Synchronous Online: Start 12pm, Wednesday 11th November

MAS2905: Probability and Bayesian Methods with R

Newcastle University
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Semester 1, 2020/21

The story so far...

- You should now have completed **all of Chapter 3**
- You should also have completed **Home Study 4**:
  - Introduction to the **Inverse CDF method**, including a proof
  - **Examples 4.1** and **4.2** completed
  - You should have also attempted **Example 4.3** – solutions to be revealed today!
- Don't forget: **Test 1** deadline tomorrow, 16:00!

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- **Aim:** To simulate from *any* probability distribution, given just realisations from $U \sim U(0, 1)$
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  - Small sample problems
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Example 4.3

Example

The **Pareto distribution**, named after the Italian civil engineer, economist, and sociologist Vilfredo Pareto, is a power-law probability distribution that is used in the description of social, scientific, geophysical, actuarial, and many other types of observable phenomena.

It was originally used to describe the distribution of wealth in a society, fitting the trend that a large portion of wealth is held by a small fraction of the population.

If $X \sim Pareto(\sigma)$,then it has cumulative distribution function

$$F_X(x;\sigma) = 1 - (\sigma/x)^{\log_4 5}, \quad x \geq \sigma, \sigma > 0.$$

Write a function in R to produce realisations from this distribution, and test your function out for $\sigma = \{1, 2, 5\}$.

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Example 4.3: Solution

Find the inverse CDF.

If $y = 1 - (\sigma/x)^{\log_4 5}$, then

$$\begin{aligned}
\left(\frac{\sigma}{x}\right)^{\log_4 5} &= 1 - y \\
\frac{\sigma}{x} &= (1-y)^{\frac{1}{\log_4 5}} \\
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and so

$$F_X^{-1}(x) = \frac{\sigma}{(1-x)^{\frac{1}{\log_4 5}}}.$$

We'll now try this out in R, for $\sigma = \{1, 2, 5\}$.

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pareto.sim = function(n, sigma){
  U = runif(n)
  answer = sigma/((1-U)^(1/(log(5, base=4))))
  return(answer)}
```

We can then test this out:

```
test_sigma1 = pareto.sim(100, 1)
test_sigma2 = pareto.sim(100, 2)
test_sigma5 = pareto.sim(100, 5)

