

Chapter 2 so far...

Model	Prior	Posterior
Binomial(k, θ)	$\theta \sim U(0, 1)$ $\theta \sim \text{Beta}(a, b)$	Beta Beta
Exponential(θ)	$\theta \sim \text{Ga}(g, h)$ $\theta \sim \text{exp}(g)$	Gamma ??
Normal(μ, σ^2)	$\mu \sim N(b, 1/d)$	Normal

Chapter 2 so far...

Model	Prior	Posterior
Binomial(k, θ)	$\theta \sim U(0, 1)$ $\theta \sim \text{Beta}(a, b)$	Beta Beta
Exponential(θ)	$\theta \sim \text{Ga}(g, h)$ $\theta \sim \exp(g)$	Gamma ??
Normal(μ, σ^2)	$\mu \sim N(b, 1/d)$	Normal

Chapter 2 so far...

Model	Prior	Posterior
Binomial(k, θ)	$\theta \sim U(0, 1)$ $\theta \sim \text{Beta}(a, b)$	Beta Beta
Exponential(θ)	$\theta \sim \text{Ga}(g, h)$ $\theta \sim \text{exp}(g)$	Gamma ??
Normal(μ, σ^2)	$\mu \sim N(b, 1/d)$	Normal

Chapter 2 so far...

Model	Prior	Posterior
Binomial(k, θ)	$\theta \sim U(0, 1)$ $\theta \sim \text{Beta}(a, b)$	Beta Beta
Exponential(θ)	$\theta \sim \text{Ga}(g, h)$ $\theta \sim \text{exp}(g)$	Gamma ??
Normal(μ, σ^2)	$\mu \sim N(b, 1/d)$	Normal

Definition (Conjugacy)

Suppose that data $\mathbf{x}$ are to be observed with distribution $f(\mathbf{x}|\theta)$.

A family $\mathfrak{F}$ of prior distributions for θ is said to be **conjugate** to $f(\mathbf{x}|\theta)$ if for every prior distribution $\pi(\theta) \in \mathfrak{F}$, the posterior distribution $\pi(\theta|\mathbf{x})$ is also in $\mathfrak{F}$.

Example 2.6

Suppose we have a random sample from a Normal distribution.

In Bayesian statistics, when dealing with the Normal distribution, the mathematics is more straightforward if we work with the **precision** ($= 1/\text{variance}$) of the distribution rather than the variance itself.

So we will assume that this population has unknown mean μ but known precision τ : $X_i|\mu \sim N(\mu, 1/\tau)$, $i = 1, 2, \dots, n$ (independent), where τ is known.

Suppose our prior beliefs about μ can be summarised by a $N(b, 1/d)$ distribution, with probability density function

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \quad (2.19)$$

Determine the posterior distribution for μ .

Example 2.6

Suppose we have a random sample from a Normal distribution.

In Bayesian statistics, when dealing with the Normal distribution, the mathematics is more straightforward if we work with the **precision** ($= 1/\text{variance}$) of the distribution rather than the variance itself.

So we will assume that this population has unknown mean μ but known precision τ : $X_i|\mu \sim N(\mu, 1/\tau)$, $i = 1, 2, \dots, n$ (independent), where τ is known.

Suppose our prior beliefs about μ can be summarised by a $N(b, 1/d)$ distribution, with probability density function

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \quad (2.19)$$

Determine the posterior distribution for μ .

Example 2.6

Suppose we have a random sample from a Normal distribution.

In Bayesian statistics, when dealing with the Normal distribution, the mathematics is more straightforward if we work with the **precision** ($= 1/\text{variance}$) of the distribution rather than the variance itself.

So we will assume that this population has unknown mean μ but known precision τ : $X_i|\mu \sim N(\mu, 1/\tau)$, $i = 1, 2, \dots, n$ (independent), where τ is known.

Suppose our prior beliefs about μ can be summarised by a $N(b, 1/d)$ distribution, with probability density function

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \quad (2.19)$$

Determine the posterior distribution for μ .

Example 2.6

Suppose we have a random sample from a Normal distribution.

