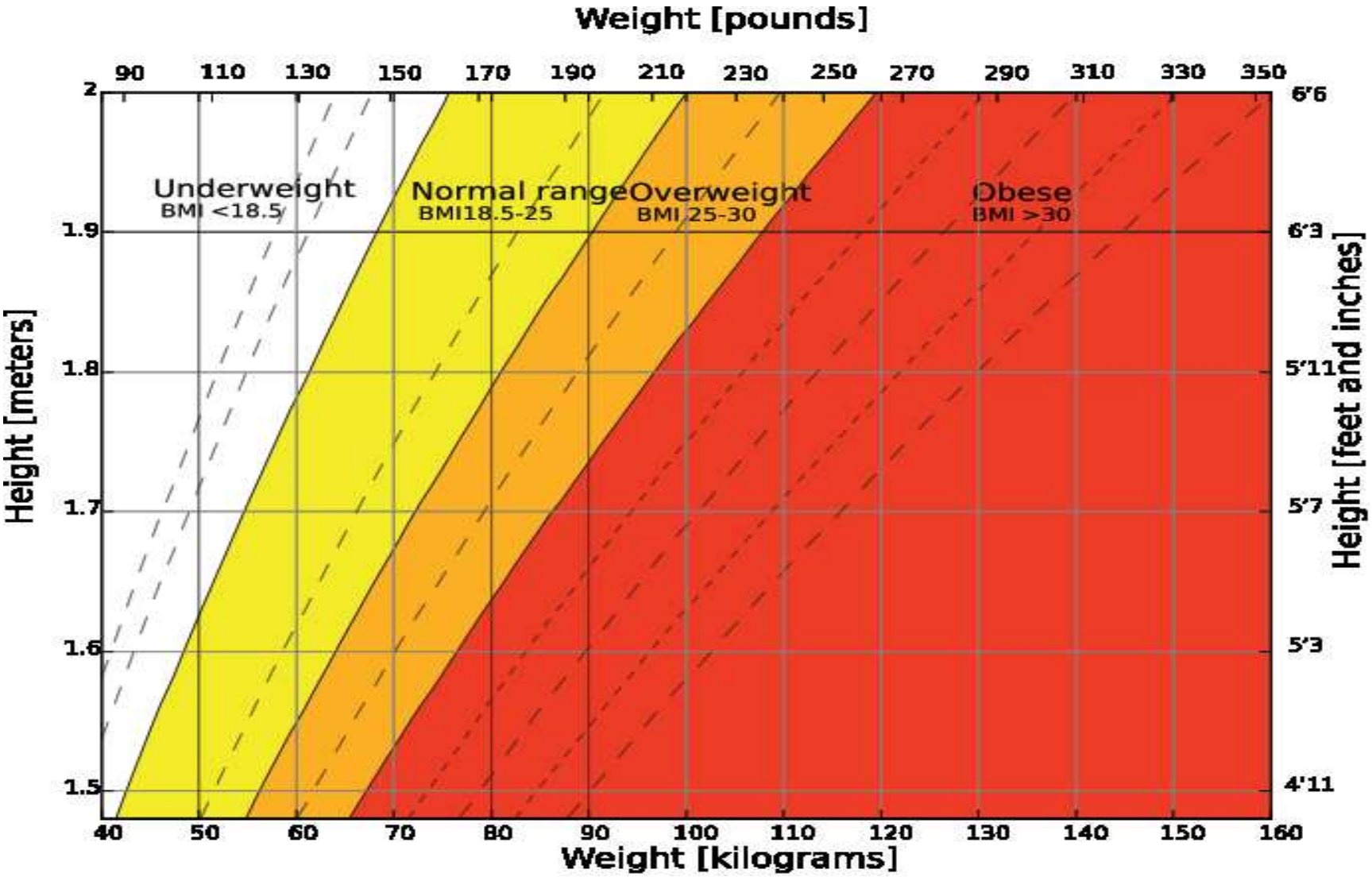
Body Mass Index



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Suppose we obtain the following BMI values for a random sample of students from two groups:

