

Example 2.8

Suppose we have a random sample from an exponential distribution with a gamma prior distribution, that is,
 $X_i|\theta \sim \text{Exp}(\theta)$, $i = 1, 2, \dots, n$ (independent) and $\theta \sim \text{Ga}(g, h)$.

Determine a sufficient statistic T for θ and verify that
 $\pi(\theta|\mathbf{x}) = \pi(\theta|t)$.

Solution to Example 2.8 (1/2)

The density of the data is

$$\begin{aligned}f_{\mathbf{X}}(\mathbf{x}|\theta) &= \prod_{i=1}^n \theta e^{-\theta x_i} \\&= \theta^n \exp\left(-\theta \sum_{i=1}^n x_i\right) \\&= 1 \times \theta^n \exp\left(-\theta \sum_{i=1}^n x_i\right) \\&= h(\mathbf{x}) g(\sum x_i, \theta)\end{aligned}$$

and therefore, by the Factorisation Theorem, $T = \sum_{i=1}^n X_i$ is sufficient for θ .

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Solution to Example 2.8 (2/2)

Now $T|\theta \sim Ga(n, \theta)$ and so

$$f_T(t|\theta) = \frac{\theta^n t^{n-1} e^{-\theta t}}{\Gamma(n)}, \quad \theta > 0.$$

Also

$$\pi(\theta) = \frac{h^g \theta^{g-1} e^{-h\theta}}{\Gamma(g)}, \quad \theta > 0.$$

Therefore, by Bayes Theorem,

$$\begin{aligned}\pi(\theta|t) &\propto \pi(\theta)f(t|\theta) \\ &\propto \frac{h^g \theta^{g-1} e^{-h\theta}}{\Gamma(g)} \times \frac{\theta^n t^{n-1} e^{-\theta t}}{\Gamma(n)}, \quad \theta > 0 \\ &\propto \theta^{g+n-1} e^{-(h+t)\theta}, \quad \theta > 0\end{aligned}$$

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