In Bayesian statistics, when dealing with the Normal distribution, the mathematics is more straightforward if we work with the **precision** ($= 1/\text{variance}$) of the distribution rather than the variance itself.

So we will assume that this population has unknown mean μ but known precision τ : $X_i|\mu \sim N(\mu, 1/\tau)$, $i = 1, 2, \dots, n$ (independent), where τ is known.

Suppose our prior beliefs about μ can be summarised by a $N(b, 1/d)$ distribution, with probability density function

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \quad (2.19)$$

Determine the posterior distribution for μ .

Solution to Example 2.6 (1/5)

The likelihood function for μ is

$$\begin{aligned}f(\mathbf{x}|\mu) &= \prod_{i=1}^n \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x_i - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n (x_i - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n ([x_i - \bar{x}] + [\bar{x} - \mu])^2\right\}\end{aligned}$$

Solution to Example 2.6 (1/5)

The likelihood function for μ is

$$\begin{aligned}f(\mathbf{x}|\mu) &= \prod_{i=1}^n \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x_i - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n (x_i - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n ([x_i - \bar{x}] + [\bar{x} - \mu])^2\right\}\end{aligned}$$

Solution to Example 2.6 (1/5)

The likelihood function for μ is

$$\begin{aligned}f(\mathbf{x}|\mu) &= \prod_{i=1}^n \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x_i - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n (x_i - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n ([x_i - \bar{x}] + [\bar{x} - \mu])^2\right\}\end{aligned}$$

Solution to Example 2.6 (1/5)

The likelihood function for μ is

$$\begin{aligned}f(\mathbf{x}|\mu) &= \prod_{i=1}^n \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x_i - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n (x_i - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n (x_i - \bar{x} + \bar{x} - \mu)^2\right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2} \sum_{i=1}^n ([x_i - \bar{x}] + [\bar{x} - \mu])^2\right\}\end{aligned}$$

Solution to Example 2.6 (2/5)

Giving

$$\begin{aligned} f(\mathbf{x}|\mu) &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n (x_i - \bar{x})^2 \right. \right. \\ &\quad \left. \left. + 2(\bar{x} - \mu) \sum_{i=1}^n (x_i - \bar{x}) + \sum_{i=1}^n (\bar{x} - \mu)^2 \right] \right\} \\ &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2 \right] \right\}, \end{aligned}$$

Let

$$s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2;$$

Then

$$f(\mathbf{x}|\mu) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{n\tau}{2} \left[s^2 + (\bar{x} - \mu)^2 \right] \right\}. \quad (2.20)$$

Solution to Example 2.6 (2/5)

Giving

$$\begin{aligned} f(\mathbf{x}|\mu) &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n (x_i - \bar{x})^2 \right. \right. \\ &\quad \left. \left. + 2(\bar{x} - \mu) \sum_{i=1}^n (x_i - \bar{x}) + \sum_{i=1}^n (\bar{x} - \mu)^2 \right] \right\} \\ &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2 \right] \right\}, \end{aligned}$$

Let

$$s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2;$$

Then

$$f(\mathbf{x}|\mu) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{n\tau}{2} \left[s^2 + (\bar{x} - \mu)^2 \right] \right\}. \quad (2.20)$$

Solution to Example 2.6 (2/5)

Giving

$$\begin{aligned}f(\mathbf{x}|\mu) &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n (x_i - \bar{x})^2 \right. \right. \\&\quad \left. \left. + 2(\bar{x} - \mu) \sum_{i=1}^n (x_i - \bar{x}) + \sum_{i=1}^n (\bar{x} - \mu)^2 \right] \right\} \\&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2 \right] \right\},\end{aligned}$$

Let

$$s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2;$$

Then

$$f(\mathbf{x}|\mu) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{n\tau}{2} \left[s^2 + (\bar{x} - \mu)^2 \right] \right\}. \quad (2.20)$$

Solution to Example 2.6 (2/5)

