

■ Course preface: What is Bayesian Statistics?

- Famous applications: Air France 447, cracking the Enigma machine
- Frequentists versus Bayesians
- History of Bayesian Statistics
- Bayesian Statistics at Newcastle

■ Different ways of calculating probabilities (Section 1.2)

■ Conditional Probability and Bayes Theorem (Section 1.3) pause

- Results and proofs
- Examples: blood test, car diagnosis, multiple choice exam

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Suppose that an experiment results in data $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ and we decide to model the data using a probability (density) function $f(\mathbf{x}|\theta)$.

This p(d)f describes how likely different data $\mathbf{x}$ are to occur given a value of the (unknown) parameter θ .

However, once we have observed the data, $f(\mathbf{x}|\theta)$ tells us how likely different values of the parameters θ are: it is then known as the **likelihood function** for θ .

In other courses you may have seen it written as $L(\theta|\mathbf{x})$ or $L(\theta)$ but, whatever the notation used for the likelihood function, it is simply the joint probability (density) function of the data, $f(\mathbf{x}|\theta)$, regarded as a function of θ rather than of $\mathbf{x}$.

The likelihood function can be simplified if we have further structure in the data.

For example, we may have independent observations, in which case

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n f_{X_i}(x_i|\theta), \quad (1.1)$$

or independent and identically distributed observations (random sample), so that

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n f_X(x_i|\theta). \quad (1.2)$$

Example 1.5

Suppose we have a random sample $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ of radioactive particle counts.

A typical model for such data would be $X_i|\theta \sim \text{Poisson}(\theta)$, usually abbreviated $X_i|\theta \sim \text{Po}(\theta)$, (independent).

Determine the likelihood function for θ .

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Determine the likelihood function for θ .

Solution to Example 1.5 (1/1)

Using Equation (1.2), the likelihood function is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \prod_{i=1}^n \frac{e^{-\theta} \theta^{x_i}}{x_i!} \\&= \frac{e^{-\theta} \times \dots \times e^{-\theta} \times \theta^{x_1} \times \dots \times \theta^{x_n}}{\prod_{i=1}^n x_i!} \\&= \frac{e^{-\theta-\theta-\dots-\theta} \theta^{x_1+\dots+x_n}}{\prod_{i=1}^n x_i!} \\&= \frac{e^{-n\theta} \theta^{\sum x_i}}{\prod_{i=1}^n x_i!} \\&= \frac{e^{-n\theta} \theta^{n\bar{x}}}{\prod_{i=1}^n x_i!}.\end{aligned}$$

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Example 1.6

Suppose we have a random sample $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ of times between radioactive particle emissions.

If the emissions occur randomly in time then a plausible model for such data would be $X_i|\theta \sim \text{Exponential}(\theta)$, usually abbreviated $X_i|\theta \sim \text{Exp}(\theta)$, (independent).

Determine the likelihood function for θ .

Solution to Example 1.6 (1/1)

Using Equation (1.2), the likelihood function is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \prod_{i=1}^n \theta e^{-\theta x_i} \\&= \theta \times \dots \times \theta \times e^{-\theta x_1} \times \dots \times e^{-\theta x_n} \\&= \theta^n e^{-\theta x_1 - \theta x_2 - \dots - \theta x_n} \\&= \theta^n e^{-\theta(x_1 + \dots + x_n)} \\&= \theta^n e^{-\theta n \bar{x}}.\end{aligned}$$

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Example 1.7

Suppose we have a random sample $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ from a Normal distribution: $X_i | \mu, \sigma \sim N(\mu, \sigma^2)$, $i = 1, 2, \dots, n$ (independent).

Determine the likelihood function for (μ, σ) .

Solution to Example 1.7 (1/1)

The (joint) probability density function is

$$\begin{aligned}f(\mathbf{x}|\mu, \sigma) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i - \mu)^2}{2\sigma^2}\right\} \\&= (2\pi)^{-n/2} \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right\} \\&= (2\pi)^{-n/2} \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2} \left(\sum x_i^2 - 2\mu \sum x_i + n\mu^2\right)\right\}.\end{aligned}$$

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Consider again the Poisson model in Example 1.5. The likelihood function is

$$\begin{aligned} f(\mathbf{x}|\theta) &= \frac{\theta^{n\bar{x}} e^{-n\theta}}{\prod_{i=1}^n x_i!} \\ &= \left(\prod_{i=1}^n x_i! \right)^{-1} \times \theta^{n\bar{x}} e^{-n\theta}. \end{aligned}$$

Notice that this likelihood function depends on the data only through $\prod_{i=1}^n (x_i!)^{-1}$ and $\bar{x}$.

Further, in $f(\mathbf{x}|\theta)$, θ only “interacts” with $\bar{x}$ — the other term simply scales $f(\mathbf{x}|\theta)$ — so that, for example, the point at which $f(\mathbf{x}|\theta)$ is maximized is determined only by $\bar{x}$.

