

Home Study 4: To be completed by Wednesday 11th November

MAS2905: Probability and Bayesian Methods with R

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Semester 1, 2020/21

What you should have done so far...

- Completed everything in the first three chapters of the course notes
- Made a start on Test 1 in Numbas (deadline 16:00 this coming Thursday)
- We'll now cover some ground on the material in **Chapter 4**, to prepare you for the synchronous online lecture this coming Wednesday

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4. The inverse CDF method

This is a method for simulating **univariate continuous** random variables.

- Let $U \sim U(0, 1)$
- Suppose $F(x)$ is a well-defined CDF which is invertible
- Then the random variable $X = F^{-1}(U)$ has CDF $F(x)$

4. The inverse CDF method

$$X = F^{-1}(U).$$

So

$$\begin{aligned}\Pr(X \leq x) &= \Pr(F^{-1}(U) \leq x) \\ &= \Pr(F(F^{-1}(U)) \leq F(x)) \\ &= \Pr(U \leq F(x)) \\ &= F(x),\end{aligned}$$

since $0 \leq F(x) \leq 1$ (see Lee's sketches too from the video "Home Study 4").

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Example 4.1

Example

Suppose we obtain two observations from a $U(0, 1)$ distribution: 0.1 and 0.85. Use these values to obtain two observations from $X \sim \text{Exp}(2)$.

Example 4.1: Solution

If $X \sim \text{Exp}(2)$ then $F_X(x) = 1 - e^{-2x}$.

If $y = 1 - e^{-2x}$, then

$$e^{-2x} = 1 - y$$

$$-2x = \log(1 - y)$$

$$x = -\frac{1}{2} \log(1 - y).$$

Therefore

$$F_X^{-1}(x) = -\frac{1}{2} \log(1 - x).$$

Plug in $x = 0.1$ and $x = 0.85$; gives realized values of 0.0527 and 0.949.

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Example 4.2

Example

Write some R code to sample n observations from an $Exp(2)$ distribution.

Do not use the built-in `rexp` function.

Check your code by comparing a histogram of some simulated realisations against the PDF for an $Exp(2)$ distribution.

Also check your code against results from the built-in `rexp` function.

Example 4.2: Solution

As we saw in Example 4.1, we have

$$F_X^{-1}(x) = -\frac{1}{2} \log(1 - x).$$

Therefore:

```
1 inv.cdf = function(n){  
2   U = runif(n)  
3   answer = -0.5*log(1-runif(U))  
4   return(answer)}
```

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$$F_X^{-1}(x) = -\frac{1}{2} \log(1 - x).$$

Therefore:

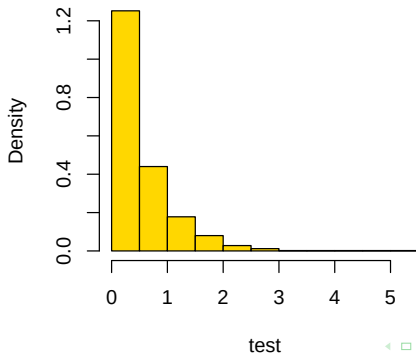
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Example 4.2: Solution

Putting the function to good use:

```
1 test = inv.cdf(1000)
2 hist(test, col="gold", main="Simulated Exp(2) random
   variates", probability=TRUE)
```

Simulated Exp(2) random variates

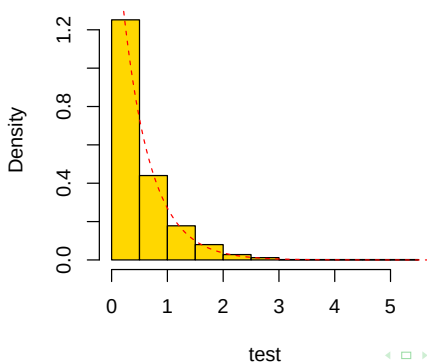


Example 4.2: Solution

Now to check it has worked!

```
1 x = seq(0, 6, 0.001)
2 y = 2*exp(-2*x)
3 lines(x, y, type="l", col="red", lty=2)
```

Simulated Exp(2) random variates



Example 4.2: Solution

Checking against R's built-in function:

```
1 test2 = rexp(1000, 2)
2 summary(test)
3      Min.   1st Qu.   Median     Mean   3rd Qu.    Max.
4 0.000064 0.147899 0.344216 0.507016 0.697876 5.889635
5 summary(test2)
6      Min.   1st Qu.   Median     Mean   3rd Qu.    Max.
7 0.000017 0.143081 0.346082 0.501171 0.693844 4.527852
```

Look very similar – we should be happy our function has worked!

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 - Make sure you're on top of EVERYTHING we've gone through in this video – take some time go over it slowly, again, should you need to
 - Try **Example 4.3** (solutions will be revealed in our lecture on Wednesday)
 - Keep chipping away at Test 1!
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