MAS2903: Introduction to Bayesian Methods

Dr. Lee Fawcett

Semester 2, 2018—2019

Contact details

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Timetable and Administrative arrangements

- **Lectures** are on Mondays at 11 (LT1) and Tuesdays at 3 (LT1). In the first two weeks there will be a third lecture on Thursdays at 2 (LT1).
- **Problems classes** are in ODD teaching weeks starting in teaching week 3. These take place on Thursdays at 2 in LT1.
- **Drop–in sessions** are at the same time and place in EVEN weeks starting in teaching week 4.
- **Computer practicals**: Fridays at 1 (Herschel full PC cluster). These happen in *some* EVEN weeks, and you will be notified of these well in advance.

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So to summarise:

What?	When?		Where?	How often?
Lecture	Monday	11–12	Herschel: LT1	Every week
	Tuesday	3–4	Herschel: LT1	Every week
Problems class/ Drop–in session	Thursday	2–3	Herschel: LT1	Every week has one or the other
Computer practical	Friday	1–2	Herschel: PC cluster	Some even weeks
Office hours	Monday	12–1	My office	Every week
	Wednesday	12–1		

Assessment

Assessment is by:

- End of semester exam in May/June (**85%**)
- In course assessment, including written solution to problems and computer practical work (**5%**)
- Mid-semester test (**10%**)
- No CBAs!

There will be two assignments, each one having a mixture of written and practical questions.

Solutions to the assignment questions should be submitted by the dates given in your notes.

We will work through the unassessed questions in problems classes and practicals.

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Recommended textbooks

- "*Bayes' Rule: A Tutorial Introduction to Bayesian Analysis*" - James Stone
- "*Doing Bayesian Data Analysis: A Tutorial with R, JAGS, and Stan*" - John Krushke
- "*Bayesian Statistics: An Introduction*" - Peter Lee

"Bayes' rule" is a good introduction to the main concepts in Bayesian statistics but doesn't cover everything in this course. The other books are broader references which go well beyond the contents of this course.

Other stuff

- Notes (with gaps) will be handed out in lectures – you should fill in the gaps during lectures.
- A summarised version of the notes will be used in lectures as slides.
  - **Listen and learn**
  - **Write down**
  - **Announcements**
- These notes and slides will be posted on the **course website** and/or BlackBoard after each topic is finished, along with any other course material – such as problems sheets, model solutions to assignment questions, supplementary handouts etc.

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  - Intellectual stimulation: 4.1/5
  - Feedback: 4.5/5
  - Satisfaction: 4.0/5
- **Teacher**
  - Stimulated me to learn: 4.0/5 (Chris: 4.6/5)
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  - Lee and Chris were brilliant, I hated computing going into this year but these two have really changed my opinion on it :-)
  - Enthusiasm of both lecturers was great :-)
  - Chris: Absolute lad, to bo honest
  - Chris: Top geez as always
  - Chris: Passionate, fun guy
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Preface to lecture notes:

What is Bayesian Statistics?

Statistics: an alternative approach

- Up until now, you have been taught Probability and Statistics according to a particular way of thinking
- The **Bayesian paradigm** offers another way of seeing things
- Your ideas about Probability and Statistics are deeply entrenched...perhaps so much so that at first, *Bayesian Statistics* might take you a while to grasp!
- Not because it's hard! Just because you have become so conditioned to think in a particular way.
- We want to broaden your horizons a bit!

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Statistics: an alternative approach

There are two main approaches to statistics: **frequentist** (or classical) statistics and **Bayesian** statistics.

All of the statistics teaching you've encountered so far is likely to be about frequentist methods. Bayesian methods are substantially different and can feel quite strange to start with.

So, before starting the main course material, this **non-examinable** preface explains the concepts behind Bayesian statistics and how they differ from frequentist approaches.

Statistics: an alternative approach

One way of defining Statistics is as a way to learn about the world from some data which is subject to random variation.

For example, in **climate science** we want to learn about the climate given some imperfect measurements.

Some statistical questions that we might ask in this field are:

- What is our best estimate of the world temperature this year? This is a point estimation problem.
- What is a plausible range of values? This is an interval estimation (or uncertainty quantification) problem.
- What will the temperature be in 100 years?This is a prediction problem.
- Is the climate warming?This is a hypothesis testing problem.

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Statistics: an alternative approach

Frequentist and Bayesian methods address all these types of problem: point estimation, interval estimation, prediction and hypothesis testing.

But they use different approaches to do so.

Some typical frequentist approaches include:

- Least squares (point estimation)
- Maximum likelihood (point estimation)
- Confidence intervals (interval estimation)
- Test statistics and p-values (hypothesis testing)

Bayesian Statistics doesn't use any of these familiar methods!

Bayes goes to war!

- The work of Thomas Bayes (see later) was published in 1764, 3 years after his death
- Over the course of the next 100–150 years, it received little attention
- In fact, some key figures in Statistics – e.g. **R.A. Fisher** – outrightly rejected the idea of Bayesian statistics
- By the start of WW2, Bayes' rule was **virtually taboo** in the world of Statistics!
- During WW2, some of the world's leading mathematicians resurrected Bayes' rule in deepest secrecy to crack the coded messages of the Germans

Bayes goes to war!

- **Alan Turing** – mathematician working at Bletchley Park
- Designed the ‘bombe’ – an electro–mechanical machine for testing every possible permutation of a message produced on the Enigma machine — could take up to **4 days** to decode a message
- New system: **Banburismus** – where Bayesian methods were used to quantify the belief in guesses of a stretch of letters in an Enigma message
- Certain permutations – unlikely to be the original message – were ‘thrown out’ before they were even tested
- Greatly reduced the time it took to crack Enigma codes

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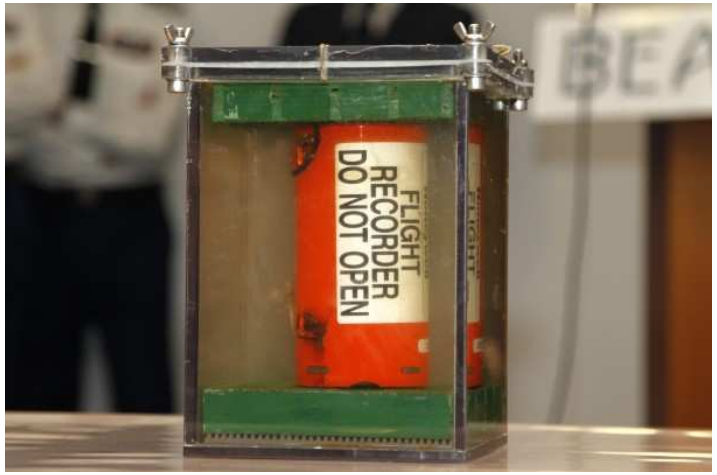
Air France 447

Heure Time	Provenance From	Vol Flight			Terminal	
1100	Los Angeles Minneapolis	AF	065	DL 8553	2E	Landed
1115	Rio Janeiro Intl	AF	447		2E	Delayed
1120	Bogota	AF	423		2E	Arrived 11:34
1125	Detroit Wayne Co	AF	377	DL 8573	2E	Arrived 11:29
1130	Damascus	AF	511		2E	Arrived 11:31
1130	Washington Dulles	AF	027	DL 8331	2E	Arrived 11:39
1135	Montreal	AF	349		2E	Arrived 11:41
1135	Sao Paulo Guarul	AF	459			Transfered 2C
1140	Istanbul	AF	2391	NW 4367	2E	Arrived 11:54
1140	Tunis	AF	1685	DL 8595	2E	Arrived 11:28
1150	Birmingham	AF	5133	MK 9331	2E	Arrived 11:42
1150	London-City	AF	5059	MK 9391	2E	Arrived 11:29
1210	Dublin	AF	5001	UX 3574	2E	Arrived 12:05



Air France 447

- Black boxes could be anywhere within an area of the South Atlantic the size of Switzerland (**6,500 square miles**)
- Mid–Atlantic ridge – between two tectonic plates – just as Mountainous as Switzerland!
- So remote – scientists have not yet charted the sea–bed
- Search method: sonar – detectors emitting sound waves which would bounce back once they hit something
- After two years of meticulously searching an area north of the plane's trajectory (after analysing debris drift) $\longrightarrow$ **nothing**

Air France 447

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- Included in the analysis were:
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  - Positions/recovery times of bodies found drifting – **expert opinions** assigned to the reliability of this data because of the turbulent equatorial waters
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Body Mass Index

Suppose we are interested in whether or not there is any real difference between the Body Mass Index (BMI) of Maths & Stats students and Food & Human Nutrition students.

BMI is often used as a measure of ‘fatness’ or ‘thinness’: it is the ratio of a person’s weight (measured in kg) to their squared height (measured in metres).

In a two–sample t–test, we often test the null hypothesis that there is no difference between the population means of the two groups:

$$H_0 : \mu_M = \mu_F,$$

How can we test this hypothesis?

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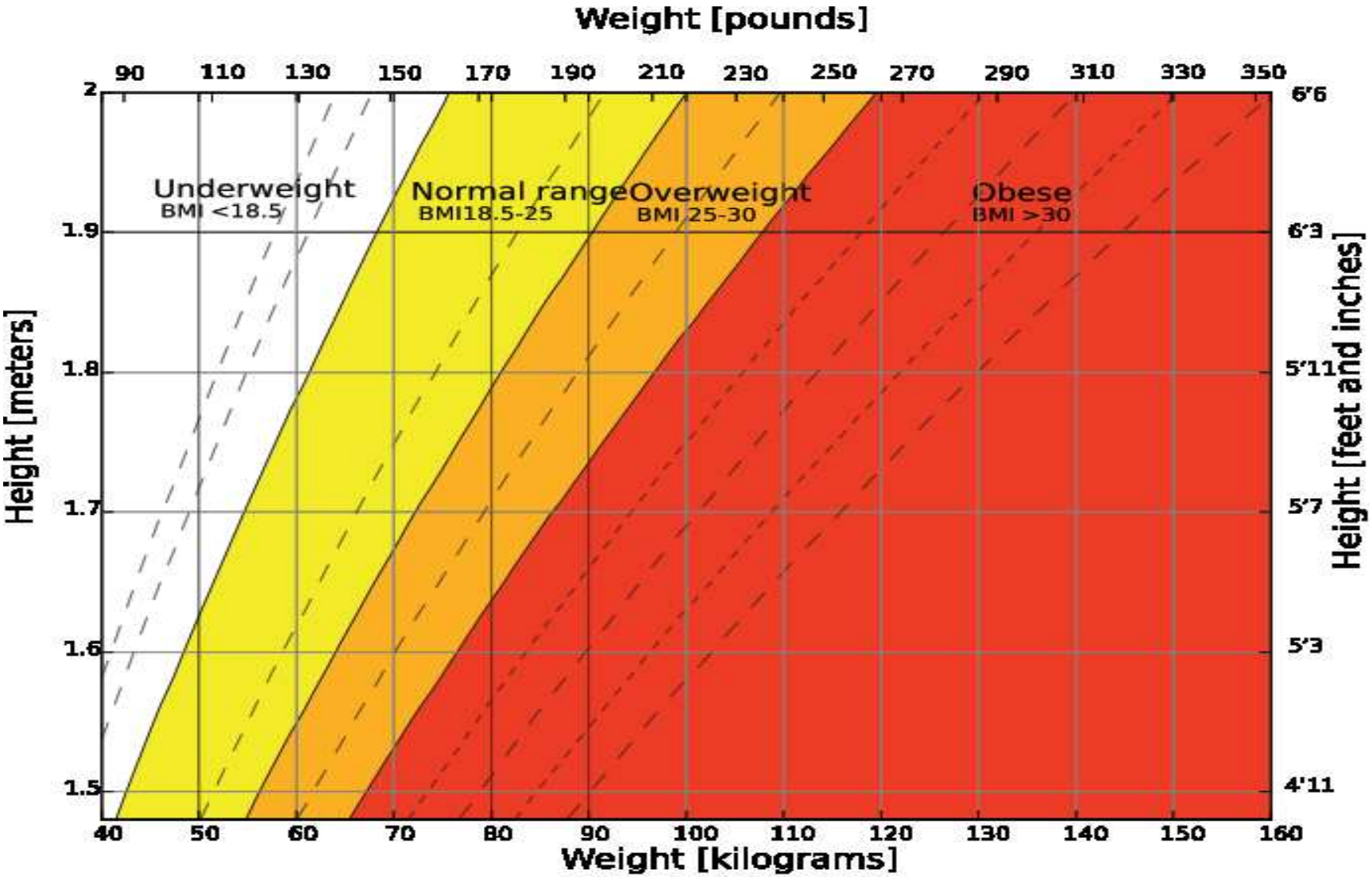