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Informally, we think of all the information about θ in the data being contained in $\bar{x}$.

More formally, we can show that the distribution of the data given the value $\bar{x}$ does not depend on θ .

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A **statistic** is any function of the data (and not of unknown parameters).

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The statistic $T(\mathbf{X})$ is **sufficient** for θ if $f(\mathbf{x} | T(\mathbf{X}) = t)$ does not depend on θ .

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Example 1.8

Consider again the Poisson model in Example 1.5.

Suppose we had just two observations.

Then $n = 2$ and $X_i|\theta \sim Po(\theta)$, $i = 1, 2$ (independent).

Show that $T = X_1 + X_2$ is sufficient for θ . Note that $T|\theta \sim Po(2\theta)$.

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Solution to Example 1.8 (1/2)

Consider the conditional distribution of the data $\mathbf{X}$ on $T = t$.

As X_1 and X_2 are discrete random variables, we have

$$\begin{aligned}f(\mathbf{x}|T(\mathbf{X}) = t) &= \frac{\Pr(X_1 = x_1, X_2 = x_2, T = t)}{\Pr(T = t)} \\&= \frac{\Pr(X_1 = x_1, X_2 = t - x_1)}{\Pr(T = t)}.\end{aligned}$$

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Solution to Example 1.8 (1/2)

Consider the conditional distribution of the data $\mathbf{X}$ on $T = t$.

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Solution to Example 1.8 (2/2)

For the denominator we know that

$$\Pr(T = t|\theta) = \frac{e^{-2\theta}(2\theta)^t}{t!} \quad (\text{sum of two Poissons}).$$

Therefore

$$\begin{aligned} f(\mathbf{x}|T(\mathbf{X}) = t) &= \frac{e^{-2\theta}\theta^t}{x_1!(t-x_1)!} \bigg/ \frac{e^{-2\theta}(2\theta)^t}{t!} \\ &= \frac{t!e^{-2\theta}\theta^t}{x_1!(t-x_1)!e^{-2\theta}(2\theta)^t} \\ &= \frac{t!}{2^t x_1!(t-x_1)!} \end{aligned}$$

which does not depend on θ . Hence, by Definition 1.4, $T = X_1 + X_2$ is sufficient for θ .

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Definition (Joint sufficiency)

The statistics $\underline{T}(\mathbf{X}) = (T_1(\mathbf{X}), T_2(\mathbf{X}), \dots, T_k(\mathbf{X}))^T$ are **jointly sufficient** for θ if $f(\mathbf{x} | \underline{T}(\mathbf{X}) = \mathbf{t})$ does not depend on θ .

Theorem (Factorisation Theorem)

Under certain regularity conditions

$$\underline{T}(\mathbf{X}) \text{ is sufficient for } \theta \iff f(\mathbf{x} | \theta) = h(\mathbf{x}) g(\underline{t}(\mathbf{x}), \theta)$$

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Example 1.9

Consider again the Poisson model in Example 1.5 with $n = 2$: $X_i|\theta \sim Po(\theta)$, $i = 1, 2$ (independent). Determine a sufficient statistic for θ .

Solution to Example 1.9 (1/1)

The (joint) probability function is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \frac{e^{-\theta}\theta^{x_1}}{x_1!} \times \frac{e^{-\theta}\theta^{x_2}}{x_2!} \\&= \frac{e^{-2\theta}\theta^{x_1+x_2}}{x_1!x_2!} \\&= \frac{1}{x_1!x_2!} \times e^{-2\theta}\theta^{x_1+x_2} \\&= h(\mathbf{x}) g(x_1 + x_2, \theta),\end{aligned}$$

where $h(\mathbf{x}) = 1/(x_1!x_2!)$ and $g(t, \theta) = e^{-2\theta}\theta^t$.

Therefore, by the Factorisation Theorem, $T = X_1 + X_2$ is sufficient for θ .

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Example 1.10

Suppose we have a random sample from an exponential distribution: $X_i|\theta \sim \text{Exp}(\theta)$, $i = 1, 2, \dots, n$ (independent).

Determine a sufficient statistic for θ .

Solution to Example 1.10 (1/1)

The (joint) probability density function is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \prod_{i=1}^n \theta e^{-\theta x_i} \\&= \theta^n \exp\left(-\theta \sum_{i=1}^n x_i\right) \\&= 1 \times \theta^n \exp\left(-\theta \sum_{i=1}^n x_i\right) \\&= h(\mathbf{x}) g(\sum x_i, \theta),\end{aligned}$$

where $h(\mathbf{x}) = 1$ and $g(t, \theta) = \theta^n e^{-\theta t}$. Therefore, by the Factorisation Theorem, $T = \sum_{i=1}^n X_i$ is sufficient for θ .

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