Maths & Stats	23.1	28.3	35.3	24.0	32.4	30.5	30.1	21.9	22.1
	29.9	27.2	33.2						
Food & Human Nutrition	25.5	27.3	21.0	23.3	19.1	22.2	29.5	27.5	23.2

From this we can obtain the summaries:

$$n_M = 12 \qquad \bar{x}_M = 28.17 \qquad s_M = 4.54$$

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The pooled standard deviation is

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\begin{aligned}
s &= \sqrt{\frac{(n_M - 1)s_M^2 + (n_F - 1)s_F^2}{n_M + n_F - 2}} \\
&= \sqrt{\frac{11 \times 4.54^2 + 8 \times 3.4^2}{12 + 9 - 2}} \\
&= 4.099
\end{aligned}
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Then the test statistic is

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\begin{aligned}
t &= \frac{|\bar{x}_M - \bar{x}_F|}{s \times \sqrt{\frac{1}{n_M} + \frac{1}{n_F}}} \\
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From t_{19}–tables, we get:

p–value	10%	5%	1%
critical value	1.729	2.093	2.861

Our test statistic $t = 2.147$ indicates that our p–value is less than 5%:

- We have a significant result at the 5% level of significance
- We **reject** H_0
- We conclude that there *is* a significant difference in BMI between M&S and FHN students

Body Mass Index

Notice from R that the *exact* p–value is 0.004488 = 4.488%.

- As we have already concluded, we would **reject** H_0 at the 5% level;
- However, if we work at the 1% level of significance, we would **retain** H_0!

Convention tells us to work at the 5% level, but why? Some practitioners work at the 1% level. Do we need to collect more data and re-analyse?

Body Mass Index

Obtain a 95% confidence interval for the population mean BMI for Maths & Stats students.

We use

$$\bar{x}_M \pm t\frac{s_M}{\sqrt{n_M}}, \qquad \text{giving}$$

$$28.17 \quad \pm \quad 2.201 \times \frac{4.54}{\sqrt{12}} \qquad \text{i.e.}$$

$$28.17 \quad \pm \quad 2.743,$$

giving (25.28, 31.05).

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NOT 0.95!! In the frequentist approach, population parameters (in this case μ_M) are NOT random variables!

This means μ_M is a fixed (but unknown) quantity. So it's either *in* the interval, or it's *not*, i.e.

$$\Pr(25.28 < \mu_M < 31.05) = 1 \qquad \text{or}$$

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Nothing else!

Equivalent intervals in the **Bayesian framework** have more natural interpretations!

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Quick quiz

Suppose A, θ and μ_M are constants and Z is a random variable, such that $A = 5$, $\theta = 10$, $\mu_M = 33.5$ and $Z \sim N(0, 1)$.

Write down

1. $\Pr(Z > 0)$
2. $\Pr(-1.96 < Z < 1.96)$
3. $\Pr(2 < A < 4)$
4. $\Pr(7 < \theta < 12)$
5. $\Pr(25.28 < \mu_M < 31.05)$

Chapter 1

Introduction

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Consider the following three experiments.

- **Experiment 1: Fisher's tea lady**
  The tea lady claims to know whether milk or tea is poured in first: for 10 pairs of cups of tea she makes the correct choice each time.
- **Experiment 2: Music expert**
  The expert claims he can distinguish between a page from a Haydn score and a page from a Mozart score: he does so correctly 10 times.
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Let $\theta = \Pr(\text{correct choice})$.

Let's suppose the tea lady, the music expert and the drunk *cannot* do as they claim.