Giving

$$\begin{aligned} f(\mathbf{x}|\mu) &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n (x_i - \bar{x})^2 \right. \right. \\ &\quad \left. \left. + 2(\bar{x} - \mu) \sum_{i=1}^n (x_i - \bar{x}) + \sum_{i=1}^n (\bar{x} - \mu)^2 \right] \right\} \\ &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{\tau}{2} \left[\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu)^2 \right] \right\}, \end{aligned}$$

Let

$$s^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2;$$

Then

$$f(\mathbf{x}|\mu) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp \left\{ -\frac{n\tau}{2} [s^2 + (\bar{x} - \mu)^2] \right\}. \quad (2.20)$$

Solution to Example 2.6 (3/5)

Applying Bayes Theorem, the posterior density function is

$$\pi(\mu|\mathbf{x}) \propto \pi(\mu)f(\mathbf{x}|\mu)$$

$$\propto \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \\ \times \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\}$$

$$= k_1 \exp\left\{-\frac{d}{2}(\mu - b)^2 - \frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\}$$

$$= k_1 \exp\left\{-\frac{1}{2}\left[d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2\right]\right\}$$

where k_1 is a constant that does not depend on μ .

Solution to Example 2.6 (3/5)

Applying Bayes Theorem, the posterior density function is

$$\pi(\mu|\mathbf{x}) \propto \pi(\mu)f(\mathbf{x}|\mu)$$

$$\begin{aligned} &\propto \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \\ &\quad \times \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\} \end{aligned}$$

$$= k_1 \exp\left\{-\frac{d}{2}(\mu - b)^2 - \frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\}$$

$$= k_1 \exp\left\{-\frac{1}{2}\left[d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2\right]\right\}$$

where k_1 is a constant that does not depend on μ .

Solution to Example 2.6 (3/5)

Applying Bayes Theorem, the posterior density function is

$$\pi(\mu|\mathbf{x}) \propto \pi(\mu)f(\mathbf{x}|\mu)$$

$$\begin{aligned} &\propto \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \\ &\quad \times \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\} \end{aligned}$$

$$= k_1 \exp\left\{-\frac{d}{2}(\mu - b)^2 - \frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\}$$

$$= k_1 \exp\left\{-\frac{1}{2}\left[d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2\right]\right\}$$

where k_1 is a constant that does not depend on μ .

Solution to Example 2.6 (4/5)

We can also simplify the exponent. We have

$$\begin{aligned}d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2 \\&= d(\mu^2 - 2b\mu + b^2) + n\tau(\bar{x}^2 - 2\bar{x}\mu + \mu^2) \\&= d\mu^2 + n\tau\mu^2 - 2db\mu - 2n\tau\bar{x}\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau)\mu^2 - 2(db + n\tau\bar{x})\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau) \left(\mu^2 - \frac{2(db + n\tau\bar{x})}{d + n\tau} \mu + c \right) \\&= (d + n\tau) \left\{ \mu - \left(\frac{db + n\tau\bar{x}}{d + n\tau} \right) \right\}^2 + c\end{aligned}$$

where c does not depend on μ .

Solution to Example 2.6 (4/5)

We can also simplify the exponent. We have

$$\begin{aligned}d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2 \\&= d(\mu^2 - 2b\mu + b^2) + n\tau(\bar{x}^2 - 2\bar{x}\mu + \mu^2) \\&= d\mu^2 + n\tau\mu^2 - 2db\mu - 2n\tau\bar{x}\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau)\mu^2 - 2(db + n\tau\bar{x})\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau) \left(\mu^2 - \frac{2(db + n\tau\bar{x})}{d + n\tau} \mu + c \right) \\&= (d + n\tau) \left\{ \mu - \left(\frac{db + n\tau\bar{x}}{d + n\tau} \right) \right\}^2 + c\end{aligned}$$

where c does not depend on μ .

Solution to Example 2.6 (4/5)

We can also simplify the exponent. We have

$$\begin{aligned}d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2 \\&= d(\mu^2 - 2b\mu + b^2) + n\tau(\bar{x}^2 - 2\bar{x}\mu + \mu^2) \\&= d\mu^2 + n\tau\mu^2 - 2db\mu - 2n\tau\bar{x}\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau)\mu^2 - 2(db + n\tau\bar{x})\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau) \left(\mu^2 - \frac{2(db + n\tau\bar{x})}{d + n\tau} \mu + c \right) \\&= (d + n\tau) \left\{ \mu - \left(\frac{db + n\tau\bar{x}}{d + n\tau} \right) \right\}^2 + c\end{aligned}$$

where c does not depend on μ .

