

MAS2903

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS, STATISTICS & PHYSICS

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SEMESTER 2 2021/22 Specimen paper

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MAS2903

Introduction to Bayesian Methods

Time allowed: 2 hours

*Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks may be adjusted in accordance with the University's Moderation and Scaling Policy.*

*There are THREE questions in Section A and THREE questions in Section B.*

*Answers to questions should be entered directly on this question paper in the spaces provided. Rough work should be done on the blank pages at the end of the paper. You must clearly indicate any rough work that you wish to be marked. This question paper must be handed in, attached inside an anonymised cover sheet, at the end of the examination.*

*Calculators may be used.*

**Beta distribution:** If  $X \sim Be(\alpha, \beta)$  then it has density

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad \alpha > 0, \beta > 0.$$

Also,  $E(X) = \alpha/(\alpha + \beta)$ ,  $Var(X) = \alpha\beta/\{(\alpha + \beta)^2(\alpha + \beta + 1)\}$  and  $Mode(X) = (\alpha - 1)/(\alpha + \beta - 2)$ . Note also that  $B(a, b) = (a - 1)!(b - 1)!/(a + b - 1)!$ ,  $a, b \in \mathbb{Z}$ .

**Beta-Binomial distribution:** If  $X \sim BetaBin(n, a, b)$  then it has density

$$f(x|n, a, b) = \binom{n}{x} \frac{B(x + a, b + n - x)}{B(a, b)}, \quad x = 0, 1, \dots, n, \quad n \in \mathbb{Z}_+, a > 0, b > 0.$$

Also,  $E(X) = na/(a + b)$  and  $Var(X) = nab(a + b + n)/(a + b)^2(a + b + 1)$ .

**Binomial distribution** If  $X|\theta \sim Bin(k, \theta)$  then it has probability function

$$f(x|\theta) = \binom{k}{x} \theta^x (1 - \theta)^{k-x}, \quad x = 0, 1, \dots, k, \quad k \in \mathbb{Z}_+, 0 < \theta < 1.$$

Also,  $E(X) = k\theta$  and  $Var(X) = k\theta(1 - \theta)$ .

**Exponential distribution:** If  $X \sim Exp(\lambda)$  then it has density

$$f(x|\lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Also,  $E(X) = 1/\lambda$  and  $Var(X) = 1/\lambda^2$ .

**Gamma distribution:** If  $X \sim Ga(\alpha, \lambda)$  then it has density

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha > 0, \lambda > 0.$$

Also,  $E(X) = \alpha/\lambda$ ,  $Var(X) = \alpha/\lambda^2$  and  $Mode(X) = (\alpha - 1)/\lambda$  when  $\alpha \geq 1$ . Note also that  $\Gamma(a) = (a - 1)!$ ,  $a \in \mathbb{Z}$ .

**Normal distribution:** If  $X \sim N(\mu, 1/\tau)$  then it has density

$$f(x|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x - \mu)^2\right\}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \tau > 0.$$

Also,  $E(X) = \mu$  and  $Var(X) = 1/\tau$ .

Note: If  $X \sim N(0, 1)$  then the following quantiles apply:

|              |       |       |       |       |       |        |       |        |       |
|--------------|-------|-------|-------|-------|-------|--------|-------|--------|-------|
| $x$          | 1.96  | 2.326 | 2.366 | 2.409 | 2.457 | 2.484  | 2.512 | 2.543  | 2.576 |
| $\Pr(X < x)$ | 0.975 | 0.99  | 0.991 | 0.992 | 0.993 | 0.9935 | 0.994 | 0.9945 | 0.995 |

**Poisson distribution:** If  $X \sim Po(\lambda)$  then it has probability function

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \dots, \quad \lambda > 0.$$

Also,  $E(X) = \lambda$  and  $Var(X) = \lambda$ .

**Continuous Uniform distribution:** If  $X \sim U(a, b)$ , then it has density

$$f(x|a, b) = \frac{1}{b - a}, \quad a < x < b, \quad -\infty < a, b < \infty.$$

Also,  $E(X) = (a + b)/2$  and  $Var(X) = (b - a)^2/12$ .