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Suppose we obtain the following BMI values for a random sample of students from both groups:

Maths & Stats	23.1	28.3	35.3	24.0	32.4	30.5	30.1	21.9	22.1
	29.9	27.2	33.2						
Food & Human Nutrition	25.5	27.3	21.0	23.3	19.1	22.2	29.5	27.5	23.2

From this we can obtain the summaries:

$$n_M = 12 \qquad \bar{x}_M = 28.17 \qquad s_M = 4.54$$

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Body Mass Index

The pooled standard deviation is

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\begin{aligned}
s &= \sqrt{\frac{(n_M - 1)s_M^2 + (n_F - 1)s_F^2}{n_M + n_F - 2}} \\
&= \sqrt{\frac{11 \times 4.54^2 + 8 \times 3.4^2}{12 + 9 - 2}} \\
&= 4.099
\end{aligned}
$$

Then the test statistic is

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\begin{aligned}
t &= \frac{|\bar{x}_M - \bar{x}_F|}{s \times \sqrt{\frac{1}{n_M} + \frac{1}{n_F}}} \\
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From t_{19}–tables, we get:

p–value	10%	5%	1%
critical value	1.729	2.093	2.861

Our test statistic $t = 2.147$ indicates that our p–value is less than 5%:

- We have a significant result at the 5% level of significance
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Notice from R that the *exact* p–value is 0.004488 = 4.488%.

- As we have already concluded, we would **reject** H_0 at the 5% level;
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What are we to do? Convention tells us to work at the 5% level, but why? Some practitioners work at the 1% level!

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In MAS1604 you were introduced to **Bayes' Theorem**.

As we shall see, Bayes' Theorem gives us a way of combining subjective assessments with observed data.

For example, what if – prior to observing the data – we believed that Food & Human Nutrition students were likely to have a **considerably lower BMI** than Maths & Stats students?

What if, from a previous study, we knew that Maths & Stats students were quite likely to have a **BMI somewhere between 25 and 33**?

Body Mass Index

Would it not be sensible to build this information into our analysis before forming our conclusions using the data alone?

This could be a good idea – we have relatively small samples, which could be biased!

A **Bayesian analysis** can do this!

Some people argue against this on the grounds that it is subjective... But as we have just seen, the usual approach to hypothesis testing is also subjective!

Not only are hypothesis tests tricky to interpret... but so are **confidence intervals**.

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Obtain a 95% confidence interval for the population mean BMI for Maths & Stats students.

We use

$$\bar{x}_M \pm t\frac{s_M}{\sqrt{n_M}}, \qquad \text{giving}$$

$$28.17 \quad \pm \quad 2.201 \times \frac{4.54}{\sqrt{12}} \qquad \text{i.e.}$$

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giving (25.28, 31.05).

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What is $\Pr(25.28 < \mu_M < 31.05)$?

NOT 0.95!! In the frequentist approach, population parameters (in this case μ_M) are NOT random variables!

This means μ_M is a fixed (but unknown) quantity. So it's either *in* the interval, or it's *not*, i.e.

$$\Pr(25.28 < \mu_M < 31.05) = 1 \qquad \text{or}$$

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Equivalent intervals in the **Bayesian framework** have more natural interpretations!

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Frequentist/Subjectivist debate

One of the main arguments *against* working within the Bayesian paradigm is that it is **subjective**.

Although we have not yet thought about *how* the Bayesian framework combines subjective assessments with the data, we have said that this is what it does.

Surely we should strive to be as *objective* as we can?

Frequentist/Subjectivist debate

Actually, to be able to incorporate a person's beliefs into an analysis of sample data is surely the **right thing to do** — especially if that person is an expert!

This is in line with how scientific experiments are carried out:

- The experimenter usually **knows something**;
- She then carries out the experiment in which **data are collected**;
- the experimenter then **updates her beliefs** from these results.

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In other words, the data are used to refine what the experimenter knows.

And that is what MAS2903 is all about – we will consider how

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Bayesian Statistics: leading players

Thomas Bayes (1702–1761)

- Presbyterian minister in Tunbridge Wells, Kent
- Bayes' solution to a problem of "inverse probability" was presented in the *Essay towards Solving a Problem in the Doctrine of Chances* (1764)
- Work published posthumously by his friend Richard Price in the *Philosophical Transactions of the Royal Society of London*.
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LII. *An Essay towards solving a Problem in the Doctrine of Chances. By the late Rev. Mr.* Bayes, *F. R. S. communicated by Mr.* Price, *in a Letter to* John Canton, *A. M. F. R. S.*

Dear Sir,

Read Dec. 23, 1763.

I Now send you an essay which I have found among the papers of our deceased friend Mr. Bayes, and which, in my opinion, has great merit, and well deserves to be preserved. Experimental philosophy, you will find, is nearly interested in the subject of it; and on this account there seems to be particular reason for thinking that a communication of it to the Royal Society cannot be improper.

He had, you know, the honour of being a member of that illustrious Society, and was much esteemed by many in it as a very able mathematician. In an introduction which he has writ to this Essay, he says, that his design at first in thinking on the subject of it was, to find out a method by which we might judge concerning the probability that an event has to happen, in given circumstances, upon supposition that we know nothing concerning it but that, under the same circum-

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Bruno di Finetti (1906–1985)

- Italian mathematical probabilist, developed his ideas on subjective probability in the 1920s.
- Famed for saying

  "Probability does not exist"

- By this, he meant it has no *objective* existence – i.e. it is not a feature of the world around us.
- It is a measure of degree–of–belief – your belief, my belief, someone else's belief – all of which could be different.

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- Worked hard to find a mathematical basis for the subject of Statistics
- With Leonard Savage, he found a deeper justification for Statistics in **Bayesian theory**
- Turned into a critic of the classical statistical inference he had hoped to justify
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Recent developments

One of the difficulties in early applications of Bayesian methodology was the maths:

Combining prior beliefs with a probability model for the data often resulted in maths that was just **too difficult/impossible to solve by hand**.

Recent developments

Then – in the early 1990s – a technique called "**Markov chain Monte Carlo**" (MCMC for short) was developed.

MCMC is a computer–intensive simulation–based procedure that gets around the problem of hard maths.

If you choose to take **MAS3902: Bayesian Inference** next year, you will be introduced to this technique.

MCMC has **revolutionised the use of Bayesian Statistics**, to the extent that Bayesian data analyses are now routinely used by non–Statisticians in fields as diverse as biology, civil engineering and sociology.

Newcastle's contribution

Number of academic staff publishing advanced research using Bayesian methods:

- 1985: **1** (Professor Boys)
- 2018: **Lots!**
  - *Systems Biology:* (Boys, Farrow, Gillespie, Golightly, Heaps, Henderson, Prangle, Wilkinson, Wilson)
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Newcastle's contribution: Environmental Extremes

Use Statistical methods to help plan for environmental extremes:

- **Hurricane–induced waves**: how high should we build a sea–wall to protect New Orleans against another *Hurricane Katrina*?
- **Extreme wind speeds**: how strong should we design buildings so they will not fail in storm–strength wind speeds?
- **Extreme cold spells**: How much fuel should we stockpile to cater for an extremely cold winter?

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We work within the Bayesian framework to combine

- expert information from hydrologists/oceanographers, with
- extreme rainfall data/hurricane–induced sea surge data,

to help estimate how likely an **extreme flooding event is to occur**, or to aid the **design of sea wall defences** in storm–prone regions.

If you're interested, you can see my web–page for more details.

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Quick quiz

Suppose A, θ and μ_M are constants and Z is a random variable, such that $A = 5$, $\theta = 10$, $\mu_M = 33.5$ and $Z \sim N(0, 1)$.

Write down

1. $\Pr(Z > 0)$
2. $\Pr(-1.96 < Z < 1.96)$
3. $\Pr(2 < A < 4)$
4. $\Pr(7 < \theta < 12)$
5. $\Pr(25.28 < \mu_M < 31.05)$

Chapter 1

Introduction

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Consider the following three experiments.

- **Experiment 1: Fisher's tea lady**
  The tea lady claims to know whether milk or tea is poured in first: for 10 pairs of cups of tea she makes the correct choice each time.
- **Experiment 2: Music expert**
  The expert claims he can distinguish between a page from a Haydn score and a page from a Mozart score: he does so correctly 10 times.
- **Experiment 3: The Drunk**
  A somewhat inebriated friend at a party claims they can predict the outcome of the toss of a coin: they do so correctly 10 times.

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Let $\theta = \Pr(\text{correct choice})$.

Let's suppose the tea lady, the music expert and the drunk *cannot* do as they claim.

Then, just by guessing, we could expect each of them to 'get it right' 5 times out of 10, i.e. $\theta = 1/2$.

