

Definition (Joint sufficiency)

The statistics $\underline{T}(\mathbf{X}) = (T_1(\mathbf{X}), T_2(\mathbf{X}), \dots, T_k(\mathbf{X}))^T$ are **jointly sufficient** for θ if $f(\mathbf{x} | \underline{T}(\mathbf{X}) = \mathbf{t})$ does not depend on θ .

Theorem (Factorisation Theorem)

Under certain regularity conditions

$$\underline{T}(\mathbf{X}) \text{ is sufficient for } \theta \iff f(\mathbf{x} | \theta) = h(\mathbf{x}) g(\underline{t}(\mathbf{x}), \theta)$$

for some functions h and g .

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Example 1.11

Suppose we have a random sample from a Normal distribution:
 $X_i | \mu, \sigma \sim N(\mu, \sigma^2), i = 1, 2, \dots, n$ (independent).

Determine sufficient statistics for (μ, σ) .

Solution to Example 1.11 (1/1)

The (joint) probability density function is

$$\begin{aligned}f(\mathbf{x}|\mu, \sigma) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x_i - \mu)^2}{2\sigma^2} \right\} \\&= (2\pi)^{-n/2} \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right\} \\&= (2\pi)^{-n/2} \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \left(\sum x_i^2 - 2\mu \sum x_i + n\mu^2 \right) \right\} \\&= h(\mathbf{x}) g(\sum x_i, \sum x_i^2, \mu, \sigma),\end{aligned}$$

where $h(\mathbf{x}) = (2\pi)^{-n/2}$ and

$g(t_1, t_2, \mu, \sigma) = \sigma^{-n} \exp\{-(t_2 - 2\mu t_1 + n\mu^2)/(2\sigma^2)\}$.

Therefore, by the Factorisation Theorem, $T_1 = \sum_{i=1}^n X_i$ and $T_2 = \sum_{i=1}^n X_i^2$ are (jointly) sufficient for μ and σ .

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If $X_1, X_2, \dots, X_n$ are independent random variables with probability density function

$$f(x|\theta) = \theta(1 - \theta)^{x-1}, \quad x = 1, 2, \dots,$$

- (a) Form the likelihood function $f(\mathbf{x}|\theta)$;
- (b) Use the Factorisation Theorem to obtain a sufficient statistic for θ .

Quick quiz: solution

The joint probability function is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \prod_{i=1}^n \theta(1-\theta)^{x_i-1} \\&= \theta^n(1-\theta)^{x_1-1}(1-\theta)^{x_2-1} \times \dots \times (1-\theta)^{x_n-1} \\&= \theta^n(1-\theta)^{\sum x_i - n} \\&= 1 \times \theta^n(1-\theta)^{\sum x_i - n} \\&= h(\mathbf{x}) g\left(\sum x_i, \theta\right)\end{aligned}$$

where $h(\mathbf{x}) = 1$ and $g(t, \theta) = \theta^n(1-\theta)^{t-n}$. Therefore, by the Factorisation Theorem, $T = \sum X_i$ is sufficient for θ .

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