

Example 2.9

Suppose we have a random sample from a normal distribution with known variance and a normal prior distribution for the mean parameter, that is, $X_i|\mu \sim N(\mu, 1/\tau)$, $i = 1, 2, \dots, n$ (independent) and $\mu \sim N(b, 1/d)$.

Determine a sufficient statistic T for μ and verify that $\pi(\mu|\mathbf{x}) = \pi(\mu|t)$.

Solution to Example 2.9 (1/3)

Recall from Equation (2.20) that

$$\begin{aligned}f_{\mathbf{X}}(\mathbf{x}|\mu) &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau s^2}{2}\right\} \times \exp\left\{-\frac{n\tau}{2}(\bar{x} - \mu)^2\right\} \\&= h(\mathbf{x}) g(\bar{x}, \mu)\end{aligned}$$

and therefore, by the Factorisation Theorem, $T = \bar{X}$ is sufficient for μ .

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Solution to Example 2.9 (2/3)

Now $T|\mu \sim N(\mu, 1/(n\tau))$ (by the Central Limit Theorem) and so

$$f_T(t|\mu) = \left(\frac{n\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{n\tau}{2}(t-\mu)^2\right\}.$$

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Solution to Example 2.9 (3/3)

Therefore, by Bayes' Theorem,

$$\pi(\mu|t) \propto \pi(\mu)f(t|\mu)$$

$$\begin{aligned} &\propto \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \\ &\quad \times \left(\frac{n_T}{2\pi}\right)^{1/2} \exp\left\{-\frac{n_T}{2}(t - \mu)^2\right\} \\ &\propto \exp\left\{-\frac{d}{2}(\mu - b)^2 - \frac{n_T}{2}(t - \mu)^2\right\} \\ &\quad \vdots \\ &\propto \exp\left\{-\frac{D}{2}(\mu - B)^2\right\} \end{aligned}$$

where B and D are as in Equation (2.21), with t replacing $\bar{x}$; that is, $\mu|t \sim N(B, 1/D)$, the same distribution as $\mu|\mathbf{x}$.

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