par(mfrow=c(1,3))
hist(test_sigma1,main="Sigma=1",col="darkgoldenrod1")
hist(test_sigma2,main="Sigma=2",col="deepskyblue2")
hist(test_sigma5,main="Sigma=5",col="yellowgreen")
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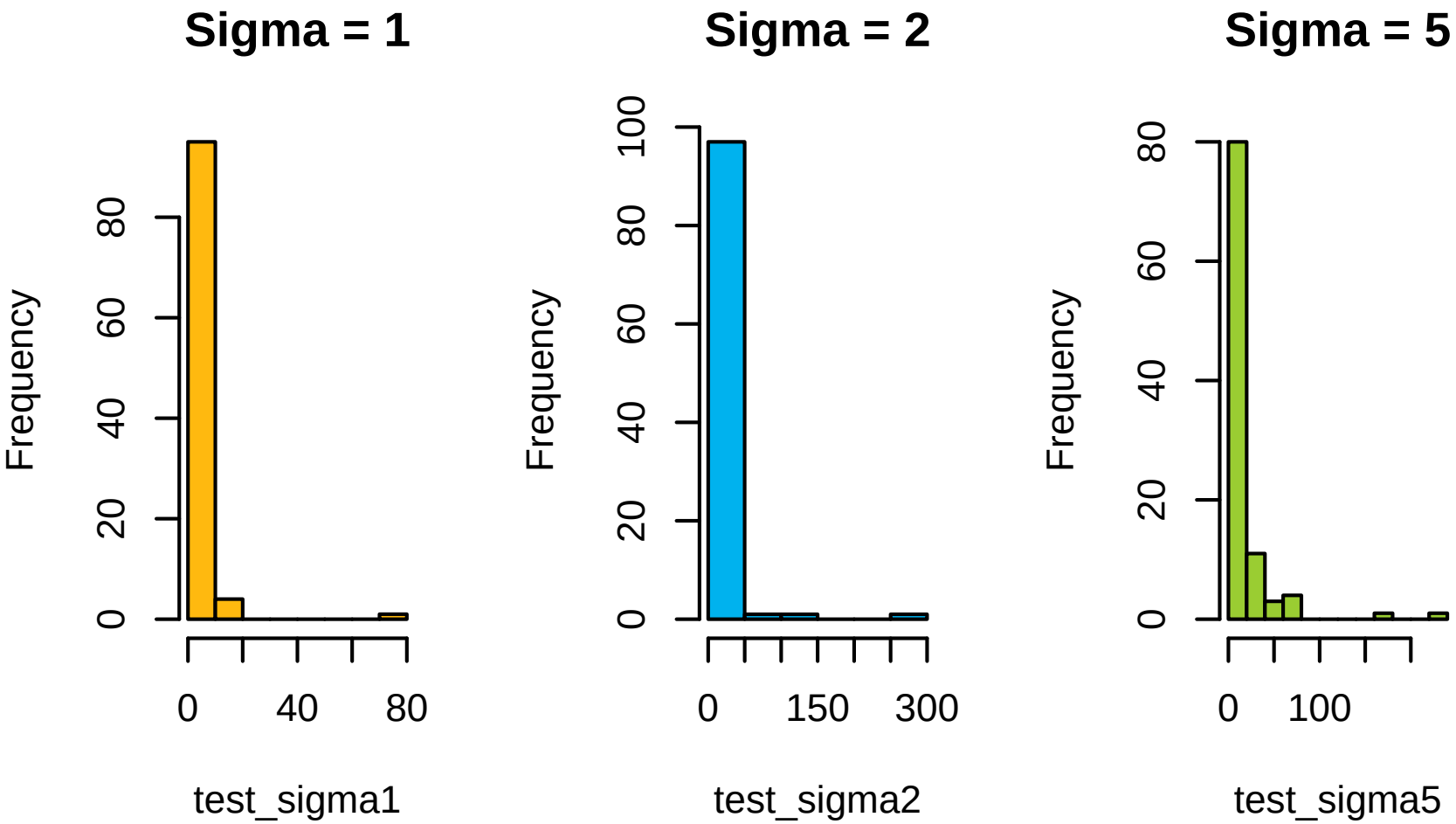

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The **Weibull distribution** is often used in survival data analysis to model lifetimes (e.g. the lifetimes of patients receiving a particular drug).

If $X \sim \mathit{Weibull}(\lambda, \kappa)$, then it has cumulative distribution function

$$F_X(x; \lambda, \kappa) = 1 - e^{-(x/\lambda)^\kappa}, \quad x \geq 0, \lambda, \kappa > 0.$$

Write a function in R to produce realisations from this distribution, and test your function out.

(A recent study suggested that, for a particular cancer drug, $\lambda = 2$ and $\kappa = 1.5$ for survival times in months after starting this drug).

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First, find the inverse CDF.

If $y = 1 - e^{-(x/\lambda)^\kappa}$, then

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and so

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In R:

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weibull.sim = function(n, lambda, kappa){
  U = runif(n)
  answer =  lambda*((-log(1-U))^(1/kappa))
  return(answer)}
```

Then, to test it out using the provided parameters:

```
test = weibull.sim(1000, 2, 1.5)
summary(test)
    Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
0.009993 0.879331 1.590474 1.779364 2.451390 6.215579

