

MAS2905: Revision

Dr. Lee Fawcett

Semesters 1 and 2, 2020—2021

The exam

- Exam start time: 09:30 Wednesday 26th May 2021
- Solutions document to be uploaded by: **09:30 Thursday 27th May 2021**
- 50% of the whole module
- **Section I:** Semester 1 material, 20 marks, 2–3 questions
- **Section II:** Semester 2 material, 80 marks, 4–5 questions
- Open-book, but formula sheet still included on page 2 for convenience
- Do NOT focus on proofs or definitions, you will NOT be asked for these...
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- Three main topics to look out for:
 - The inverse CDF method (Chapter 4)
 - Transformations (Chapter 5)
 - Bivariate distributions and the bivariate Normal distribution (Chapter 6)
- No R code in the exam, no R programming necessary to answer the questions
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The inverse CDF method

Basic recipe

1. You'll either be given the PDF or the CDF. Make sure you know what you've been given! You'll either be told:
 - “The cumulative distribution function is...”, “The CDF is...”, “ $F_X(x) = \dots$ ”, “ $F(x) = \dots$ ”, OR
 - “The probability density function is...”, “The PDF is...”, “ $f_X(x) = \dots$ ”, “ $f(x) = \dots$ ”
2. If you've been given the PDF, integrate over a dummy variable to get the CDF! If you've been given the CDF already, you're good to go to the next step...
3. Invert your CDF to find $F_X^{-1}(x)$
4. Find $F_X^{-1}(u)$, where $u \sim U(0, 1)$. Job done!

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Exam-style question (10 marks)

The random variable X has probability density function (PDF)

$$f_X(x) = \begin{cases} \left(\frac{2}{x}\right)^3 & \text{when } x > 2, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose RStudio has been used to generate $u = 0.540$ from $U(0, 1)$. Use u to generate a single realisation from X .

Basic recipe

1. Decide whether you'll use the 'informal' (Examples 5.1-5.3 in the notes) or the 'formal' (Examples 5.4–5.7) approach. It doesn't matter, either can be used to gain full marks. The proof (page 42) is unnecessary!
2. Suppose X is the 'original' variable and Y the transformed variable
3. Substitute your transformation $Y = \dots$ into $P(Y \leq y)$ and rearrange to find $P(X \leq \dots) = F_X(\dots)$
4. Use the chain rule to differentiate your result in (3), with respect to y , to obtain $f_Y(y)$
5. Initially, your answer in (4) will be in terms of the $f_X(\dots)$; use the known PDF for X to write down your final answer for $f_Y(y)$, simplifying where possible
6. Does your PDF in (5) as come from a known distribution?

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Exam-style question (12 marks)

Suppose X follows a standard Normal distribution.

- (a) What is the probability density function (PDF) of $Y = X^2$?
- (b) Find $E(Y)$ and $Var(Y)$. [Note that $\Gamma(0.5) \equiv \sqrt{\pi}$.]

Bivariate distributions

In your revision, focus on properties of the bivariate Normal distribution – most of the other stuff was R-based simulation.

If $(X, Y)^T \sim N_2(\underline{\mu}, \Sigma)$, then

- 1 The marginal distributions are normal:

$$X \sim N(\mu_x, \sigma_x^2) \quad \text{and} \quad Y \sim N(\mu_y, \sigma_y^2).$$

- 2 The conditional distributions are normal:

$$X|Y=y \sim N(\mu_x + \rho \frac{\sigma_x}{\sigma_y}(y - \mu_y), \sigma_x^2(1 - \rho^2)) \quad \text{and}$$

$$Y|X=x \sim N(\mu_y + \rho \frac{\sigma_y}{\sigma_x}(x - \mu_x), \sigma_y^2(1 - \rho^2)).$$

- 3 When $\rho = 0$, X and Y are independent.

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- 3 When $\rho = 0$, X and Y are independent.

- 4 Linear combinations of X and Y are also normally distributed:

$$aX + bY \sim N(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho\sigma_x\sigma_y)$$

where a, b are constants. Thus a linear combination of normal random variables is also normally distributed itself.

And don't forget:

$$\underline{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix},$$

and

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}.$$

Exam-style question (7)

Suppose

$$(X, Y)^T \sim N_2 \left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix} \right).$$

Determine the distribution of $Z = 3X - 2Y$, and sketch the probability density function (PDF) for Z .