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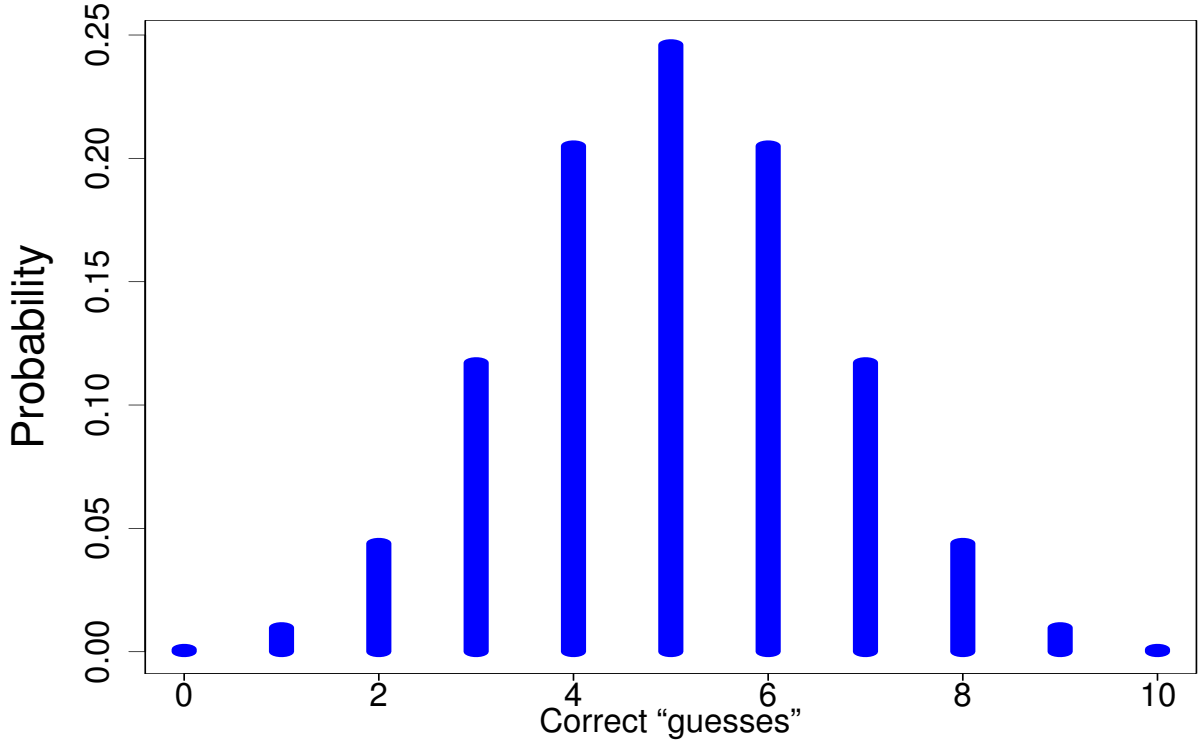
We could then test the **null hypothesis**

$$H_0 : \text{they were guessing}, \qquad \text{i.e.}$$

$$H_0 : \theta = 1/2;$$

Graphically, assuming H_0 is true, this gives:



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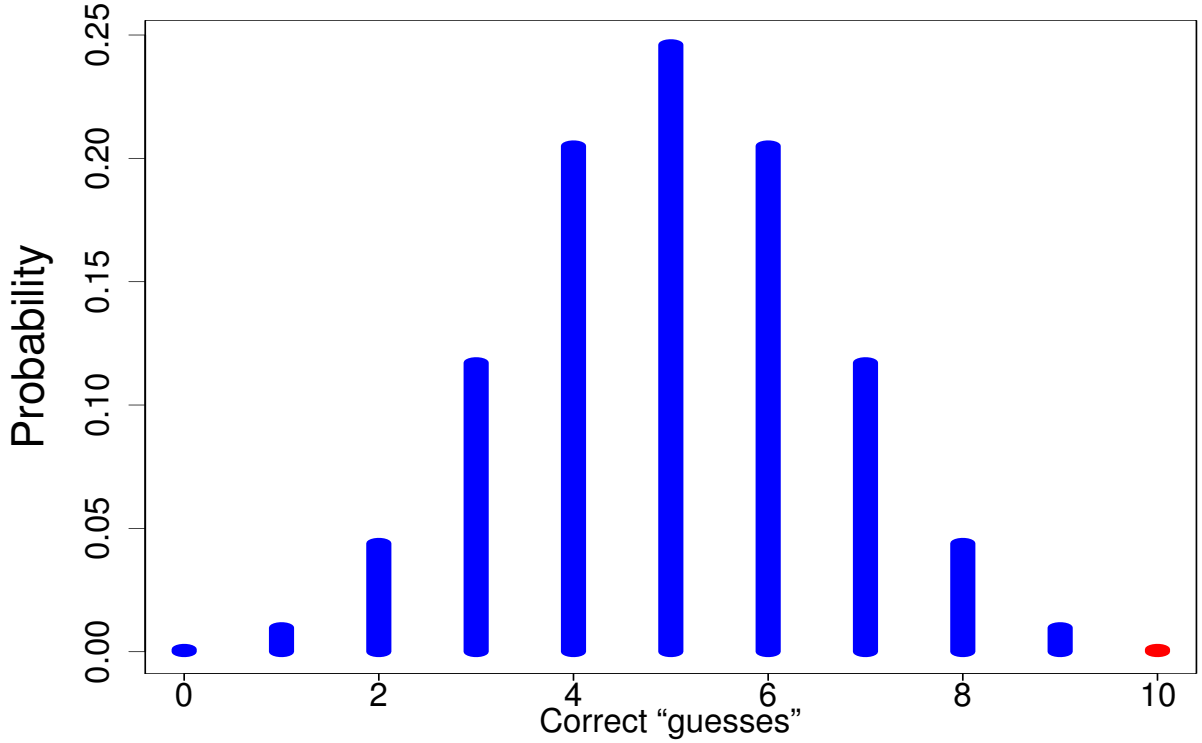

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This gives us:

$$p\text{−value} = \left(\frac{1}{2}\right)^{10} = 0.00098 < 0.1\%.$$

From this $p-$value, we would:

- conclude that we had **very strong evidence** against H_0
- conclude that the choices were **not just guesses**
- perhaps feel justified in **validating each claim**

But does this make sense?

Surely, we have some additional information about what values of θ are **plausible** for each experiment?

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- **Experiment 1:** $\theta > 0.5$ – folklore (and science!) suggests this may be possible;
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The traditional approach to Statistics, sometimes called **Frequentist Statistics** or **Classical Statistics**, may try to take this prior information into account:

For example, in Experiment 2, the test may be of $H_0 : \theta \geq 0.9$ against $H_1 : \theta < 0.9$.

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The key ingredients of a Bayesian analysis are

- a statistical model for the experimental data;
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Probability

The concept of probability (chance) has been around for a very long time, particularly in the area of **gambling**.

Games of chance have been played since about **3500 B.C.**

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The mathematical theory of probability was started around the 17th century by **Galilei**, **Pascal** and **Fermat** to solve (again) gambling problems.

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Frequency interpretation

The probability of an outcome is the **relative frequency** with which the outcome would be obtained if the experiment were repeated a large number of times *under similar conditions*.

For example, if a coin is tossed 1,000,000 times and a head appears n times then

$$\Pr(\text{Head}) = \frac{n}{1,000,000}.$$

We would expect this probability to be about 0.5.

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Your **subjective probability** for an outcome A represents your own judgement of the likelihood that the outcome will occur.

This judgement will be based on the beliefs and information H you have at the time.

One way of determining (or quantifying) a subjective value for $\Pr_H(A)$ is to consider a series of possible bets with outcome

$$\text{win £}c \text{ if } A \text{ occurs and £0 if } A^c \text{ occurs.}$$

How much would you be prepared to pay (stake) for placing such a bet?

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Subjective interpretation (1/1)

In terms of expected winnings, you should be prepared to stake £cp if you believe that $\Pr_H(A) = p$. Why?

Suppose you would win £10 if A occurs and £0 if it does not. Suppose further that you believe A is likely to occur with probability $p = 0.3$.

Then the **expected monetary value** of the bet is
£10 $\times$ 0.3 = £3;

that is, on average, we can expect to win £3 from this bet. Thus, staking any more than £3 would not be wise!

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In terms of expected winnings, you should be prepared to stake £cp if you believe that $\Pr_H(A) = p$. Why?

Suppose you would win £10 if A occurs and £0 if it does not. Suppose further that you believe A is likely to occur with probability $p = 0.3$.

Then the **expected monetary value** of the bet is
£$10 \times 0.3 =$ £3;

that is, on average, we can expect to win £3 from this bet. Thus, staking any more than £3 would not be wise!

Subjective interpretation

One problem with this approach is that, in general, p will depend on c.

A person who is willing to bet £1 on the spin of a coin to win £2 if it lands heads may refuse to bet if the stakes are raised to £1000: most people are **risk–averse**.

Therefore, we shall restrict our attention to the $c = 1$ case: pay £p for the bet

$$\text{win £1 if } A \text{ occurs and £0 if } A^c \text{ occurs.}$$

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Subjective interpretation

You can make sure that your bet is "honest" by randomising between whether you "**host**" the bet or "**place**" the bet.

For example, suppose you believe that $\Pr_H(A) = 0.5$.

An "honest" bet would mean that you would buy the bet for a maximum stake of £0.50.

However, if you weren't honest you might try to buy the bet for any amount less than £0.50, say £0.20.

If you were hosting the bet, you would take the bet for any amount more than £p, say for £0.80.

These conflicting interests can be offset if, when choosing p, it is equally likely that you are hosting the bet or placing the bet.

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Frequency interpretation: drawbacks

1. It does not say how many times the experiment should be repeated
2. "Similar conditions" is a vague concept
3. It is not appropriate for many probability calculations of one-off events
4. Standard statistical methods using the frequentist approach are not totally objective since they require subjective judgements about the validity of probability models, choice of hypotheses and interpretation of results (for instance, see BMI example in the preface to these lecture notes)

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1. It is not objective – but perhaps it is more obvious (honest) about when subjective beliefs are used
2. It requires people to be **coherent**: they will not make any wagers which they are certain to lose; also, they will not prefer to suffer a given penalty when there is the option of another penalty which is certainly smaller

   Being coherent results in, *inter alia*, that

$$\Pr(A_1|H) > \Pr(A_2|H) \quad \text{and} \quad \Pr(A_2|H) > \Pr(A_3|H) \implies \Pr(A_1|H) > \Pr(A_3|H).$$

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Probability

Each of these interpretations use quite different methods of reasoning.

In this course – unlike any other course you have taken so far – we will concentrate on the subjective interpretation and describe how, if carefully used, it can be a more useful approach than the other two methods.

Everything we do rests on **Bayes' Theorem**, and we review this now.

Bayes' Theorem

Before we state **Bayes' Theorem**, we need a recap of **conditional probability**.

Definition (Conditional Probability)

Consider two events E and F, where $\Pr(F) > 0$.