Solution to Example 2.6 (4/5)

We can also simplify the exponent. We have

$$\begin{aligned}d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2 \\&= d(\mu^2 - 2b\mu + b^2) + n\tau(\bar{x}^2 - 2\bar{x}\mu + \mu^2) \\&= d\mu^2 + n\tau\mu^2 - 2db\mu - 2n\tau\bar{x}\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau)\mu^2 - 2(db + n\tau\bar{x})\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau) \left(\mu^2 - \frac{2(db + n\tau\bar{x})}{d + n\tau} \mu + c \right) \\&= (d + n\tau) \left\{ \mu - \left(\frac{db + n\tau\bar{x}}{d + n\tau} \right) \right\}^2 + c\end{aligned}$$

where c does not depend on μ .

Solution to Example 2.6 (4/5)

We can also simplify the exponent. We have

$$\begin{aligned}d(\mu - b)^2 &+ n\tau(\bar{x} - \mu)^2 \\&= d(\mu^2 - 2b\mu + b^2) + n\tau(\bar{x}^2 - 2\bar{x}\mu + \mu^2) \\&= d\mu^2 + n\tau\mu^2 - 2db\mu - 2n\tau\bar{x}\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau)\mu^2 - 2(db + n\tau\bar{x})\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau) \left(\mu^2 - \frac{2(db + n\tau\bar{x})}{d + n\tau} \mu + c \right) \\&= (d + n\tau) \left\{ \mu - \left(\frac{db + n\tau\bar{x}}{d + n\tau} \right) \right\}^2 + c\end{aligned}$$

where c does not depend on μ .

Solution to Example 2.6 (4/5)

We can also simplify the exponent. We have

$$\begin{aligned}d(\mu - b)^2 &+ n\tau(\bar{x} - \mu)^2 \\&= d(\mu^2 - 2b\mu + b^2) + n\tau(\bar{x}^2 - 2\bar{x}\mu + \mu^2) \\&= d\mu^2 + n\tau\mu^2 - 2db\mu - 2n\tau\bar{x}\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau)\mu^2 - 2(db + n\tau\bar{x})\mu + db^2 + n\tau\bar{x}^2 \\&= (d + n\tau) \left(\mu^2 - \frac{2(db + n\tau\bar{x})}{d + n\tau} \mu + c \right) \\&= (d + n\tau) \left\{ \mu - \left(\frac{db + n\tau\bar{x}}{d + n\tau} \right) \right\}^2 + c\end{aligned}$$

where c does not depend on μ .

Solution to Example 2.6 (5/5)

Let

$$B = \frac{db + n\tau\bar{x}}{d + n\tau} \quad \text{and} \quad D = d + n\tau. \quad (2.21)$$

Then

$$\begin{aligned} \pi(\mu|\mathbf{x}) &= k_1 \exp \left\{ -\frac{D}{2}(\mu - B)^2 - \frac{c}{2} \right\} \\ &= k \exp \left\{ -\frac{D}{2}(\mu - B)^2 \right\}, \end{aligned} \quad (2.22)$$

where k is a constant that does not depend on μ .

Therefore, the posterior distribution takes the form $k \exp\{-D(\mu - B)^2/2\}$, $-\infty < \mu < \infty$ and so must be a normal distribution: we have $\mu|\mathbf{x} \sim N(B, 1/D)$.

Solution to Example 2.6 (5/5)

Let

$$B = \frac{db + n\tau\bar{x}}{d + n\tau} \quad \text{and} \quad D = d + n\tau. \quad (2.21)$$

Then

$$\begin{aligned} \pi(\mu|\mathbf{x}) &= k_1 \exp \left\{ -\frac{D}{2}(\mu - B)^2 - \frac{c}{2} \right\} \\ &= k \exp \left\{ -\frac{D}{2}(\mu - B)^2 \right\}, \end{aligned} \quad (2.22)$$

where k is a constant that does not depend on μ .