**SECTION A**

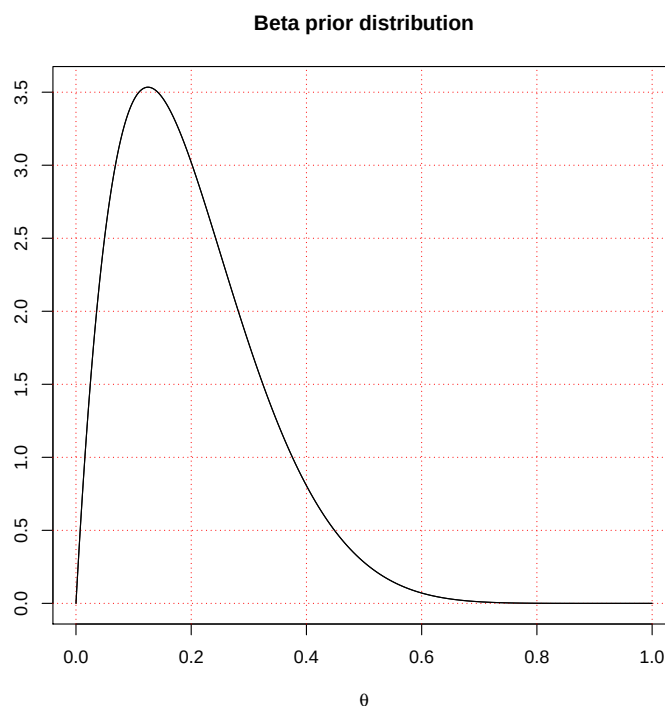
- A1.** A *SARS-CoV-2 Antigen Rapid Test* for Covid-19 yields a false positive result for 0.3% of healthy people tested. The test is 96% effective in detecting the Covid-19 virus when it is present. In a particular town, suppose one in a thousand people currently has Covid-19. Of 500 people who received a positive test result, how many might we expect will actually have the virus?

**Answer:**

**[5 marks]**

**A2.** As a Statistician working for the Canadian betting company *ZenTrader* you are interested in  $\theta$ , the probability that the Montreal Canadiens ice hockey team will win their next match against the Boston Bruins.

- (a) After extensive discussions with fans of the Boston Bruins, you choose a Beta distribution  $\theta \sim \text{Beta}(a, b)$ , for certain values of  $a$  and  $b$ , to represent your prior beliefs about  $\theta$ . A plot of this prior distribution is shown below; look at it, and then answer questions (i)–(v) underneath.



- (i) In your solutions, write down a suitable label for the  $y$ -axis. [1 mark]

**Answer:**

- (ii) What is the mode of your prior? Choose from: [1 mark]

**0.125, 0.2, 0.5 or 3.534263.**

**Answer:**

(iii) What is  $E(\theta)$ ? Choose from: [1 mark]

**0.125, 0.2, 0.5 or 3.534263.**

**Answer:**

(iv) What is  $Var(\theta)$ ? Choose from the values below, and explain your choice: [2 marks]

**$\frac{1}{4}$ ,  $\frac{21}{50}$  or  $\frac{4}{275}$ .**

**Answer:**

(v) In a single sentence, explain what this prior distribution tells us about  $\theta$ . [2 marks]

**Answer:**

(b) The Montreal Canadiens have won two of their last five matches against the Boston Bruins. Write down the likelihood  $f(x = 2|\theta)$ , assuming that the outcome of each of these matches is independent of the outcomes of all others. [3 marks]

**Answer:**

- (c) Why might you be concerned about the way the prior distribution was chosen in part (a)? [2 marks]

**Answer:**

- (d) (i) Worried about the way the prior was chosen in part (a), you decide to use a prior with the following density instead:

$$\pi(\theta) \propto \frac{1}{\theta(1-\theta)}.$$

Use this prior (and *not* the prior from part (a)) to find the posterior distribution for  $\theta$ . Make sure you identify the posterior fully. [4 marks]

**Answer:**

- (ii) Find  $E(\theta|x=2)$  and  $Var(\theta|x=2)$ . [3 marks]

**Answer:**

- (e) Briefly explain why the prior used in part (d) might be termed *vague*. Can you suggest another prior for  $\theta$  that might be useful in the absence of any prior knowledge about  $\theta$ ? [4 marks]

**Answer:**

[23 marks]

**A3.** A marine biologist is interested in the mean birth weight,  $\mu$  grams, of turtles on a beach on the Greek island of Kefalonia. She assumes the birth weights of these turtles to be Normally distributed with standard deviation 3 grams.

The marine biologist uses past studies to justify a Normal prior for  $\mu$  such that  $\mu \sim N(25, 1)$ .