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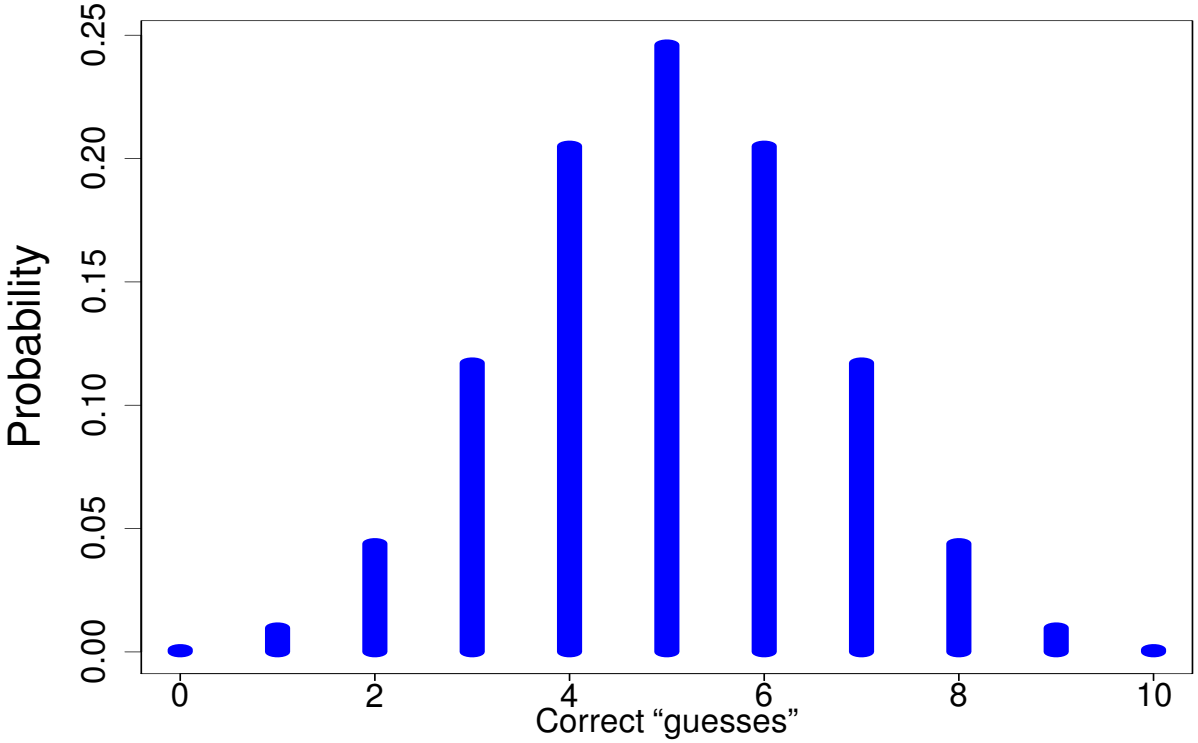
Graphically, assuming H_0 is true, this gives:



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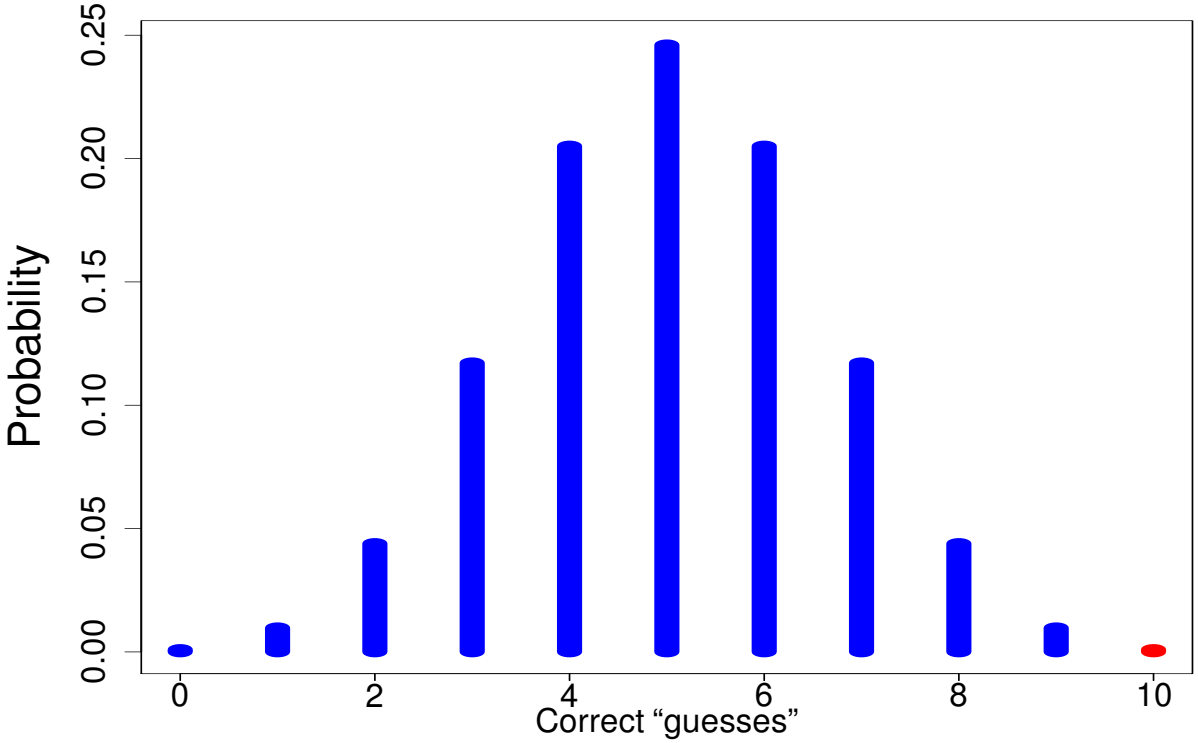