Therefore, the posterior distribution takes the form $k \exp\{-D(\mu - B)^2/2\}$, $-\infty < \mu < \infty$ and so must be a normal distribution: we have $\mu|\mathbf{x} \sim N(B, 1/D)$.

Solution to Example 2.6 (5/5)

Let

$$B = \frac{db + n\tau\bar{x}}{d + n\tau} \quad \text{and} \quad D = d + n\tau. \quad (2.21)$$

Then

$$\begin{aligned} \pi(\mu|\mathbf{x}) &= k_1 \exp \left\{ -\frac{D}{2}(\mu - B)^2 - \frac{c}{2} \right\} \\ &= k \exp \left\{ -\frac{D}{2}(\mu - B)^2 \right\}, \end{aligned} \quad (2.22)$$

where k is a constant that does not depend on μ .

Therefore, the posterior distribution takes the form $k \exp\{-D(\mu - B)^2/2\}$, $-\infty < \mu < \infty$ and so must be a normal distribution: we have $\mu|\mathbf{x} \sim N(B, 1/D)$.

Example 2.6: Summary

If we have a random sample from a $N(\mu, 1/\tau)$ distribution (with τ known) and our prior beliefs about μ follow a $N(b, 1/d)$ distribution then, after incorporating the data, our (posterior) beliefs about μ follow a $N(B, 1/D)$ distribution.

Example 2.6: Summary

If we have a random sample from a $N(\mu, 1/\tau)$ distribution (with τ known) and our prior beliefs about μ follow a $N(b, 1/d)$ distribution then, after incorporating the data, our (posterior) beliefs about μ follow a $N(B, 1/D)$ distribution.

Example 2.6: Summary

The changes in our beliefs about μ are summarised in Table 2.6.

	Prior (2.19)	Likelihood (2.20)	Posterior (2.21)
$Mode(\mu)$	b	$\bar{x}$	$(db + n\tau\bar{x})/(d + n\tau)$
$E(\mu)$	b	—	$(db + n\tau\bar{x})/(d + n\tau)$
$Precision(\mu)$	d	—	$d + n\tau$

Example 2.6: Summary

The changes in our beliefs about μ are summarised in Table 2.6.

	Prior (2.19)	Likelihood (2.20)	Posterior (2.21)
$Mode(\mu)$	b	$\bar{x}$	$(db + n\tau\bar{x})/(d + n\tau)$
$E(\mu)$	b	—	$(db + n\tau\bar{x})/(d + n\tau)$
$Precision(\mu)$	d	—	$d + n\tau$

Example 2.6: Summary

Notice that the way prior information and observed data combine is through the parameters of the normal distribution:

$$b \rightarrow \frac{db + n\tau\bar{x}}{d + n\tau} \quad \text{and} \quad d \rightarrow d + n\tau.$$

Example 2.6: Summary

Notice that the way prior information and observed data combine is through the parameters of the normal distribution:

$$b \rightarrow \frac{db + n_{\tau} \bar{x}}{d + n_{\tau}} \quad \text{and} \quad d \rightarrow d + n_{\tau}.$$

Example 2.6: Summary

Notice also that the posterior variance (and precision) does not depend on the data, and the posterior mean is a convex combination of the prior and sample means, that is,

$$\begin{aligned} B &= \frac{db + n\tau\bar{x}}{d + n\tau} \\ &= \frac{db}{d + n\tau} + \frac{n\tau\bar{x}}{d + n\tau} \\ &= \frac{d}{d + n\tau}b + \frac{n\tau}{d + n\tau}\bar{x} \\ &= \left(\frac{d}{d + n\tau}\right)b + \left(\frac{d + n\tau - d}{d + n\tau}\right)\bar{x} \\ &= \alpha b + (1 - \alpha)\bar{x}, \quad \alpha = \frac{d}{d + n\tau}. \end{aligned}$$