The **conditional probability** of E given that F has occurred is

$$\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$

Bayes' Theorem

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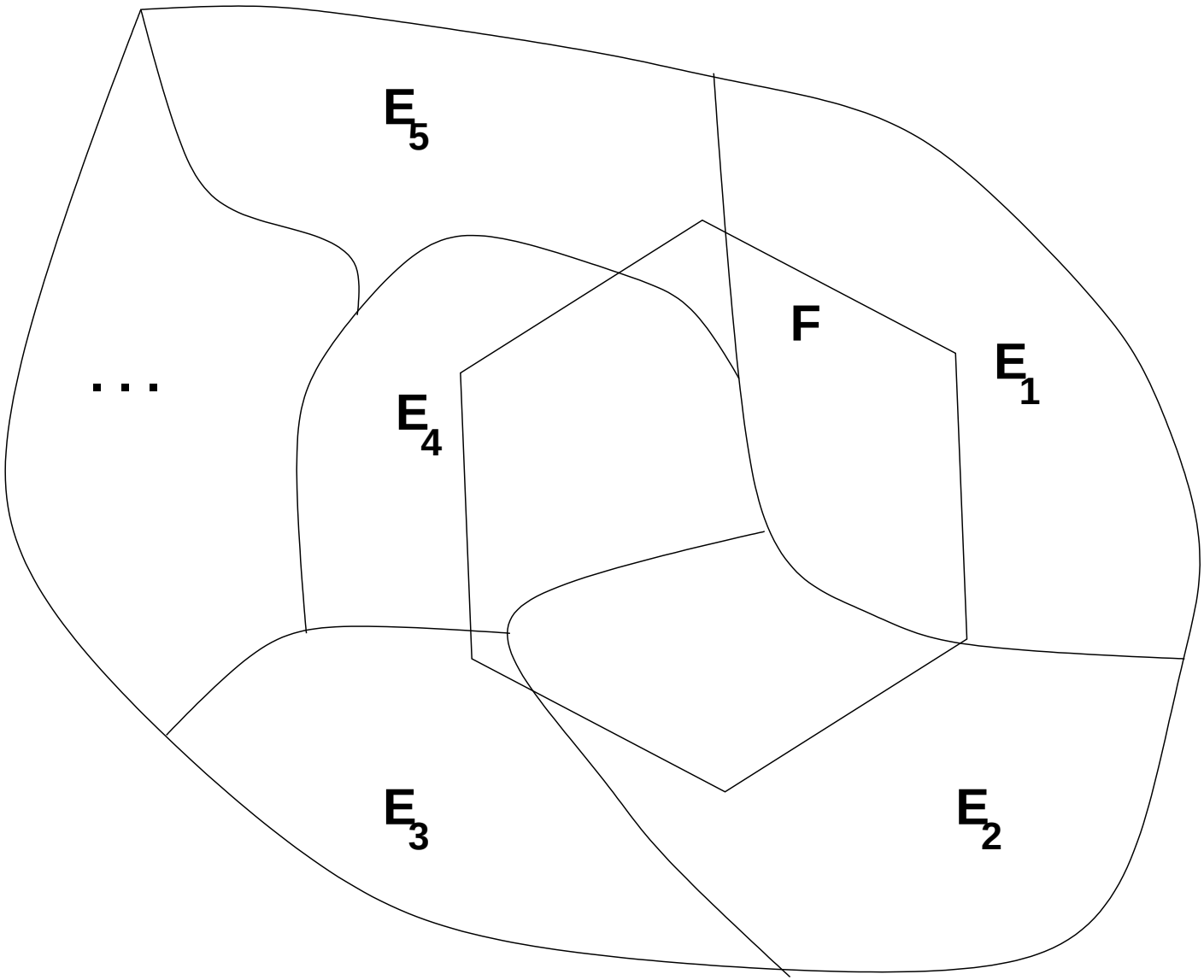
Bayes' Theorem

Definition (Parition)

The events $E_1, E_2, \ldots, E_n$ form a **partition** of the sample space $\mathcal{S}$ if they are disjoint events ($E_i \cap E_j = \emptyset$, $i \neq j$) with $\Pr(E_i) > 0$, $i = 1, 2, \ldots, n$, and $\cup_{i=1}^{n} E_i = \mathcal{S}$.

Figure 1.1 gives a diagram of a typical partition with an additional event F.

Bayes' Theorem



Bayes' Theorem

Fact (Law of Total Probability)

If $E_1, E_2, \ldots, E_n$ are a **partition** *of $\mathcal{S}$ and F is any event then*

$$\Pr(F) = \sum_{i=1}^{n} \Pr(F|E_i)\Pr(E_i).$$

Law of Total Probability: Proof

As $E_1, E_2, \ldots, E_n$ are a **partition** of $\mathcal{S}$, we have

$$\Pr(F) = \Pr(F \cap E_1) + \Pr(F \cap E_2) + \ldots + \Pr(F \cap E_n)$$

$$= \Pr(F|E_1)\Pr(E_1) + \Pr(F|E_2)\Pr(E_2) + \ldots + \Pr(F|E_n)\Pr(E_n)$$

(by conditional probability)

$$= \sum_{i=1}^{n} \Pr(F|E_i)\Pr(E_i).$$

Bayes' Theorem

Theorem (Bayes' Theorem)

If $E_1, E_2, \ldots, E_n$ *are a* **partition** *of* $\mathcal{S}$ *and* F *is any event with* $\Pr(F) > 0$ *then*

$$\Pr(E_i|F) = \frac{\Pr(F|E_i)\Pr(E_i)}{\sum_{j=1}^{n} \Pr(F|E_j)\Pr(E_j)}, \qquad i = 1, 2, \ldots, n.$$

Bayes' Theorem: Proof (slide 1/1)

Using the definition of conditional probability, for $i = 1, 2, \ldots, n$,

$$\Pr(E_i|F) = \frac{\Pr(E_i \cap F)}{\Pr(F)}$$

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Example 1.2

A laboratory blood test is 95% effective in detecting a certain disease when it is present.

However, the test also yields a “false positive” result for 1% of healthy people tested.

Also, 0.5% of the population actually have the disease.

(a) Calculate the probability that a person who tests positive actually has the disease.

(b) Find the probability that a person who tests negative does *not* have the disease.

Solution to Example 1.2 (slide 1/3)

Let D = person has the disease, +ve = result is postive and –ve = result is negative.

We require (a) $\Pr(D|\text{+ve})$, and (b) $\Pr(D^c|\text{–ve})$.

From the question we have

- $\Pr(\text{+ve}|D) = 0.95$
- $\Pr(\text{+ve}|D^c) = 0.01$
- $\Pr(D) = 0.005$ and
- $\Pr(D^c) = 0.995$

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Solution to Example 1.2 (slide 2/3)

Also, D and D^c form a partition of $\mathcal{S}$. Therefore, by Bayes' Theorem

$$\Pr(D|+\text{ve}) = \frac{\Pr(+\text{ve}|D)\Pr(D)}{\Pr(+\text{ve}|D)\Pr(D) + \Pr(+\text{ve}|D^c)\Pr(D^c)}$$

$$= \frac{0.95 \times 0.005}{0.95 \times 0.005 \; + \; 0.01 \times 0.995}$$

$$\simeq 0.323.$$

Thus, only 32.3% of people who test positive actually have the disease – 67.7% of people who test positive do *not* have the disease!

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Solution to Example 1.2 (slide 3/3)

Now for part (b),

$$\Pr(D^c|\text{–ve}) = \frac{\Pr(\text{–ve}|D^c)\Pr(D^c)}{\Pr(\text{–ve}|D^c)\Pr(D^c) + \Pr(\text{–ve}|D)\Pr(D)}$$

$$= \frac{0.99 \times 0.995}{0.99 \times 0.995 \ + \ 0.05 \times 0.005}$$

$$\simeq 0.9997.$$

Therefore, nearly everyone who tests negative does not have the disease – very few people who test negative *will* have the disease.

Solution to Example 1.2 (slide 3/3)

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Quick recap

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  - Deciphering Enigma messages
  - Air France 447
  - Body Mass Index
- Recap of probability
  - Frequency, classical and subjective interpretations
  - Limitations
  - Rules: conditional probability, total probability, Bayes
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Example 1.3

Suppose that your car suffers from two intermittent problems, one caused by a fault in the engine (θ_1) and the other due to a fault in the gearbox (θ_2).

These occur with probabilities 0.4 and 0.6 respectively.

When examined your car exhibits one of the following symptoms

$$
\begin{aligned}
x_1 &: \text{overheating only,} \\
x_2 &: \text{irregular traction only,} \\
x_3 &: \text{both symptoms.}
\end{aligned}
$$

Example 1.3

Suppose it is known in the garage trade that these symptoms occur with probabilities that depend on the fault.

The probabilities $\Pr(X = x|\theta)$ are given in Table 1.1.

	O/H x_1	I/T x_2	Both x_3
θ_1: fault in engine	0.1	0.4	0.5
θ_2: fault in gearbox	0.5	0.3	0.2

Construct a diagnostic rule for these symptoms and determine the probability of misdiagnosis.

Solution to Example 1.3 (1/3)

First we must calculate the **posterior probabilities** $\Pr(\theta_1|x)$ and $\Pr(\theta_2|x)$ for $x = x_1, x_2, x_3$.

Since θ_1 and θ_2 form a partition, we can use Bayes' Theorem as follows.

We have

$$\Pr(\theta_1|x_1) = \frac{\Pr(X = x_1|\theta_1)\Pr(\theta_1)}{\Pr(X = x_1|\theta_1)\Pr(\theta_1) + \Pr(X = x_1|\theta_2)\Pr(\theta_2)}$$

$$= \frac{0.1 \times 0.4}{0.1 \times 0.4 + 0.5 \times 0.6}$$

$$= \frac{4}{34} = 0.118.$$

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Solution to Example 1.3 (2/3)

Also,

$$\Pr(\theta_1|x_2) = \frac{\Pr(X = x_2|\theta_1)\Pr(\theta_1)}{\Pr(X = x_2|\theta_1)\Pr(\theta_1) + \Pr(X = x_2|\theta_2)\Pr(\theta_2)}$$

$$= \frac{0.4 \times 0.4}{0.4 \times 0.4 + 0.3 \times 0.6}$$

$$= \frac{16}{34} = 0.471.$$

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$$= \frac{16}{34} = 0.471.$$

Solution to Example 1.3 (3/3)

And

$$\Pr(\theta_1|x_3) = \frac{\Pr(X = x_3|\theta_1)\Pr(\theta_1)}{\Pr(X = x_3|\theta_1)\Pr(\theta_1) + \Pr(X = x_3|\theta_2)\Pr(\theta_2)}$$

$$= \frac{0.5 \times 0.4}{0.5 \times 0.4 + 0.2 \times 0.6}$$

$$= \frac{20}{32} = 0.625.$$

Also, $\Pr(\theta_2|x_i) = 1 - \Pr(\theta_1|x_i)$, $i = 1, 2, 3$, and so we obtain the **posterior distributions** $\Pr(\theta|x)$ given in Table 1.2.

Solution to Example 1.3 (3/3)

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Example 1.3

	O/H x_1	I/T x_2	Both x_3
θ_1: fault in engine	0.118	0.471	0.625
θ_2: fault in gearbox	0.882	0.529	0.375

This table is very informative. For example, it shows that if both symptoms (x_3) are observed, then the probability that the fault is in the engine (θ_1) changes from 0.4 to 0.625.

