = b$ and $Var(\mu) = \frac{ac}{a-2}$, if $a \geq 2$.

Normal distribution: If $\mu \sim N(b, 1/d)$ then it has density

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\}, \quad -\infty < \mu < \infty$$

where $-\infty < b < \infty$ and $c > 0$. Also, $E(\mu) = b$ and $Var(\mu) = 1/d$.

Normal-gamma distribution: If $\begin{pmatrix} \mu \\ \tau \end{pmatrix} \sim NGa(b, c, g, h)$ then it has density

$$\pi(\mu, \tau) \propto \tau^{g-\frac{1}{2}} \exp\left\{-\frac{\tau}{2} [c(\mu - b)^2 + 2h]\right\}, \quad -\infty < \mu < \infty, \tau > 0$$

where $-\infty < b < \infty$ and $c, g, h > 0$. Also, $\mu|\tau \sim N\left(b, \frac{1}{c\tau}\right)$, $\tau \sim Ga(g, h)$

and $\mu \sim t_{2g}\left(b, \frac{h}{gc}\right)$.

Uniform distribution: If $\theta \sim U(a, b)$ then it has density

$$\pi(\theta) = \frac{1}{b-a}, \quad a < \theta < b,$$

where $a < b$. Also, $E(\theta) = (a+b)/2$ and $Var(\theta) = (b-a)^2/12$.

THE END