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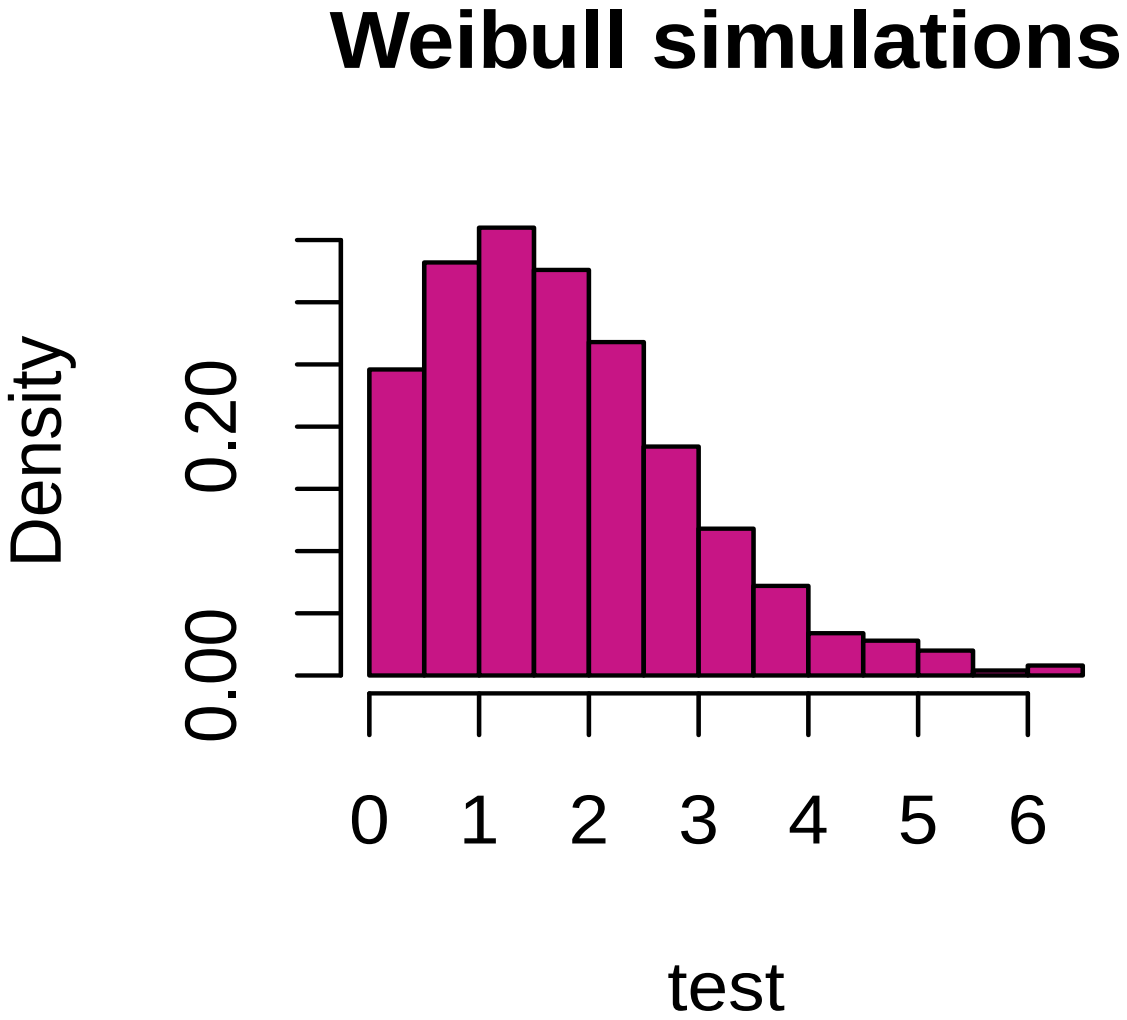

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We can check our work, as there is a built-in function in R to simulate from a Weibull!

We must note that $\lambda = 2$ is the **scale parameter** and $\kappa = 1.5$ is the **shape parameter**:

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test2 = rweibull(1000, scale=2, shape=1.5)
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Note that these are **similar** to the summaries from our own function. However, a more conclusive check might be to overlay the PDF.