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This gives us:

$$p\text{−value} = \left(\frac{1}{2}\right)^{10} = 0.00098 < 0.1\%.$$

From this $p-$value, we would:

- conclude that we had **very strong evidence** against H_0
- conclude that the choices were **not just guesses**
- perhaps feel justified in **validating each claim**

But does this make sense?

Surely, we have some additional information about what values of θ are **plausible** for each experiment?

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The traditional approach to Statistics, sometimes called **Frequentist Statistics** or **Classical Statistics**, may try to take this prior information into account:

For example, in Experiment 2, the test may be of $H_0 : \theta \geq 0.9$ against $H_1 : \theta < 0.9$.

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The key ingredients of a Bayesian analysis are

- a statistical model for the experimental data;
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Probability

The concept of probability (chance) has been around for a very long time, particularly in the area of **gambling**.

Games of chance have been played since about **3500 B.C.**

The Egyptians started using cubical dice around **2000 B.C.**

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The probability of an outcome is the **relative frequency** with which the outcome would be obtained if the experiment were repeated a large number of times *under similar conditions*.

For example, if a coin is tossed 1,000,000 times and a head appears n times then

$$\Pr(\text{Head}) = \frac{n}{1,000,000}.$$

We would expect this probability to be about 0.5.

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Your **subjective probability** for an outcome A represents your own judgement of the likelihood that the outcome will occur.

This judgement will be based on the beliefs and information H you have at the time.

One way of determining (or quantifying) a subjective value for $\Pr_H(A)$ is to consider a series of possible bets with outcome

$$\text{win £}c \text{ if } A \text{ occurs and £0 if } A^c \text{ occurs.}$$

How much would you be prepared to pay (stake) for placing such a bet?

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Subjective interpretation (1/1)

In terms of expected winnings, you should be prepared to stake £cp if you believe that $\Pr_H(A) = p$. Why?

Suppose you would win £10 if A occurs and £0 if it does not. Suppose further that you believe A is likely to occur with probability $p = 0.3$.

Then the **expected monetary value** of the bet is
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One problem with this approach is that, in general, p will depend on c.

A person who is willing to bet £1 on the spin of a coin to win £2 if it lands heads may refuse to bet if the stakes are raised to £1000: most people are **risk–averse**.

Therefore, we shall restrict our attention to the $c = 1$ case: pay £p for the bet

$$\text{win £1 if } A \text{ occurs and £0 if } A^c \text{ occurs.}$$

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You can make sure that your bet is "honest" by randomising between whether you "**host**" the bet or "**place**" the bet.

For example, suppose you believe that $\Pr_H(A) = 0.5$.

An "honest" bet would mean that you would buy the bet for a maximum stake of £0.50.

However, if you weren't honest you might try to buy the bet for any amount less than £0.50, say £0.20.

If you were hosting the bet, you would take the bet for any amount more than £p, say for £0.80.

These conflicting interests can be offset if, when choosing p, it is equally likely that you are hosting the bet or placing the bet.