In terms of **odds**:

$$\text{Prior odds}: \frac{\Pr(\theta_1)}{\Pr(\theta_2)} = \frac{0.4}{0.6} = \frac{2}{3} \quad \text{or 3:2 in favour of } \theta_2$$

$$\text{Posterior odds}: \frac{\Pr(\theta_1|x_3)}{\Pr(\theta_2|x_3)} = \frac{0.625}{0.375} = \frac{5}{3} \quad \text{or 5:3 in favour of } \theta_1.$$

Example 1.3

	O/H x_1	I/T x_2	Both x_3
θ_1: fault in engine	0.118	0.471	0.625
θ_2: fault in gearbox	0.882	0.529	0.375

This table is very informative. For example, it shows that if both symptoms (x_3) are observed, then the probability that the fault is in the engine (θ_1) changes from 0.4 to 0.625.

In terms of **odds**:

$$\text{Prior odds}: \frac{\Pr(\theta_1)}{\Pr(\theta_2)} = \frac{0.4}{0.6} = \frac{2}{3} \quad \text{or 3:2 in favour of } \theta_2$$

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Example 1.3

We are now in a position to design our **diagnostic rule**.

This is simply a rule which diagnoses a symptom (x) as being due to some particular fault (θ).

Consider first that we observe overheating only (x_1). The posterior probabilities are in favour of declaring the fault as in the gearbox (θ_2) since $\Pr(\theta_2|x_1) > \Pr(\theta_1|x_1)$.

In the same way, we can determine the most likely diagnosis having observed irregular traction only (x_2) and both symptoms (x_3), giving the diagnostic rule in Table 1.3.

Symptom	Diagnosis
overheating only (x_1)	fault in gearbox (θ_2)
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Solution to Example 1.3 (1/1)

If we were to carry out this diagnostic rule repeatedly then the probability of misdiagnosing a fault is

$$\begin{aligned}
&\Pr(\text{Misdiagnosis}) \\
&= \Pr(\theta_1, X = x_1) + \Pr(\theta_1, X = x_2) + \Pr(\theta_2, X = x_3) \\
&= \Pr(X = x_1|\theta_1)\Pr(\theta_1) + \Pr(X = x_2|\theta_1)\Pr(\theta_1) \\
&\qquad + \Pr(X = x_3|\theta_2)\Pr(\theta_2) \\
&= (0.1 \times 0.4) + (0.4 \times 0.4) + (0.2 \times 0.6) \\
&= 0.32.
\end{aligned}$$

Therefore, in repeated use of this rule, around a third of the diagnoses will be wrong.

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If we were to carry out this diagnostic rule repeatedly then the probability of misdiagnosing a fault is

$$\Pr(\text{Misdiagnosis})$$

$$= \Pr(\theta_1, X = x_1) + \Pr(\theta_1, X = x_2) + \Pr(\theta_2, X = x_3)$$

$$= \Pr(X = x_1|\theta_1)\Pr(\theta_1) + \Pr(X = x_2|\theta_1)\Pr(\theta_1) + \Pr(X = x_3|\theta_2)\Pr(\theta_2)$$

$$= (0.1 \times 0.4) + (0.4 \times 0.4) + (0.2 \times 0.6)$$

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&= (0.1 \times 0.4) + (0.4 \times 0.4) + (0.2 \times 0.6) \\
&= 0.32.
\end{aligned}$$

Therefore, in repeated use of this rule, around a third of the diagnoses will be wrong.

Example 1.4

A student sits a multiple choice exam in which there are m alternative answers to each question.

The student either knows the answer (with probability θ) or guesses randomly (with probability $1 - \theta$).

What is the probability that the student actually knew the answer to a question they answered correctly?

Solution to Example 1.4 (1/2)

Let

$$C = \text{student answers question correctly}$$
$$K = \text{student knows answer.}$$

We require $\Pr(K|C)$. From the question we have

- $\Pr(K) = \theta$
- $\Pr(K^c) = 1 - \theta$
- $\Pr(C|K) = 1$ and
- $\Pr(C|K^c) = \frac{1}{m}$.

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- $\Pr(C|K^c) = \frac{1}{m}$.

Solution to Example 1.4 (2/2)

Also, K and K^c form a partition of $\mathcal{S}$. Therefore, by Bayes' Theorem,

$$\begin{aligned}
\Pr(K|C) &= \frac{\Pr(C|K)\Pr(K)}{\Pr(C|K)\Pr(K) + \Pr(C|K^c)\Pr(K^c)} \\
&= \frac{1 \times \theta}{1 \times \theta \; + \; \frac{1}{m} \times (1-\theta)} \\
&= \frac{\theta}{\frac{m\theta + (1-\theta)}{m}} \\
&= \frac{m\theta}{1 + (m-1)\theta}.
\end{aligned}$$

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\end{aligned}$$

Example 1.4

Suppose that there are $m = 5$ alternative answers for each question. Then

$$\Pr(K|C) = \frac{5\theta}{1 + 4\theta}.$$

We can see the effect of observing a correct answer on our belief that the student actually knows the answer by calculating $\Pr(K|C)$ for various θ – see Table 1.4.

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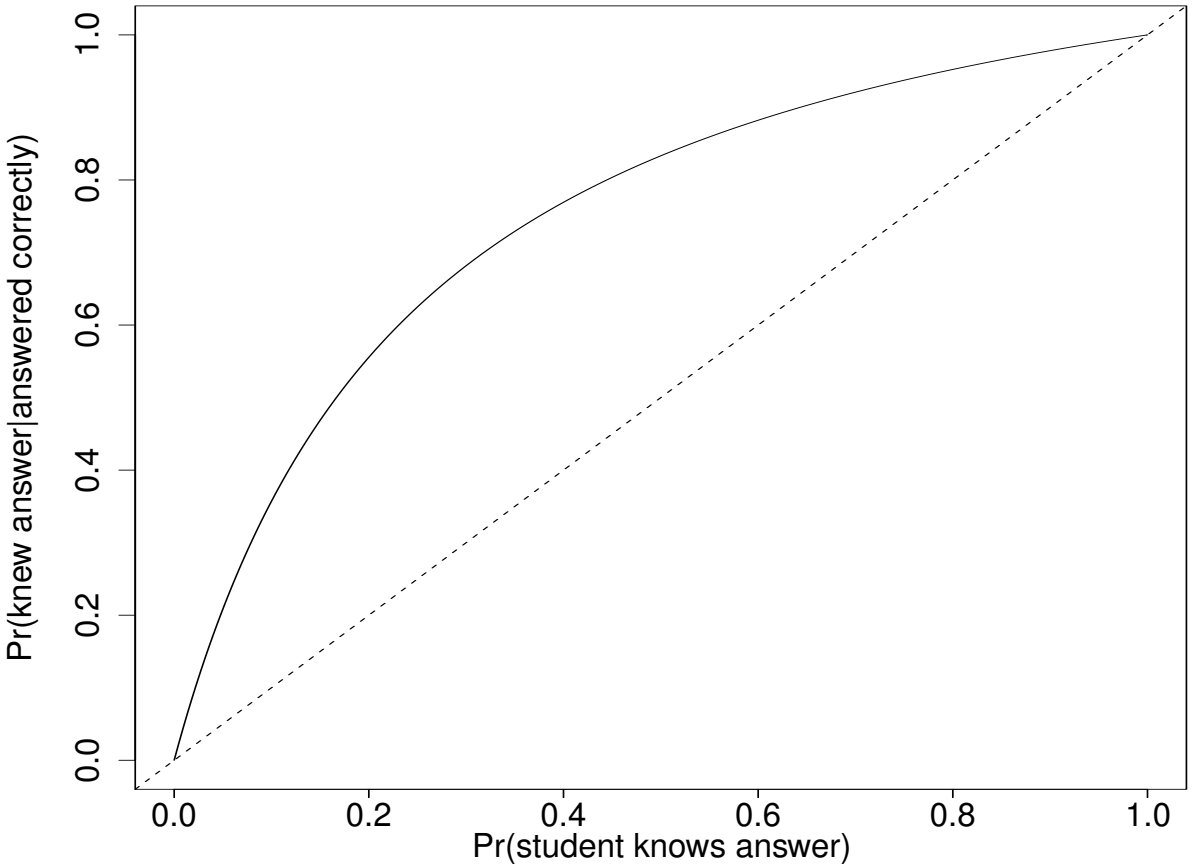
Example 1.4

$\Pr(K)$ $= \theta$	$\Pr(K\|C)$ $= 5\theta/(1+4\theta)$
0.0	0.000
0.1	0.357
0.2	0.556
0.3	0.682
0.4	0.769
0.5	0.833
0.6	0.882
0.7	0.921
0.8	0.952
0.9	0.978
1.0	1.000

Example 1.4


Pr(knew answer|answered correctly)
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.2
0.4
0.6
0.8
1.0
Pr(student knows answer)

Likelihood

Suppose that an experiment results in data $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ and we decide to model the data using a probability (density) function $f(\boldsymbol{x}|\theta)$.

This p(d)f describes how likely different data $\boldsymbol{x}$ are to occur given a value of the (unknown) parameter θ.

However, once we have observed the data, $f(\boldsymbol{x}|\theta)$ tells us how likely different values of the parameters θ are: it is then known as the **likelihood function** for θ.

In other courses you may have seen it written as $L(\theta|\boldsymbol{x})$ or $L(\theta)$ but, whatever the notation used for the likelihood function, it is simply the joint probability (density) function of the data, $f(\boldsymbol{x}|\theta)$, regarded as a function of θ rather than of $\boldsymbol{x}$.

Likelihood

The likelihood function can be simplified if we have further structure in the data.

For example, we may have independent observations, in which case

$$f(\boldsymbol{x}|\theta) = \prod_{i=1}^{n} f_{X_i}(x_i|\theta), \tag{1.1}$$

or independent and identically distributed observations (random sample), so that

$$f(\boldsymbol{x}|\theta) = \prod_{i=1}^{n} f_{X}(x_i|\theta). \tag{1.2}$$

Example 1.5

Suppose we have a random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ of radioactive particle counts.

A typical model for such data would be $X_i|\theta \sim Poisson(\theta)$, usually abbreviated $X_i|\theta \sim Po(\theta)$, (independent).

Determine the likelihood function for θ.

Example 1.5

Suppose we have a random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ of radioactive particle counts.

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Determine the likelihood function for θ.

Solution to Example 1.5 (1/1)

Using Equation (1.2), the likelihood function is

$$
\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} \\
&= \frac{e^{-\theta} \times \ldots \times e^{-\theta} \times \theta^{x_1} \times \ldots \times \theta^{x_n}}{\prod_{i=1}^{n} x_i!} \\
&= \frac{e^{-\theta-\theta-\ldots-\theta}\theta^{x_1+\ldots+x_n}}{\prod_{i=1}^{n} x_i!} \\
&= \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!} \\
&= \frac{e^{-n\theta}\theta^{n\bar{x}}}{\prod_{i=1}^{n} x_i!}.
\end{aligned}
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&= \frac{e^{-n\theta}\theta^{n\bar{x}}}{\prod_{i=1}^{n} x_i!}.
\end{aligned}
$$

Example 1.6

Suppose we have a random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ of times between radioactive particle emissions.