Example 2.6: Summary

Notice also that the posterior variance (and precision) does not depend on the data, and the posterior mean is a convex combination of the prior and sample means, that is,

$$\begin{aligned} B &= \frac{db + n\tau\bar{x}}{d + n\tau} \\ &= \frac{db}{d + n\tau} + \frac{n\tau\bar{x}}{d + n\tau} \\ &= \frac{d}{d + n\tau}b + \frac{n\tau}{d + n\tau}\bar{x} \\ &= \left(\frac{d}{d + n\tau}\right)b + \left(\frac{d + n\tau - d}{d + n\tau}\right)\bar{x} \\ &= \alpha b + (1 - \alpha)\bar{x}, \quad \alpha = \frac{d}{d + n\tau}. \end{aligned}$$

Example 2.6: Summary

Notice also that the posterior variance (and precision) does not depend on the data, and the posterior mean is a convex combination of the prior and sample means, that is,

$$\begin{aligned} B &= \frac{db + n\tau\bar{x}}{d + n\tau} \\ &= \frac{db}{d + n\tau} + \frac{n\tau\bar{x}}{d + n\tau} \\ &= \frac{d}{d + n\tau}b + \frac{n\tau}{d + n\tau}\bar{x} \\ &= \left(\frac{d}{d + n\tau}\right)b + \left(\frac{d + n\tau - d}{d + n\tau}\right)\bar{x} \\ &= \alpha b + (1 - \alpha)\bar{x}, \quad \alpha = \frac{d}{d + n\tau}. \end{aligned}$$

Example 2.6: Summary

Notice also that the posterior variance (and precision) does not depend on the data, and the posterior mean is a convex combination of the prior and sample means, that is,

$$\begin{aligned} B &= \frac{db + n\tau\bar{x}}{d + n\tau} \\ &= \frac{db}{d + n\tau} + \frac{n\tau\bar{x}}{d + n\tau} \\ &= \frac{d}{d + n\tau}b + \frac{n\tau}{d + n\tau}\bar{x} \\ &= \left(\frac{d}{d + n\tau}\right)b + \left(\frac{d + n\tau - d}{d + n\tau}\right)\bar{x} \\ &= \alpha b + (1 - \alpha)\bar{x}, \quad \alpha = \frac{d}{d + n\tau}. \end{aligned}$$

Example 2.6: Summary

Notice also that the posterior variance (and precision) does not depend on the data, and the posterior mean is a convex combination of the prior and sample means, that is,

$$\begin{aligned} B &= \frac{db + n_{\tau} \bar{x}}{d + n_{\tau}} \\ &= \frac{db}{d + n_{\tau}} + \frac{n_{\tau} \bar{x}}{d + n_{\tau}} \\ &= \frac{d}{d + n_{\tau}} b + \frac{n_{\tau}}{d + n_{\tau}} \bar{x} \\ &= \left(\frac{d}{d + n_{\tau}} \right) b + \left(\frac{d + n_{\tau} - d}{d + n_{\tau}} \right) \bar{x} \\ &= \alpha b + (1 - \alpha) \bar{x}, \quad \alpha = \frac{d}{d + n_{\tau}}. \end{aligned}$$

Example 2.6: Summary

This equation for the posterior mean, which can be rewritten as

$$E(\mu|\mathbf{x}) = \alpha E(\mu) + (1 - \alpha)\bar{x},$$

arises in other models and is known as the **Bayes linear rule**.

Example 2.6: Summary

Notice that the posterior mean is greater than the prior mean if and only if the likelihood mode (sample mean) is greater than the prior mean, that is

$$E(\mu|\mathbf{x}) > E(\mu) \iff \text{Mode}[f(\mathbf{x}|\mu)] > E(\mu).$$

Also, the standard deviation of the posterior distribution is smaller than that of the prior distribution.

Example 2.6: Summary

Notice that the posterior mean is greater than the prior mean if and only if the likelihood mode (sample mean) is greater than the prior mean, that is

$$E(\mu|\mathbf{x}) > E(\mu) \quad \Longleftrightarrow \quad \text{Mode}[f(\mathbf{x}|\mu)] > E(\mu).$$

Also, the standard deviation of the posterior distribution is smaller than that of the prior distribution.