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Example 4.4: Solution

Recall that to get from the CDF to the PDF, we differentiate; thus,

$$\begin{aligned} f_X(x) &= \frac{d}{dx} F_X(x) \\ &= \frac{d}{dx}\left(1 - e^{-(x/\lambda)^\kappa}\right). \end{aligned}$$

After careful (and multiple!) use of the chain rule, you should find that

$$f_X(x) = \frac{\kappa}{\lambda}\left(\frac{x}{\lambda}\right)^{\kappa-1} e^{-(x/\lambda)^\kappa}.$$

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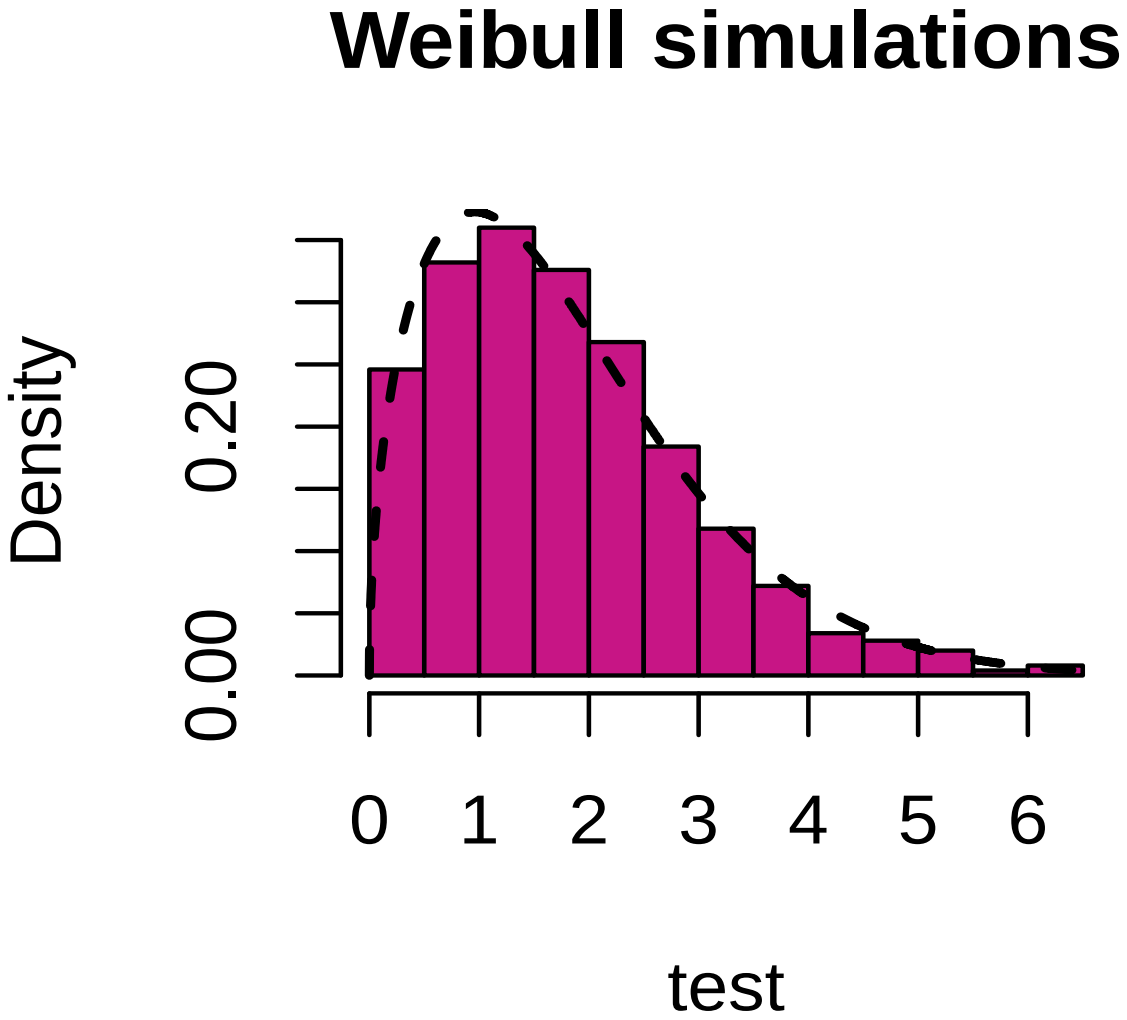
Example 4.4: Solution

So, we can add this to our plot in R:

```
weibull.pdf = function(x,lambda,kappa){
  answer = (kappa/lambda)*((x/lambda)^(kappa-1))*exp
    (-((x/lambda)^kappa))
  return(answer)}

x = seq(0, 7, 0.001)
y = weibull.pdf(x, 2, 1.5)
lines(x, y, type='l', lwd=2, lty=2)
```

Example 4.4: Solution



Example 4.5

- Patients arriving at a doctor's surgery have appointment times 15 minutes apart.
- The random variable L measures how late a patient is for his/her appointment, with negative values denoting early arrivals.

