

Home Study 6: To be completed by 12:00, Wednesday 2nd December

MAS2905: Probability and Bayesian Methods with R

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Semester 1, 2020/21

The story so far...

- You should now have completed **all of Chapter 4**
- This includes videos for **Home Study 4** and **Home Study 5** and the **Computer Practical** from last week
- Please now make sure you have the course notes for Chapter 5 in front of you
- Don't forget: We will be making a start on the project this week

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Recap Quiz

1. Write down R code that simulates 100 observations from the $U(0, 1)$ distribution and stores the output in the vector `U`.
2. Suppose X has CDF

$$F_X(x) = 1 - \frac{1}{x}, \quad x \geq 1.$$

Find $F_X^{-1}(x)$.

3. Write an R function called `inv.cdf` to generate values from X .
4. Produce a histogram of your realizations of X , coloured yellow and with a suitable title.

5.1 Transforming a normal random variable

- Consider how we might model a **continuous positive** random quantity e.g. blood pressure or height
- The Normal distribution is not suitable!

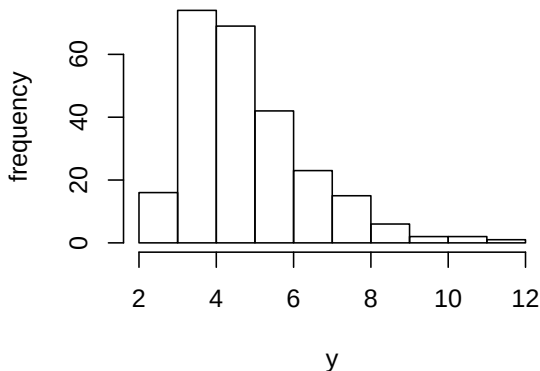
One way to obtain a positive random variable is to define

$$Y = \exp(X), \quad \text{where } X \sim N(\mu, \sigma^2).$$

Then Y is a random variable which is a **transformation** of X .

5.1 Simulation to explore transformation

```
1 y = exp(rnorm(250, mean = 1.5, sd = 0.3))  
2 hist(y, xlab = 'Y', ylab = 'frequency')
```



5.1 Calculating PDFs

- This looks like a useful distribution – what is its PDF?
- $Y = \exp(X)$ so $X = \log(Y)$
- So $f_Y(y) = f_X(\log(y))$ is a good guess, but is **NOT** the right answer!

Example

If $X \sim U(0, 1)$, what is the PDF of $Y = X^3$?

Example 5.1: Solution

$$\begin{aligned}F_Y(y) &= \Pr(Y \leq y) \\&= \Pr(X^3 \leq y) \\&= \Pr(X \leq y^{1/3}) \\&= F_X(y^{1/3}).\end{aligned}$$

Therefore

$$\begin{aligned}f_Y(y) &= \frac{d}{dy} F_X(y^{1/3}) \\&= f_X(y^{1/3}) \times \frac{1}{3} y^{-2/3} \quad \text{by chain rule} \\&= \frac{1}{3} y^{-2/3}, \quad 0 \leq y \leq 1 \text{ (and zero otherwise).}\end{aligned}$$

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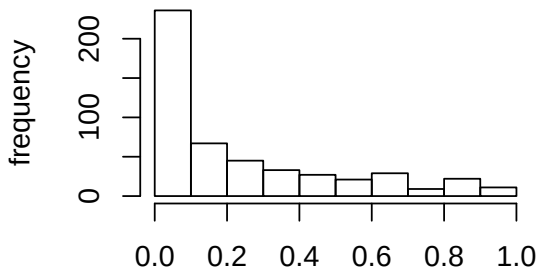
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Example 5.1: Checking via simulation

To confirm this result, we can simulate Y :

```
1 y = (runif(500))^3
2 hist(y, xlab = 'Y', ylab = 'frequency')
```



Example 5.2

Example

If $X \sim N(0, 1)$, what is the PDF of $Y = X^2$?

$$F_Y(y) = \Pr(Y \leq y) = \Pr(X \leq y^{1/2}) = F_X(y^{1/2}).$$

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$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_X(y^{1/2}) \\ &= f_X(y^{1/2}) \times \frac{1}{2} y^{-1/2} \quad \text{by chain rule} \\ &= \frac{1}{\sqrt{2\pi}} e^{-y/2} \times \frac{1}{2} y^{-1/2} \\ &= \frac{1}{2\sqrt{2\pi y}} e^{-y/2} \quad \text{for } y \geq 0, \text{ zero otherwise.} \end{aligned}$$

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Example 5.3: To try in your own time!

Example

If $X \sim N(\mu, \sigma^2)$, what is the PDF of $Y = \exp(X)$?

This is the example used at the start of this section of the notes.

We say Y has the **log-normal** distribution and write $Y \sim LN(\mu, \sigma^2)$.

$$F_Y(y) = F_X(\log y) \quad \text{by usual argument with CDF}$$

$$f_Y(y) = \frac{d}{dy} F_X(\log y)$$

$$= f_X(\log y) \times \frac{1}{y} \quad \text{for } y \geq 0$$

$$= \frac{1}{\sigma y \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left(\frac{(\log y - \mu)^2}{\sigma^2} \right) \right\}.$$

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5.3 The transformation method

If X has PDF $f_X(x)$ and $h(x)$ is a monotonically increasing or monotonically decreasing function then

- h is invertible and
- the PDF of $Y = h(X)$ is

$$f_Y(y) = f_X(h^{-1}(y)) \times \left| \frac{d(h^{-1}(y))}{dy} \right|$$

when y is in the range of $h(x)$ and zero otherwise.

This result allows direct calculation of $f_Y(y)$ and is called the **univariate transformation method**.

See Lee's scribbles for Proof.