

# *Home Study 6: To be completed by 12:00, Wednesday 2nd December*

## MAS2905: Probability and Bayesian Methods with R

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Semester 1, 2020/21

# The story so far...

- You should now have completed **all of Chapter 4**
- This includes videos for **Home Study 4** and **Home Study 5** and the **Computer Practical** from last week
- Please now make sure you have the course notes for
- Don't forget: You should now be making some inroads into the project!

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# Recap Quiz

1. Write down R code that simulates 100 observations from the  $U(0, 1)$  distribution and stores the output in the vector `U`.
2. Suppose  $X$  has CDF

$$F_X(x) = 1 - \frac{1}{x}, \quad x \geq 1.$$

Find  $F_X^{-1}(x)$ .

3. Write an R function called `inv.cdf` to generate values from  $X$ .
4. Produce a histogram of your realizations of  $X$ , coloured yellow and with a suitable title.

## 5.1 Transforming a normal random variable

- Consider how we might model a **continuous positive** random quantity e.g. blood pressure or height
- The Normal distribution is not suitable!

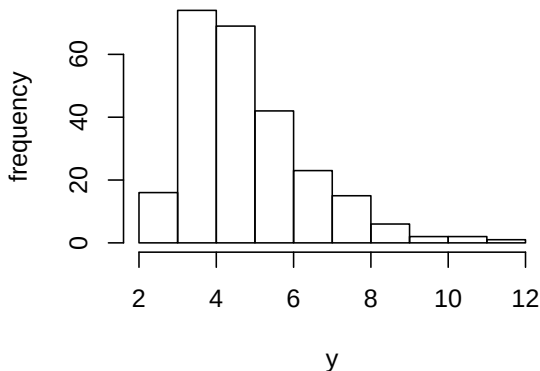
One way to obtain a positive random variable is to define

$$Y = \exp(X), \quad \text{where } X \sim N(\mu, \sigma^2).$$

Then  $Y$  is a random variable which is a **transformation** of  $X$ .

## 5.1 Simulation to explore transformation

```
1 y = exp(rnorm(250, mean = 1.5, sd = 0.3))  
2 hist(y, xlab = 'Y', ylab = 'frequency')
```



## 5.1 Calculating PDFs

- This looks like a useful distribution – what is its PDF?
- $Y = \exp(X)$  so  $X = \log(Y)$
- So  $f_Y(y) = f_X(\log(y))$  is a good guess, but is **NOT** the right answer!

### Example

If  $X \sim U(0, 1)$ , what is the PDF of  $Y = X^3$ ?

## Example 5.1: Solution

$$\begin{aligned}F_Y(y) &= \Pr(Y \leq y) \\&= \Pr(X^3 \leq y) \\&= \Pr(X \leq y^{1/3}) \\&= F_X(y^{1/3}).\end{aligned}$$

Therefore

$$\begin{aligned}f_Y(y) &= \frac{d}{dy} F_X(y^{1/3}) \\&= f_X(y^{1/3}) \times \frac{1}{3} y^{-2/3} \quad \text{by chain rule} \\&= \frac{1}{3} y^{-2/3}, \quad 0 \leq y \leq 1 \text{ (and zero otherwise).}\end{aligned}$$

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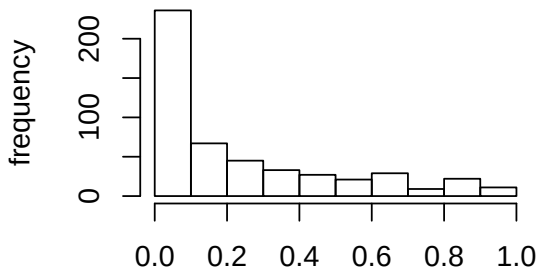
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## Example 5.1: Checking via simulation

To confirm this result, we can simulate  $Y$ :

```
1 y = (runif(500))^3
2 hist(y, xlab = 'Y', ylab = 'frequency')
```



## Example 5.2

### Example

If  $X \sim N(0, 1)$ , what is the PDF of  $Y = X^2$ ?

$$F_Y(y) = \Pr(Y \leq y) = \Pr(X \leq y^{1/2}) = F_X(y^{1/2}).$$

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## Example 5.3: To try in your own time!

### Example

If  $X \sim N(\mu, \sigma^2)$ , what is the PDF of  $Y = \exp(X)$ ?

This is the example used at the start of this section of the notes.

We say  $Y$  has the **log-normal** distribution and write

$Y \sim LN(\mu, \sigma^2)$ .

$$F_Y(y) = F_X(\log y) \quad \text{by usual argument with CDF}$$

$$f_Y(y) = \frac{d}{dy} F_X(\log y)$$

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## 5.3 The transformation method

If  $X$  has PDF  $f_X(x)$  and  $h(x)$  is a monotonically increasing or monotonically decreasing function then

- $h$  is invertible and
- the PDF of  $Y = h(X)$  is

$$f_Y(y) = f_X(h^{-1}(y)) \times \left| \frac{d(h^{-1}(y))}{dy} \right|$$

when  $y$  is in the range of  $h(x)$  and zero otherwise.

This result allows direct calculation of  $f_Y(y)$  and is called the **univariate transformation method**.

**See Lee's scribbles for Proof.**