The PDF of L is

$$f_L(\ell) = \begin{cases} 0.0025(\ell + 20), & -20 \leq \ell \leq 0, \\ 0.05e^{-\ell/10}, & \ell > 0. \end{cases}$$

Example 4.5: Questions

a. Sketch $f_L(\ell)$.

b. Show that the CDF of L is

$$F_L(\ell) = \begin{cases} 0, & \ell < -20, \\ 0.00125(\ell + 20)^2, & -20 \leq \ell \leq 0, \\ 1 - \frac{e^{-\ell/10}}{2}, & \ell > 0. \end{cases}$$

c. Suppose that 0.205 and 0.713 are two realized values of $U \sim U(0, 1)$. Show that the corresponding realized values of L are -7.19 and 5.55 under the inverse CDF simulation method.

Example 4.5: Solution (a)

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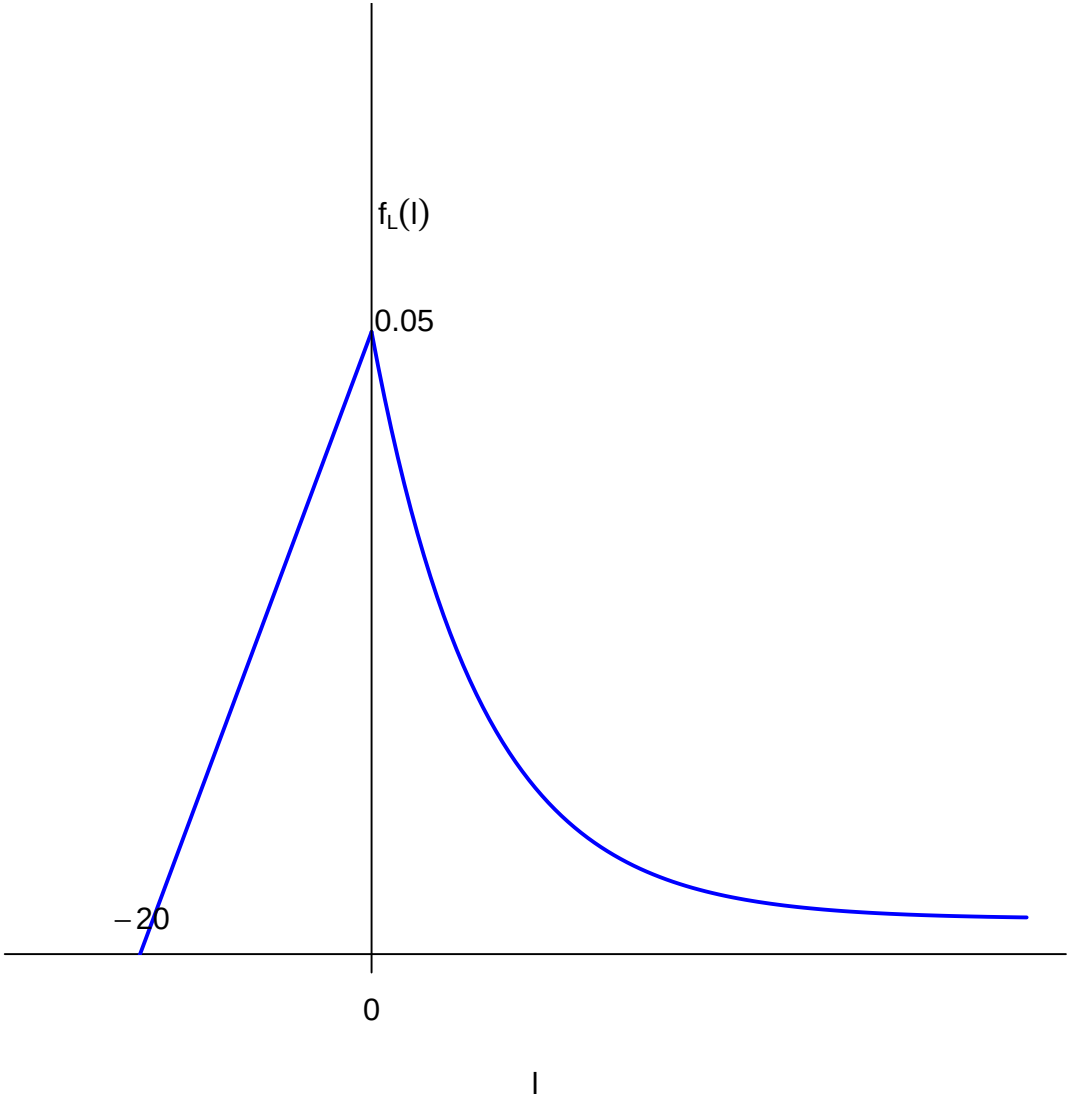