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Frequency interpretation: drawbacks

1. It does not say how many times the experiment should be repeated
2. "Similar conditions" is a vague concept
3. It is not appropriate for many probability calculations of one-off events
4. Standard statistical methods using the frequentist approach are not totally objective since they require subjective judgements about the validity of probability models, choice of hypotheses and interpretation of results (for instance, see BMI example in the preface to these lecture notes)

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1. It is not objective – but perhaps it is more obvious (honest) about when subjective beliefs are used
2. It requires people to be **coherent**: they will not make any wagers which they are certain to lose; also, they will not prefer to suffer a given penalty when there is the option of another penalty which is certainly smaller

   Being coherent results in, *inter alia*, that

$$\Pr(A_1|H) > \Pr(A_2|H) \quad \text{and} \quad \Pr(A_2|H) > \Pr(A_3|H) \implies \Pr(A_1|H) > \Pr(A_3|H).$$

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Probability

Each of these interpretations use quite different methods of reasoning.

In this course – unlike any other course you have taken so far – we will concentrate on the subjective interpretation and describe how, if carefully used, it can be a more useful approach than the other two methods.

Everything we do rests on **Bayes' Theorem**, and we review this now.

Example 1.1

1. The probability
   $\Pr(\text{Newcastle Utd win the Championship this season})$
   could only be determined using a subjective assessment.
2. The probability
   $\Pr(\text{M\&S student chosen at random was born in January})$
   could be determined using a frequency or classical interpretation (with a list of all M&S students and their birth dates) or a subjective interpretation.
3. The probability
   $\Pr(\text{England win the toss at a given Test Match})$ could be determined using either the classical or subjective interpretation.

Bayes' Theorem

Before we state **Bayes' Theorem**, we need a recap of **conditional probability**.

Definition (Conditional Probability)

Consider two events E and F, where $\Pr(F) > 0$.

The **conditional probability** of E given that F has occurred is

$$\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$

Bayes' Theorem

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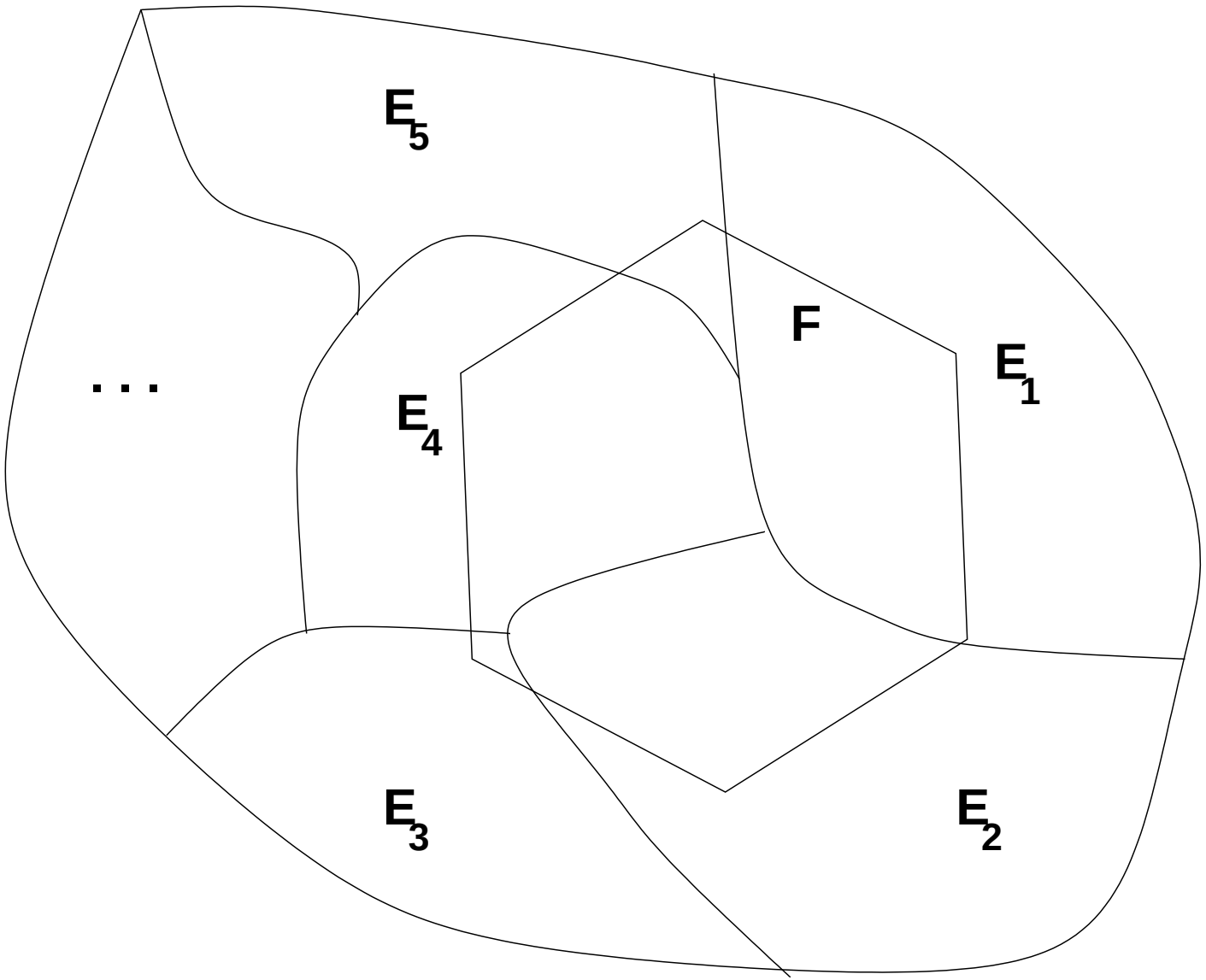
Bayes' Theorem