If the emissions occur randomly in time then a plausible model for such data would be $X_i|\theta \sim \mathit{Exponential}(\theta)$, usually abbreviated $X_i|\theta \sim \mathit{Exp}(\theta)$, (independent).

Determine the likelihood function for θ.

Solution to Example 1.6 (1/1)

Using Equation (1.2), the likelihood function is

$$
\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta e^{-\theta x_i} \\
&= \theta \times \ldots \times \theta \times e^{-\theta x_1} \times \ldots \times e^{-\theta x_n} \\
&= \theta^n e^{-\theta x_1 - \theta x_2 - \ldots - \theta x_n} \\
&= \theta^n e^{-\theta(x_1 + \ldots + x_n)} \\
&= \theta^n e^{-\theta n \bar{x}}.
\end{aligned}
$$

Solution to Example 1.6 (1/1)

Using Equation (1.2), the likelihood function is

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\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta e^{-\theta x_i} \\
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&= \theta^n e^{-\theta(x_1 + \ldots + x_n)} \\
&= \theta^n e^{-\theta n \bar{x}}.
\end{aligned}
$$

Example 1.7

Suppose we have a random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ from a Normal distribution: $X_i|\mu, \sigma \sim N(\mu, \sigma^2)$, $i = 1, 2, \ldots, n$ (independent).

Determine the likelihood function for (μ, σ).

Solution to Example 1.7 (1/1)

The (joint) probability density function is

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\begin{aligned}
f(\boldsymbol{x}|\mu,\sigma) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\} \\
&= (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \\
&= (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum x_i^2 - 2\mu\sum x_i + n\mu^2\right)\right\}.
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Sufficiency

Consider again the Poisson model in Example 1.5. The likelihood function is

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\begin{aligned}
f(\boldsymbol{x}|\theta) &= \frac{\theta^{n\bar{x}} e^{-n\theta}}{\prod_{i=1}^{n} x_i!} \\
&= \left(\prod_{i=1}^{n} x_i! \right)^{-1} \times \theta^{n\bar{x}} e^{-n\theta}.
\end{aligned}
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Notice that this likelihood function depends on the data only through $\prod_{i=1}^{n} (x_i!)^{-1}$ and $\bar{x}$.

Further, in $f(\boldsymbol{x}|\theta)$, θ only "interacts" with $\bar{x}$ — the other term simply scales $f(\boldsymbol{x}|\theta)$ — so that, for example, the point at which $f(\boldsymbol{x}|\theta)$ is maximized is determined only by $\bar{x}$.

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Informally, we think of all the information about θ in the data being contained in $\bar{x}$.

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A **statistic** is any function of the data (and not of unknown parameters).

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Example 1.8

Consider again the Poisson model in Example 1.5.

Suppose we had just two observations.

Then $n = 2$ and $X_i|\theta \sim Po(\theta)$, $i = 1, 2$ (independent).

Show that $T = X_1 + X_2$ is sufficient for θ. Note that $T|\theta \sim Po(2\theta)$.

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Solution to Example 1.8 (1/2)

Consider the conditional distribution of the data $\boldsymbol{X}$ on $T = t$.

As X_1 and X_2 are discrete random variables, we have

$$\begin{aligned} f(\boldsymbol{x}|T(\boldsymbol{X}) = t) &= \frac{\Pr(X_1 = x_1, X_2 = x_2, T = t)}{\Pr(T = t)} \\ &= \frac{\Pr(X_1 = x_1, X_2 = t - x_1)}{\Pr(T = t)}. \end{aligned}$$

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For the denominator we know that

$$\Pr(T = t|\theta) = \frac{e^{-2\theta}(2\theta)^t}{t!} \qquad \text{(sum of two Poissons).}$$

Therefore

$$\begin{aligned} f(\boldsymbol{x}|T(\boldsymbol{X}) = t) &= \frac{e^{-2\theta}\theta^t}{x_1!(t-x_1)!} \Big/ \frac{e^{-2\theta}(2\theta)^t}{t!} \\ &= \frac{t!e^{-2\theta}\theta^t}{x_1!(t-x_1)!e^{-2\theta}(2\theta)^t} \\ &= \frac{t!}{2^t x_1!(t-x_1)!} \end{aligned}$$

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Sufficiency

Definition (Joint sufficiency)

The statistics $\underline{\boldsymbol{T}}(\boldsymbol{X}) = \big(T_1(\boldsymbol{X}), T_2(\boldsymbol{X}), \ldots, T_k(\boldsymbol{X})\big)^T$ are **jointly sufficient** for θ if $f(\boldsymbol{x}|\boldsymbol{T}(\boldsymbol{X}) = \boldsymbol{t})$ does not depend on θ.

Theorem (Factorisation Theorem)

Under certain regularity conditions

$$\boldsymbol{T}(\boldsymbol{X}) \text{ is sufficient for } \theta \iff f(\boldsymbol{x}|\theta) = h(\boldsymbol{x})\, g(\boldsymbol{t}(\boldsymbol{x}), \theta)$$

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Example 1.9

Consider again the Poisson model in Example 1.5 with $n = 2$: $X_i|\theta \sim Po(\theta)$, $i = 1, 2$ (independent). Determine a sufficient statistic for θ.

Solution to Example 1.9 (1/1)

The (joint) probability function is

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\begin{aligned}
f(\boldsymbol{x}|\theta) &= \frac{e^{-\theta}\theta^{x_1}}{x_1!} \times \frac{e^{-\theta}\theta^{x_2}}{x_2!} \\
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where $h(\boldsymbol{x}) = 1/(x_1!x_2!)$ and $g(t, \theta) = e^{-2\theta}\theta^t$.

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The (joint) probability function is

$$\begin{aligned} f(\boldsymbol{x}|\theta) &= \frac{e^{-\theta}\theta^{x_1}}{x_1!} \times \frac{e^{-\theta}\theta^{x_2}}{x_2!} \\ &= \frac{e^{-2\theta}\theta^{x_1+x_2}}{x_1!x_2!} \\ &= \frac{1}{x_1!x_2!} \times e^{-2\theta}\theta^{x_1+x_2} \\ &= h(\boldsymbol{x})\, g(x_1 + x_2, \theta), \end{aligned}$$

where $h(\boldsymbol{x}) = 1/(x_1!x_2!)$ and $g(t, \theta) = e^{-2\theta}\theta^t$.

Therefore, by the Factorisation Theorem, $T = X_1 + X_2$ is sufficient for θ.

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Therefore, by the Factorisation Theorem, $T = X_1 + X_2$ is sufficient for θ.

Example 1.10

Suppose we have a random sample from an exponential distribution: $X_i|\theta \sim Exp(\theta),\ i = 1, 2, \ldots, n$ (independent).

Determine a sufficient statistic for θ.

Solution to Example 1.10 (1/1)

The (joint) probability density function is

$$
\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta e^{-\theta x_i} \\
&= \theta^n \exp\left(-\theta \sum_{i=1}^{n} x_i\right) \\
&= 1 \times \theta^n \exp\left(-\theta \sum_{i=1}^{n} x_i\right) \\
&= h(\boldsymbol{x})\, g(\Sigma\, x_i, \theta),
\end{aligned}
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where $h(\boldsymbol{x}) = 1$ and $g(t, \theta) = \theta^n e^{-\theta t}$. Therefore, by the Factorisation Theorem, $T = \sum_{i=1}^{n} X_i$ is sufficient for θ.

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Example 1.11

Suppose we have a random sample from a Normal distribution: $X_i|\mu, \sigma \sim N(\mu, \sigma^2)$, $i = 1, 2, \ldots, n$ (independent).

Determine sufficient statistics for (μ, σ).

Solution to Example 1.11 (1/1)

The (joint) probability density function is

$$
\begin{aligned}
f(\boldsymbol{x}|\mu,\sigma) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i-\mu)^2}{2\sigma^2}\right\} \\
&= (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \\
&= (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\left(\sum x_i^2 - 2\mu\sum x_i + n\mu^2\right)\right\} \\
&= h(\boldsymbol{x})\, g(\Sigma\, x_i, \Sigma\, x_i^2, \mu, \sigma),
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where $h(\boldsymbol{x}) = (2\pi)^{-n/2}$ and
$g(t_1, t_2, \mu, \sigma) = \sigma^{-n}\exp\{-(t_2 - 2\mu t_1 + n\mu^2)/(2\sigma^2)\}$.

Therefore, by the Factorisation Theorem, $T_1 = \sum_{i=1}^{n} X_i$ and $T_2 = \sum_{i=1}^{n} X_i^2$ are (jointly) sufficient for μ and σ.

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Quick quiz

If $X_1, X_2, \ldots, X_n$ are independent random variables with probability density function

$$f(x|\theta) = \theta(1-\theta)^{x-1}, \quad x = 1, 2, \ldots,$$

(a) Form the likelihood function $f(\boldsymbol{x}|\theta)$;

(b) Use the Factorisation Theorem to obtain a sufficient statistic for θ.

Quick quiz: solution

The joint probability function is

$$
\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} \\
&= \theta^n(1-\theta)^{x_1-1}(1-\theta)^{x_2-1} \times \ldots \times (1-\theta)^{x_n-1} \\
&= \theta^n(1-\theta)^{\sum x_i - n} \\
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where $h(\boldsymbol{x}) = 1$ and $g(t, \theta) = \theta^n(1-\theta)^{t-n}$. Therefore, by the Factorisation Theorem, $T = \sum X_i$ is sufficient for θ.

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Chapter 2

Bayes' Theorem for Distributions

Bayes' Theorem for Distributions

Suppose we have data $\boldsymbol{x}$ which we model using the probability (density) function $f(\boldsymbol{x}|\theta)$, which depends on a single parameter θ.

Once we have observed the data, $f(\boldsymbol{x}|\theta)$

- is the **likelihood function** for θ
- is a function of θ (for fixed $\boldsymbol{x}$) *rather than*
- a function of $\boldsymbol{x}$ (for fixed θ).

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Bayes' Theorem for Distributions

Also, suppose we have prior beliefs about likely values of θ expressed by a probability (density) function $\pi(\theta)$.

We can combine both pieces of information using the following version of Bayes Theorem.

The resulting distribution for θ is called the **posterior distribution** for θ as it expresses our beliefs about θ *after* seeing the data.

It summarises all our current knowledge about the parameter θ.