Example 4.5: Solution (b)

Now to find the CDF: L lies between -20 and $+\infty$, so $F_L(\ell) = 0$ when $\ell < -20$.

For $-20 \leq \ell \leq 0$:

$$\begin{aligned} F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\ &= \int_{-20}^{\ell} 0.0025(s+20)ds \\ &= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\ &= 0.00125(\ell+20)^2 \quad \text{for } -20 \leq \ell \leq 0. \end{aligned}$$

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$$\begin{aligned} F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\ &= \int_{-20}^{\ell} 0.0025(s+20)ds \\ &= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\ &= 0.00125(\ell+20)^2 \quad \text{for } -20 \leq \ell \leq 0. \end{aligned}$$

Also, $F_L(0) = 0.5$.

Example 4.5: Solution (b)

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Example 4.5: Solution (b)

Now

$$
\begin{aligned}
F_L(\ell) &= \int_{-20}^{0} f_L(s)ds + \int_{0}^{\ell} f_L(s)ds \quad \text{for } \ell > 0 \\
&= 0.5 + \int_{0}^{\ell} 0.05e^{-s/10}ds \\
&= 0.5 + \left[0.05 \times -10e^{-s/10}\right]_0^{\ell} \\
&= 0.5 + 0.5 - 0.5e^{-\ell/10} \\
&= 1 - \frac{e^{-\ell/10}}{2}, \quad \text{as required.}
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Example 4.5: Solution (c)

Now for some realized values:

$u = 0.205 < 0.5$, so realized value ℓ lies between -20 and 0. Thus

$$F_L(\ell) = 0.00125(\ell + 20)^2.$$

Find the inverse, and evaluate at $u = 0.205$:

$$\begin{aligned} y &= 0.00125(\ell + 20)^2, \quad \text{so} \\ \ell &= \pm\sqrt{\frac{y}{0.00125}} - 20, \end{aligned}$$

giving

$$\ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2.$$

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Similarly, $u = 0.713 > 0.5$ so realized value > 0. Thus

$$F_L(\ell) = 1 - \frac{e^{-\ell/10}}{2}.$$

Find the inverse, and evaluate at $u = 0.713$:

$$\begin{aligned} y &= 1 - \frac{e^{-\ell/10}}{2} \quad \text{so} \\ \ell &= -10 \log(2 - 2y), \end{aligned}$$

giving

$$\ell = -10 \log(2 - 2 \times 0.713) = 5.55.$$

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Synchronous Online: End 1pm, Wednesday 11th November

- That's us done for today, thanks for coming along!
- Non-Synchronous activities to be completed this week, in the following order:
  1. Complete **Test 1**: Deadline 16:00 Thursday 12th November (tomorrow!)
  2. R - E - L - A - X... next week is **Buffer Week**!
  3. See you in our computer practical sessions on **Tuesday 24th November**

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