Definition (Partition)

The events $E_1, E_2, \ldots, E_n$ form a **partition** of the sample space $\mathcal{S}$ if they are disjoint events ($E_i \cap E_j = \emptyset$, $i \neq j$) with $\Pr(E_i) > 0$, $i = 1, 2, \ldots, n$, and $\cup_{i=1}^{n} E_i = \mathcal{S}$.

Figure 1.1 gives a diagram of a typical partition with an additional event F.

Bayes' Theorem



Bayes' Theorem

Theorem (Law of Total Probability)

If $E_1, E_2, \ldots, E_n$ are a **partition** *of $\mathcal{S}$ and F is any event then*

$$\Pr(F) = \sum_{i=1}^{n} \Pr(F|E_i)\Pr(E_i).$$

Law of Total Probability: Proof

As $E_1, E_2, \ldots, E_n$ are a **partition** of $\mathcal{S}$, we have

$$\Pr(F) = \Pr(F \cap E_1) + \Pr(F \cap E_2) + \ldots + \Pr(F \cap E_n)$$

$$= \Pr(F|E_1)\Pr(E_1) + \Pr(F|E_2)\Pr(E_2) + \ldots + \Pr(F|E_n)\Pr(E_n)$$

(by conditional probability)

$$= \sum_{i=1}^{n} \Pr(F|E_i)\Pr(E_i).$$

Bayes' Theorem

Theorem (Bayes' Theorem)

If $E_1, E_2, \ldots, E_n$ *are a* **partition** *of* $\mathcal{S}$ *and* F *is any event with* $\Pr(F) > 0$ *then*

$$\Pr(E_i|F) = \frac{\Pr(F|E_i)\Pr(E_i)}{\sum_{j=1}^{n} \Pr(F|E_j)\Pr(E_j)}, \qquad i = 1, 2, \ldots, n.$$

Bayes' Theorem: Proof (slide 1/1)

Using the definition of conditional probability, for $i = 1, 2, \ldots, n$,

$$\Pr(E_i|F) = \frac{\Pr(E_i \cap F)}{\Pr(F)}$$

$$= \frac{\Pr(F \cap E_i)}{\Pr(F)}$$

$$= \frac{\Pr(F|E_i)\Pr(E_i)}{\Pr(F)} \quad \text{again using the definition}$$

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Example 1.2

A laboratory blood test is 95% effective in detecting a certain disease when it is present.