- (a) Show that the marine biologist must take a sample of at least  $n = 27$  new-born turtles to ensure that the posterior standard deviation is no more than 0.5.

**Answer:**

- (b) Find the posterior distribution for  $\mu$ , given a mean birth weight of  $\bar{x} = 30.116$  grams from a random sample of 27 turtles on this beach.

**Answer:**

- (c) Find both the prior and posterior probabilities that the mean birth weight is greater than 30 grams.

**Answer:**

**[12 marks]**

**SECTION B**

**B4.** Suppose we have a random sample  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$  from the distribution with probability density function (PDF)

$$f(x|\beta) = \beta x^{-(\beta+1)}, \quad x > 1, \quad \beta > 0.$$

(a) A Normal distribution is proposed as a prior for  $\beta$ , that is,  $\beta \sim N(b, 1/d)$ .

(i) Why would this choice of prior be inappropriate?

**Answer:**

(ii) Propose a more suitable prior for  $\beta$ .

**Answer:**

(b) Determine the asymptotic posterior distribution for  $\beta$ . Specify the parameters of the posterior in terms of the sample size and  $\bar{x}_{\log} = \sum_{i=1}^n \log x_i / n$ .

**Answer:**

**[20 marks]**

**B5.** A probability model with probability density function (PDF)

$$f(x|\sigma) = \frac{3x^2}{\sigma} \exp \left\{ -\frac{x^3}{\sigma} \right\}, \quad x > 0,$$

where  $\sigma > 0$  is a scale parameter, is used to model river discharges.

- (a) Suppose we observe a series of river discharges  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$  (in thousand m<sup>3</sup>/s). Assuming we have a sample of independent observations, obtain the likelihood function  $f(\mathbf{x}|\sigma)$ . [3 marks]

**Answer:**

- (b) The *inverse gamma distribution* (abbreviated as *InvGa*) is often used as a prior distribution for  $\sigma$ . If  $\sigma \sim \text{InvGa}(a, b)$ , then

$$\pi(\sigma) = \frac{b^{a-1} e^{-b/\sigma}}{\Gamma(a-1) \sigma^a}, \quad \sigma > 0,$$

for some parameters  $a, b > 0$ .

Show that this prior is *conjugate*. [7 marks]

**Answer:**

- (c) For three rivers in England, we observe discharges of 1000, 1000, and 2000 m<sup>3</sup>/s. If  $\sigma \sim \text{InvGa}(2, 2)$ , find the posterior for  $\sigma$ . [3 marks]

**Answer:**

(d) For a 95% Bayesian confidence interval  $(\ell, u)$  for  $\sigma$ :

(i) Show that

$$\int_{\ell}^u \sigma^{-5} e^{-12/\sigma} d\sigma = \frac{19}{69120}. \quad [6 \text{ marks}]$$

**Answer:**

(ii) Write down another condition that must hold for this Bayesian confidence interval to be one of *highest density*. [2 marks]

**Answer:**

(e) Use the table below to describe how our beliefs about  $\sigma$  have changed in light of the data. [4 marks]

|         | Prior        | Likelihood | Posterior    |
|---------|--------------|------------|--------------|
| Mode    | 0.98         | 10/3       | 2.41         |
| 95% HDI | (0.05, 8.15) | —          | (1.15, 7.85) |

**Answer:**

[25 marks]

**B6.** A meteorologist is interested in predicting temperature extremes for a location in the southern United States. From daily temperature records for this location she extracts the annual maximum temperature over a 10 year period. The mean of the observations is 98.2°F. She assumes that each observation is independently distributed as  $N(\mu, 3^2)$ . A historical study suggests that  $\mu \sim N(90, 10^2)$ .

- (a) From results in the course, it can be shown that the posterior for  $\mu$  is  $N(98.1, 0.94^2)$ . Find the values  $a$  and  $b$ , where  $(a, b)$  is a 95% posterior *Highest Density Interval* for  $\mu$ .

**Answer:**

- (b) The meteorologist says:

“There is a probability of 0.95 that  $\mu$  lies between  $a$  and  $b$ .”

Is her claim correct? In a single sentence, justify your answer.

**Answer:**

- (c) In the context of this example, briefly describe why a Bayesian analysis might be preferable to a frequentist analysis of these annual maximum temperature data.

**Answer:**

[15 marks]

**THE END**







**Rough Work – Work submitted for marking must be clearly indicated**

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