

MAS2903

Problems Class 1

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- Next week: computer practical on Friday at 1pm, to try out some of the R questions in Chapter 5

Question 5

An experiment can result in four possible outcomes x_1 , x_2 , x_3 and x_4 .

The probability of each of these outcomes is affected by a parameter θ which can take one of three values: θ_1 , θ_2 and θ_3 :

$p(x \theta)$	x_1	x_2	x_3	x_4
θ_1	0.1	0.2	0.3	0.4
θ_2	0.3	0.3	0.2	0.2
θ_3	0.5	0.1	0.1	0.3

Question 5

The prior plausibilities attaching to θ_1 , θ_2 and θ_3 are 0.1, 0.4 and 0.5.

Find the posterior probability function for θ after observing x_1 . Repeat the calculations for x_2 , x_3 and x_4 .

Solution to Question 5

The **posterior probabilities** given x_1 are

$$Pr(\theta_1|x_1) = \frac{Pr(x_1|\theta_1)Pr(\theta_1)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)}$$

Solution to Question 5

The **posterior probabilities** given x_1 are

$$\begin{aligned}Pr(\theta_1|x_1) &= \frac{Pr(x_1|\theta_1)Pr(\theta_1)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)} \\&= \frac{0.1 \times 0.1}{0.1 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.5}\end{aligned}$$

Solution to Question 5

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$$Pr(\theta_2|x_1) = \frac{Pr(x_1|\theta_2)Pr(\theta_2)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)}$$

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$$\begin{aligned}Pr(\theta_2|x_1) &= \frac{Pr(x_1|\theta_2)Pr(\theta_2)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)} \\&= \frac{0.3 \times 0.4}{0.1 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.5} \\&= 0.31579,\end{aligned}$$

Solution to Question 5

and

$$Pr(\theta_3|x_1) = 1 - Pr(\theta_1|x_1) - Pr(\theta_2|x_1)$$

Solution to Question 5

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$$\begin{aligned}Pr(\theta_3|x_1) &= 1 - Pr(\theta_1|x_1) - Pr(\theta_2|x_1) \\&= 1 - 0.02632 - 0.31579 \\&= 0.65789.\end{aligned}$$

Thus, after observing x_1 , we have the following probability function for θ :

Solution to Question 5

and

$$\begin{aligned}Pr(\theta_3|x_1) &= 1 - Pr(\theta_1|x_1) - Pr(\theta_2|x_1) \\&= 1 - 0.02632 - 0.31579 \\&= 0.65789.\end{aligned}$$

Thus, after observing x_1 , we have the following probability function for θ :

θ	θ_1	θ_2	θ_3
$Pr(\theta = \theta_i x_1)$	0.02632	0.31579	0.65789

Same for x_2 , x_3 and x_4 .

Question 6

Extensive manuscripts of two scribes show that they differ on the relative frequencies with which they use two alternative vowel forms *e* and *œ*, as in the following table:

	<i>e</i>	<i>œ</i>
Scribe A	0.3	0.7
Scribe B	0.5	0.5

Question 6

A new manuscript fragment, known to be by one of the scribes, uses the vowel form five times, three times as e and twice as œ .

If an independent assessment of similar fragments favours Scribe A as author with probability 0.7, show that the evidence from the vowel form is sufficient to turn the odds in favour of Scribe B.

Solution to Question 6

Let X = number of e 's observed.

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Then

$$Pr(\theta_A) = 0.7, \quad Pr(\theta_B) = 0.3,$$

$$X|\theta_A \sim \text{Bin}(5, 0.3) \quad \text{and} \quad X|\theta_B \sim \text{Bin}(5, 0.5).$$

Hence

$$Pr(X = 3|\theta_A) = \binom{5}{3} 0.3^3 \times 0.7^2 = 0.1323$$

Solution to Question 6

Hence

$$Pr(X = 3|\theta_A) = \binom{5}{3} 0.3^3 \times 0.7^2 = 0.1323$$

$$Pr(X = 3|\theta_B) = \binom{5}{3} 0.5^3 \times 0.5^2 = 0.3125.$$

Solution to Question 6

Therefore, using Bayes' Theorem

$$Pr(\theta_A|X=3) = \frac{Pr(X=3|\theta_A)Pr(\theta_A)}{Pr(X=3|\theta_A)Pr(\theta_A) + Pr(X=3|\theta_B)Pr(\theta_B)}$$

Solution to Question 6

Therefore, using Bayes' Theorem

$$\begin{aligned}Pr(\theta_A|X = 3) &= \frac{Pr(X = 3|\theta_A)Pr(\theta_A)}{Pr(X = 3|\theta_A)Pr(\theta_A) + Pr(X = 3|\theta_B)Pr(\theta_B)} \\&= \frac{0.1323 \times 0.7}{0.1323 \times 0.7 + 0.3125 \times 0.3}\end{aligned}$$

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$$Pr(\theta_B|x = 3) = 1 - Pr(\theta_A|x = 3) = 0.503 \text{ (3 d.p.)}.$$

Solution to Question 6

Therefore, using Bayes' Theorem

$$\begin{aligned}Pr(\theta_A|x = 3) &= \frac{Pr(X = 3|\theta_A)Pr(\theta_A)}{Pr(X = 3|\theta_A)Pr(\theta_A) + Pr(X = 3|\theta_B)Pr(\theta_B)} \\&= \frac{0.1323 \times 0.7}{0.1323 \times 0.7 + 0.3125 \times 0.3} \\&= 0.497 \text{ (3 d.p.)}\end{aligned}$$

$$Pr(\theta_B|x = 3) = 1 - Pr(\theta_A|x = 3) = 0.503 \text{ (3 d.p.)}.$$

Thus, the odds have turned from 7 : 3 in favour of Scribe A to 503 : 497 in favour of Scribe B.

Question 7

If $X_1, X_2, \dots, X_n$ are independent random variables with probability function

$$f(x|\theta) = \theta(1 - \theta)^{x-1}, \quad x = 1, 2, \dots,$$

show that $T = \sum_{i=1}^n X_i$ is sufficient for θ .

Solution to Question 7

The joint probability function is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n \theta(1 - \theta)^{x_i - 1}$$

Solution to Question 7

The joint probability function is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \prod_{i=1}^n \theta(1 - \theta)^{x_i-1} \\&= \theta^n (1 - \theta)^{x_1-1} (1 - \theta)^{x_2-1} \times \dots \times (1 - \theta)^{x_n-1}\end{aligned}$$

Solution to Question 7

The joint probability function is

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The joint probability function is

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where $h(\mathbf{x}) = 1$ and $g(t, \theta) = \theta^n(1 - \theta)^{t-n}$. Therefore, by the Factorisation Theorem, $T = \sum X_i$ is sufficient for θ .

Question 8

If $X_1, X_2, \dots, X_n$ are independent $N(5, \sigma^2)$ random variables, show that

$$T = \sum_{i=1}^n (X_i - 5)^2$$

is sufficient for σ .

Solution to Question 8

The joint probability density function is

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i - 5)^2}{2\sigma^2}\right\}$$

Solution to Question 8

The joint probability density function is

$$\begin{aligned}f(\mathbf{x}|\theta) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i - 5)^2}{2\sigma^2}\right\} \\&= (2\pi)^{-n/2} \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - 5)^2\right\}\end{aligned}$$

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where $h(\mathbf{x}) = 1$ and $g(t, \sigma) = (2\pi)^{-n/2} \sigma^{-n} e^{-t/(2\sigma^2)}$.

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Therefore, by the Factorisation Theorem, $T = \sum (X_i - 5)^2$ is sufficient for σ .