However, the test also yields a “false positive” result for 1% of healthy people tested.

Also, 0.5% of the population actually have the disease.

(a) Calculate the probability that a person who tests positive actually has the disease.

(b) Find the probability that a person who tests negative does *not* have the disease.

Solution to Example 1.2 (slide 1/3)

Let D = person has the disease, +ve = result is postive and –ve = result is negative.

We require (a) $\Pr(D|\text{+ve})$, and (b) $\Pr(D^c|\text{–ve})$.

From the question we have

- $\Pr(\text{+ve}|D) = 0.95$
- $\Pr(\text{+ve}|D^c) = 0.01$
- $\Pr(D) = 0.005$ and
- $\Pr(D^c) = 0.995$

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Solution to Example 1.2 (slide 2/3)

Also, D and D^c form a partition of $\mathcal{S}$. Therefore, by Bayes' Theorem

$$\Pr(D|+\text{ve}) = \frac{\Pr(+\text{ve}|D)\Pr(D)}{\Pr(+\text{ve}|D)\Pr(D) + \Pr(+\text{ve}|D^c)\Pr(D^c)}$$

$$= \frac{0.95 \times 0.005}{0.95 \times 0.005 \; + \; 0.01 \times 0.995}$$

$$\simeq 0.323.$$

Thus, only 32.3% of people who test positive actually have the disease – 67.7% of people who test positive do *not* have the disease!

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Solution to Example 1.2 (slide 3/3)

Now for part (b),

$$\Pr(D^c|\text{–ve}) = \frac{\Pr(\text{–ve}|D^c)\Pr(D^c)}{\Pr(\text{–ve}|D^c)\Pr(D^c) + \Pr(\text{–ve}|D)\Pr(D)}$$

$$= \frac{0.99 \times 0.995}{0.99 \times 0.995 \ + \ 0.05 \times 0.005}$$

$$\simeq 0.9997.$$

Therefore, nearly everyone who tests negative does not have the disease – very few people who test negative *will* have the disease.

Solution to Example 1.2 (slide 3/3)

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Example 1.3

Suppose that your car suffers from two intermittent problems, one caused by a fault in the engine (θ_1) and the other due to a fault in the gearbox (θ_2).

These occur with probabilities 0.4 and 0.6 respectively.

When examined your car exhibits one of the following symptoms

$$
\begin{aligned}
x_1 &: \text{overheating only,} \\
x_2 &: \text{irregular traction only,} \\
x_3 &: \text{both symptoms.}
\end{aligned}
$$

Example 1.3

Suppose it is known in the garage trade that these symptoms occur with probabilities that depend on the fault.

The probabilities $\Pr(X = x|\theta)$ are given in Table 1.1.

	O/H x_1	I/T x_2	Both x_3
θ_1: fault in engine	0.1	0.4	0.5
θ_2: fault in gearbox	0.5	0.3	0.2

Construct a diagnostic rule for these symptoms and determine the probability of misdiagnosis.

Solution to Example 1.3 (1/3)

First we must calculate the **posterior probabilities** $\Pr(\theta_1|x)$ and $\Pr(\theta_2|x)$ for $x = x_1, x_2, x_3$.

Since θ_1 and θ_2 form a partition, we can use Bayes' Theorem as follows.

We have

$$\Pr(\theta_1|x_1) = \frac{\Pr(X = x_1|\theta_1)\Pr(\theta_1)}{\Pr(X = x_1|\theta_1)\Pr(\theta_1) + \Pr(X = x_1|\theta_2)\Pr(\theta_2)}$$

$$= \frac{0.1 \times 0.4}{0.1 \times 0.4 + 0.5 \times 0.6}$$

$$= \frac{4}{34} = 0.118.$$

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Solution to Example 1.3 (2/3)

Also,

$$\Pr(\theta_1|x_2) = \frac{\Pr(X = x_2|\theta_1)\Pr(\theta_1)}{\Pr(X = x_2|\theta_1)\Pr(\theta_1) + \Pr(X = x_2|\theta_2)\Pr(\theta_2)}$$

$$= \frac{0.4 \times 0.4}{0.4 \times 0.4 + 0.3 \times 0.6}$$

$$= \frac{16}{34} = 0.471.$$

Solution to Example 1.3 (2/3)