Bayes' Theorem for Distributions

The posterior probability (density) function for θ is

$$\pi(\theta|\boldsymbol{x}) = \frac{\pi(\theta)\, f(\boldsymbol{x}|\theta)}{f(\boldsymbol{x})}$$

where

$$f(\boldsymbol{x}) = \begin{cases} \int_{\Theta} \pi(\theta)\, f(\boldsymbol{x}|\theta)\, d\theta & \text{if } \theta \text{ is continuous,} \\ \sum_{\Theta} \pi(\theta)\, f(\boldsymbol{x}|\theta) & \text{if } \theta \text{ is discrete.} \end{cases}$$

Notice that, as $f(\boldsymbol{x})$ is not a function of θ, Bayes Theorem can be rewritten as

$$\pi(\theta|\boldsymbol{x}) \propto \pi(\theta) \times f(\boldsymbol{x}|\theta)$$

i.e. posterior $\propto$ prior $\times$ likelihood.

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Bayes' Theorem for Distributions

Thus, to obtain the posterior distribution, we need:

(1) data, from which we can form the **likelihood** $f(\boldsymbol{x}|\theta)$, and

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Some important continuous distributions

Definition (Continuous Uniform distribution)

The random variable Y follows a Uniform $U(a, b)$ distribution if it has probability density function

$$f(y|a, b) = \frac{1}{b-a}, \qquad a \leq y \leq b.$$

This form of probability density function ensures that all values in the range $[a, b]$ are **equally likely**, hence the name "uniform".

This distribution is sometimes called the **rectangular distribution** because of its shape.

You should remember from MAS1604 that

$$E(Y) = \frac{a+b}{2} \quad \text{and} \quad Var(Y) = \frac{(b-a)^2}{12}.$$

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Definition (Beta distribution)

The random variable Y follows a Beta $Be(a, b)$ distribution ($a > 0$, $b > 0$) if it has probability density function

$$f(y|a, b) = \frac{y^{a-1}(1-y)^{b-1}}{B(a, b)}, \qquad 0 < y < 1. \tag{2.1}$$

The constant term $B(a, b)$, also known as the **beta function**, ensures that the density integrates to one. Therefore

$$B(a, b) = \int_0^1 y^{a-1}(1-y)^{b-1}\, dy. \tag{2.2}$$

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It can be shown that the beta function can be expressed in terms of another function, called the **gamma function** $\Gamma(\cdot)$, as

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where

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Some important continuous distributions

Tables are available for both $B(a, b)$ and $\Gamma(a)$.

However, these functions are very simple to evaluate when a and b are integers since the gamma function is a generalisation of the factorial function.

In particular, when a and b are integers, we have

$$\Gamma(a) = (a-1)! \quad \text{and} \quad B(a,b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}.$$

Some important continuous distributions

Tables are available for both $B(a, b)$ and $\Gamma(a)$.

However, these functions are very simple to evaluate when a and b are integers since the gamma function is a generalisation of the factorial function.

In particular, when a and b are integers, we have

$$\Gamma(a) = (a-1)! \quad \text{and} \quad B(a, b) = \frac{(a-1)!(b-1)!}{(a+b-1)!}.$$

Some important continuous distributions

It can be shown, using the identity $\Gamma(a) = (a-1)\Gamma(a-1)$, that

$$E(Y) = \frac{a}{a+b}, \qquad \text{and} \qquad Var(Y) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Also

$$Mode(Y) = \frac{a-1}{a+b-2}, \quad \text{if } a > 1 \text{ and } b > 1.$$

Some important continuous distributions

Definition (Gamma distribution)

The random variable Y follows a Gamma $Ga(a, b)$ distribution ($a > 0$, $b > 0$) if it has probability density function

$$f(y|a, b) = \frac{b^a y^{a-1} e^{-by}}{\Gamma(a)}, \qquad y > 0,$$

where $\Gamma(a)$ is the gamma function defined in Equation (2.3).

It can be shown that

$$E(Y) = \frac{a}{b} \qquad \text{and} \qquad Var(Y) = \frac{a}{b^2}.$$

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Bayes Theorem for distributions in action

We will now see Bayes' Theorem for distributions "in action".

Recall that we have

$$\pi(\theta|\boldsymbol{x}) \quad \propto \quad \pi(\theta) \times f(\boldsymbol{x}|\theta), \qquad \text{i.e.}$$

$$\textbf{posterior} \quad \propto \quad \textbf{prior} \times \textbf{likelihood}$$

Recall also that, for now, we will assume our prior for θ has been given to us – we will look at how such priors are constructed, or **elicited**, in the next chapter.

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Example 2.1

Consider an experiment with a possibly biased coin. Let $\theta = \Pr(\text{Head})$.

Suppose that, before conducting the experiment, we believe that all values of θ are equally likely.

This gives a prior distribution $\theta \sim U(0, 1)$, and so

$$\pi(\theta) = 1, \qquad 0 < \theta < 1. \tag{2.4}$$

Note that with this prior distribution $E(\theta) = 0.5$.

We now toss the coin 5 times and observe 1 head. Determine the posterior distribution for θ given this data.

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“Full” solution to Example 2.1 (1/4)

The data is an observation on the random variable $X|\theta \sim Bin(5, \theta)$. This gives a likelihood function

$$f(x = 1|\theta) = {}^5\mathrm{C}_1\theta^1(1-\theta)^4$$

$$= 5\theta(1-\theta)^4 \tag{2.5}$$

which favours values of θ near its maximum $\theta = 0.2$ (we observed 1 head out of 5 tosses).

Therefore, we have a conflict of opinions: the prior distribution (2.4) suggests that θ is probably around 0.5 and the data (2.5) suggest that it is around 0.2.

We can use Bayes’ Theorem to combine these two sources of information in a coherent way.

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First the denominator:

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\begin{aligned}
f(x=1) &= \int_{\Theta} \pi(\theta) f(x=1|\theta)\, d\theta \qquad (2.6)\\
&= \int_0^1 1 \times 5\theta(1-\theta)^4\, d\theta \\
&= \int_0^1 \theta \times 5(1-\theta)^4\, d\theta
\end{aligned}
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which, using integration by parts, gives

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f(x=1) &= \left[-(1-\theta)^5\,\theta\right]_0^1 + \int_0^1 (1-\theta)^5\, d\theta \\
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\pi(\theta|x=1) &= \frac{\pi(\theta)f(x=1|\theta)}{f(x=1)} \\
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&= 30\,\theta(1-\theta)^4, \qquad 0 < \theta < 1 \\
&= \frac{\theta^{2-1}(1-\theta)^{5-1}}{B(2,5)}, \qquad 0 < \theta < 1,
\end{aligned}$$

as

$$B(2,5) = \frac{(2-1)!(5-1)!}{(2+5-1)!} = \frac{24}{720} = \frac{1}{30},$$

and so the posterior distribution is $\theta|x=1 \sim Be(2,5)$ – see Equation 2.1. This distribution has its mode at $\theta = 0.2$, and mean at $E[\theta|x=1] = 2/7 = 0.286$.

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Example 2.1: “Quick” solution

However, in many cases we can recognise the posterior distribution without the need to calculate $f(x)$.

In this example, we can calculate the posterior distribution as

$$\begin{aligned}\pi(\theta|\boldsymbol{x}) &\propto \pi(\theta)f(x=1|\theta) \\ &\propto 1 \times 5\theta(1-\theta)^4, \qquad 0<\theta<1 \\ &= k\theta(1-\theta)^4, \qquad 0<\theta<1.\end{aligned}$$

You should be able to recognise this as a $Be(2,5)$ distribution – we can re–write the above as

$$\pi(\theta|\boldsymbol{x}) = k\theta^{2-1}(1-\theta)^{5-1},$$

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It is possible that we have a biased coin.

Recall that if $Y \sim U(a, b)$,

$$E(Y) = \frac{a+b}{2} \qquad \text{and} \qquad Var(Y) = \frac{(b-a)^2}{12}.$$

Also, if $Y \sim Be(a, b)$,

$$E(Y) = \frac{a}{a+b} \qquad \text{and} \qquad Var(Y) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Thus:

	Prior $U(0,1)$	Observed $\theta = 0.2$	Posterior $Be(2,5)$
Mean	0.5	$\longmapsto$	0.286
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$$E(Y) = \frac{a+b}{2} \qquad \text{and} \qquad Var(Y) = \frac{(b-a)^2}{12}.$$

Also, if $Y \sim Be(a, b)$,

$$E(Y) = \frac{a}{a+b} \qquad \text{and} \qquad Var(Y) = \frac{ab}{(a+b)^2(a+b+1)}.$$

Thus:

	Prior $U(0,1)$	Observed $\theta = 0.2$	Posterior $Be(2,5)$
Mean	0.5	$\longmapsto$	0.286
St. dev.	0.289	$\longmapsto$	0.160

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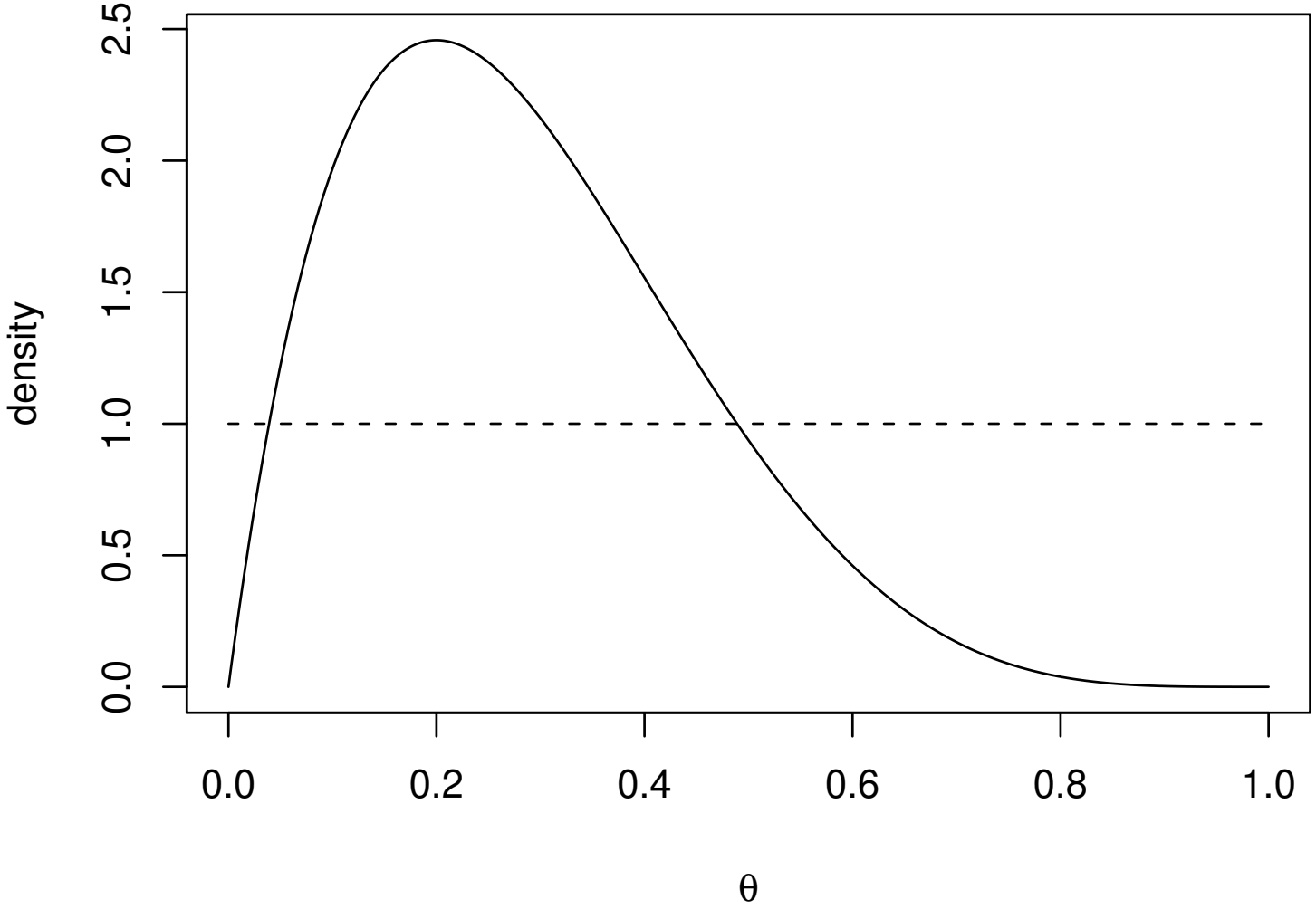

Example 2.2

Consider an experiment to determine how good a music expert is at distinguishing between pages from Haydn and Mozart scores. Let $\theta = \Pr(\text{correct choice})$.