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Solution to Example 1.3 (3/3)

And

$$\Pr(\theta_1|x_3) = \frac{\Pr(X = x_3|\theta_1)\Pr(\theta_1)}{\Pr(X = x_3|\theta_1)\Pr(\theta_1) + \Pr(X = x_3|\theta_2)\Pr(\theta_2)}$$

$$= \frac{0.5 \times 0.4}{0.5 \times 0.4 + 0.2 \times 0.6}$$

$$= \frac{20}{32} = 0.625.$$

Also, $\Pr(\theta_2|x_i) = 1 - \Pr(\theta_1|x_i)$, $i = 1, 2, 3$, and so we obtain the **posterior distributions** $\Pr(\theta|x)$ given in Table 1.2.

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Example 1.3

	O/H x_1	I/T x_2	Both x_3
θ_1: fault in engine	0.118	0.471	0.625
θ_2: fault in gearbox	0.882	0.529	0.375

This table is very informative. For example, it shows that if both symptoms (x_3) are observed, then the probability that the fault is in the engine (θ_1) changes from 0.4 to 0.625.

In terms of **odds**:

$$\text{Prior odds}: \frac{\Pr(\theta_1)}{\Pr(\theta_2)} = \frac{0.4}{0.6} = \frac{2}{3} \quad \text{or 3:2 in favour of } \theta_2$$

$$\text{Posterior odds}: \frac{\Pr(\theta_1|x_3)}{\Pr(\theta_2|x_3)} = \frac{0.625}{0.375} = \frac{5}{3} \quad \text{or 5:3 in favour of } \theta_1.$$

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$$\text{Posterior odds}: \frac{\Pr(\theta_1|x_3)}{\Pr(\theta_2|x_3)} = \frac{0.625}{0.375} = \frac{5}{3} \quad \text{or 5:3 in favour of } \theta_1.$$

Example 1.3

	O/H x_1	I/T x_2	Both x_3
θ_1: fault in engine	0.118	0.471	0.625
θ_2: fault in gearbox	0.882	0.529	0.375

This table is very informative. For example, it shows that if both symptoms (x_3) are observed, then the probability that the fault is in the engine (θ_1) changes from 0.4 to 0.625.

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Example 1.3

We are now in a position to design our **diagnostic rule**.

This is simply a rule which diagnoses a symptom (x) as being due to some particular fault (θ).

Consider first that we observe overheating only (x_1). The posterior probabilities are in favour of declaring the fault as in the gearbox (θ_2) since $\Pr(\theta_2|x_1) > \Pr(\theta_1|x_1)$.

In the same way, we can determine the most likely diagnosis having observed irregular traction only (x_2) and both symptoms (x_3), giving the diagnostic rule in Table 1.3.

Symptom	Diagnosis
overheating only (x_1)	fault in gearbox (θ_2)
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Solution to Example 1.3 (1/1)

If we were to carry out this diagnostic rule repeatedly then the probability of misdiagnosing a fault is

$$\begin{aligned}
&\Pr(\text{Misdiagnosis}) \\
&= \Pr(\theta_1, X = x_1) + \Pr(\theta_1, X = x_2) + \Pr(\theta_2, X = x_3) \\
&= \Pr(X = x_1|\theta_1)\Pr(\theta_1) + \Pr(X = x_2|\theta_1)\Pr(\theta_1) \\
&\qquad + \Pr(X = x_3|\theta_2)\Pr(\theta_2) \\
&= (0.1 \times 0.4) + (0.4 \times 0.4) + (0.2 \times 0.6) \\
&= 0.32.
\end{aligned}$$

Therefore, in repeated use of this rule, around a third of the diagnoses will be wrong.

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Example 1.4

A student sits a multiple choice exam in which there are m alternative answers to each question.

The student either knows the answer (with probability θ) or guesses randomly (with probability $1 - \theta$).

What is the probability that the student actually knew the answer to a question they answered correctly?