Suppose that, before conducting the experiment, we have been told that the expert is very competent.

In fact, it is suggested that we should have a prior distribution which has a mode around $\theta = 0.95$ and for which $\Pr(\theta < 0.8)$ is very small.

We choose $\theta \sim Be(77, 5)$ (see Example 3.2, Chapter 3), with probability density function

$$\begin{aligned} \pi(\theta) &= \frac{\theta^{76}(1-\theta)^4}{B(77,5)}, \qquad 0 < \theta < 1 \\ &= 128107980\theta^{76}(1-\theta)^4, \qquad 0 < \theta < 1. \end{aligned} \tag{2.7}$$

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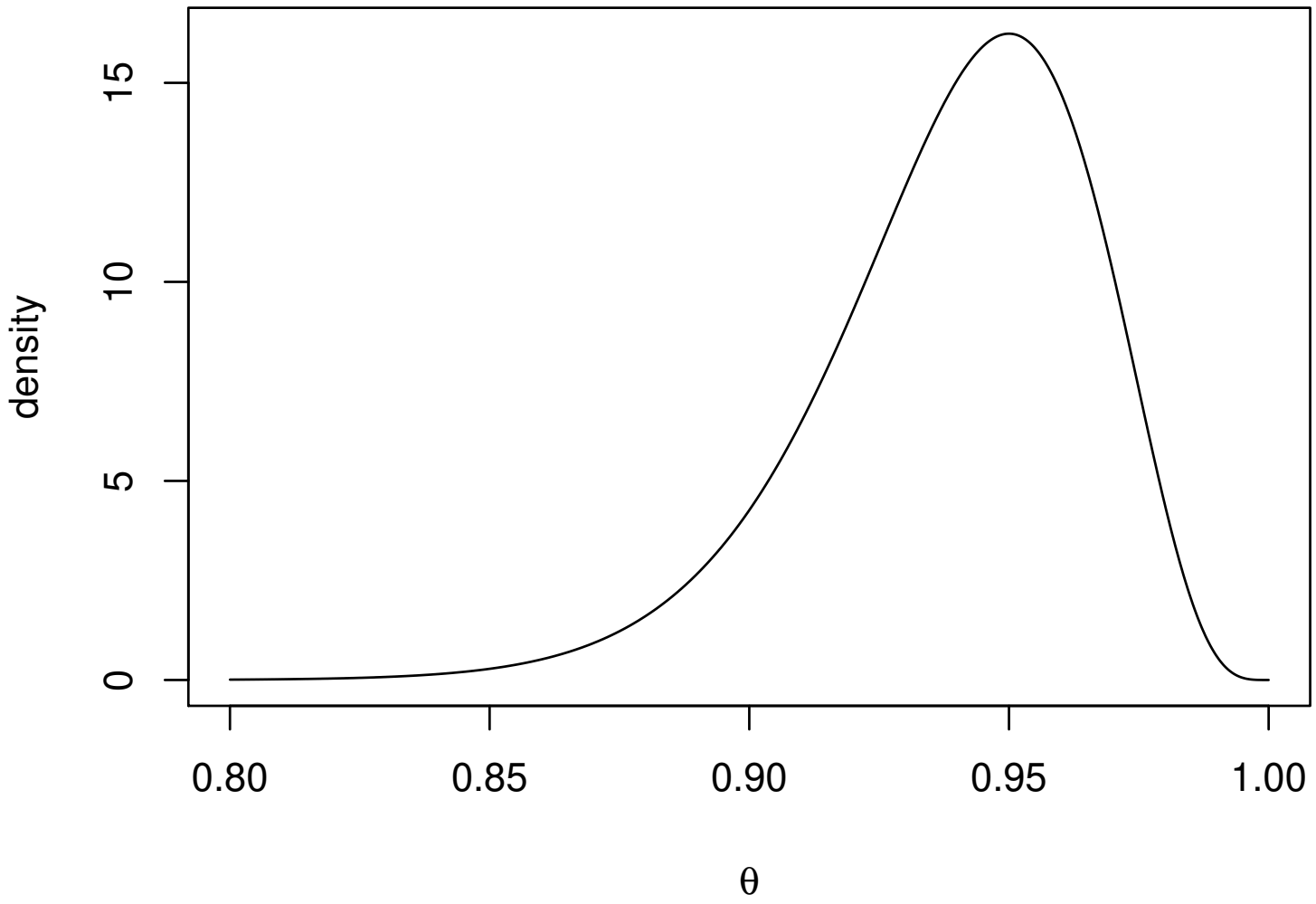
A graph of this prior density is given in Figure 2.4.



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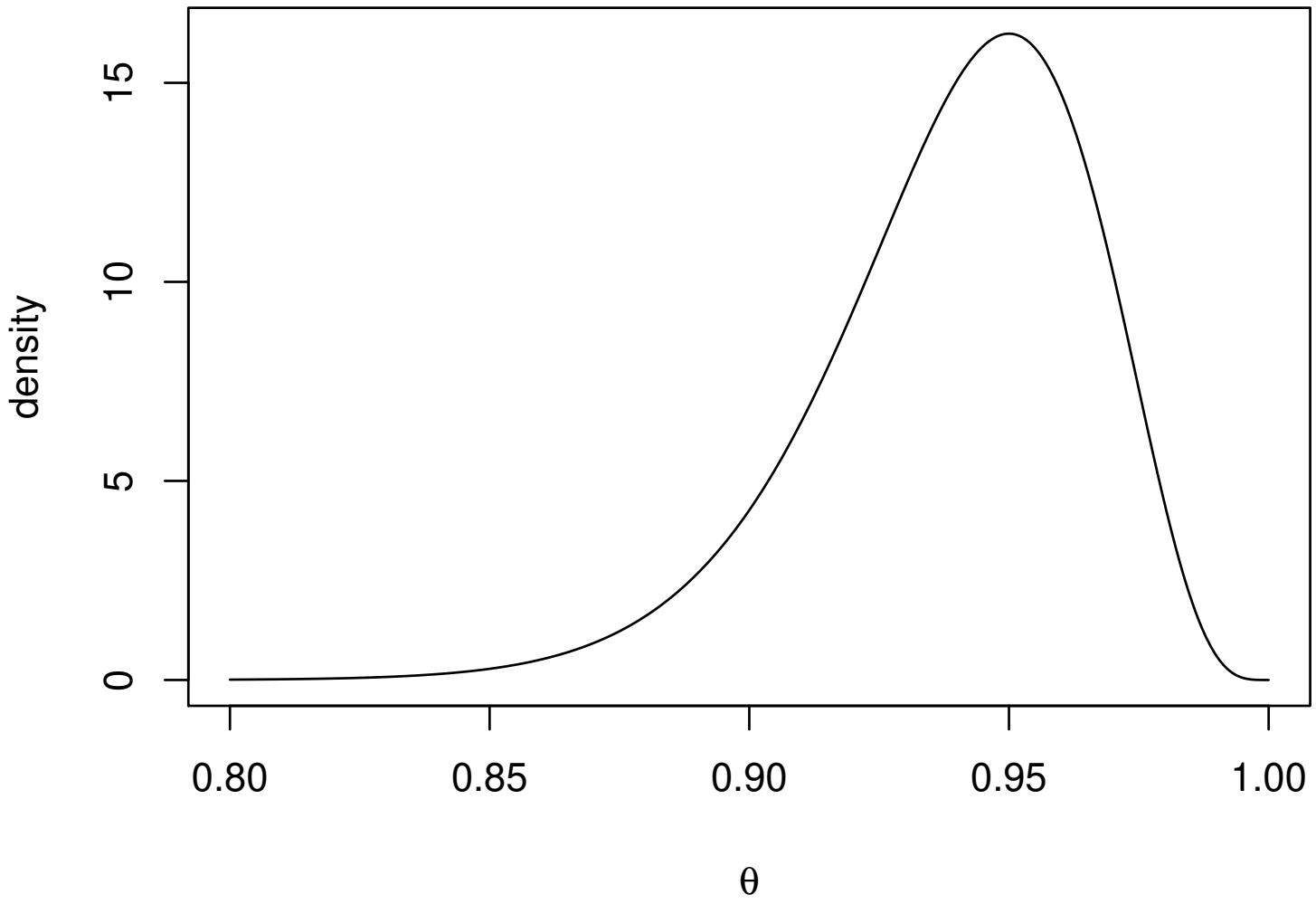

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Solution to Example 2.2 (1/2)

We have an observation on the random variable $X|\theta \sim Bin(10, \theta)$.

This gives a likelihood function of

$$f(x = 9|\theta) = {}^{10}C_9 \theta^9 (1-\theta)^1$$

$$= 10\theta^9(1-\theta) \tag{2.8}$$

which favours values of θ near its maximum $\theta = 0.9$.

We combine these two sources of information using Bayes' Theorem.

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Solution to Example 2.2 (2/2)

The posterior density function is

$$\pi(\theta|x=9) \propto \pi(\theta) f(x=9|\theta)$$

$$\propto \frac{\theta^{76}(1-\theta)^4}{B(77,5)} \times 10\,\theta^9(1-\theta), \qquad 0<\theta<1$$

$$= 128107980\theta^{76}(1-\theta)^4 \times 10\,\theta^9(1-\theta), \qquad 0<\theta<1$$

$$= k\theta^{85}(1-\theta)^5, \qquad 0<\theta<1. \tag{2.9}$$

We can recognise this density function as one from the Beta family. In fact, the posterior distribution is $\theta|x=9 \sim Be(86,6)$.

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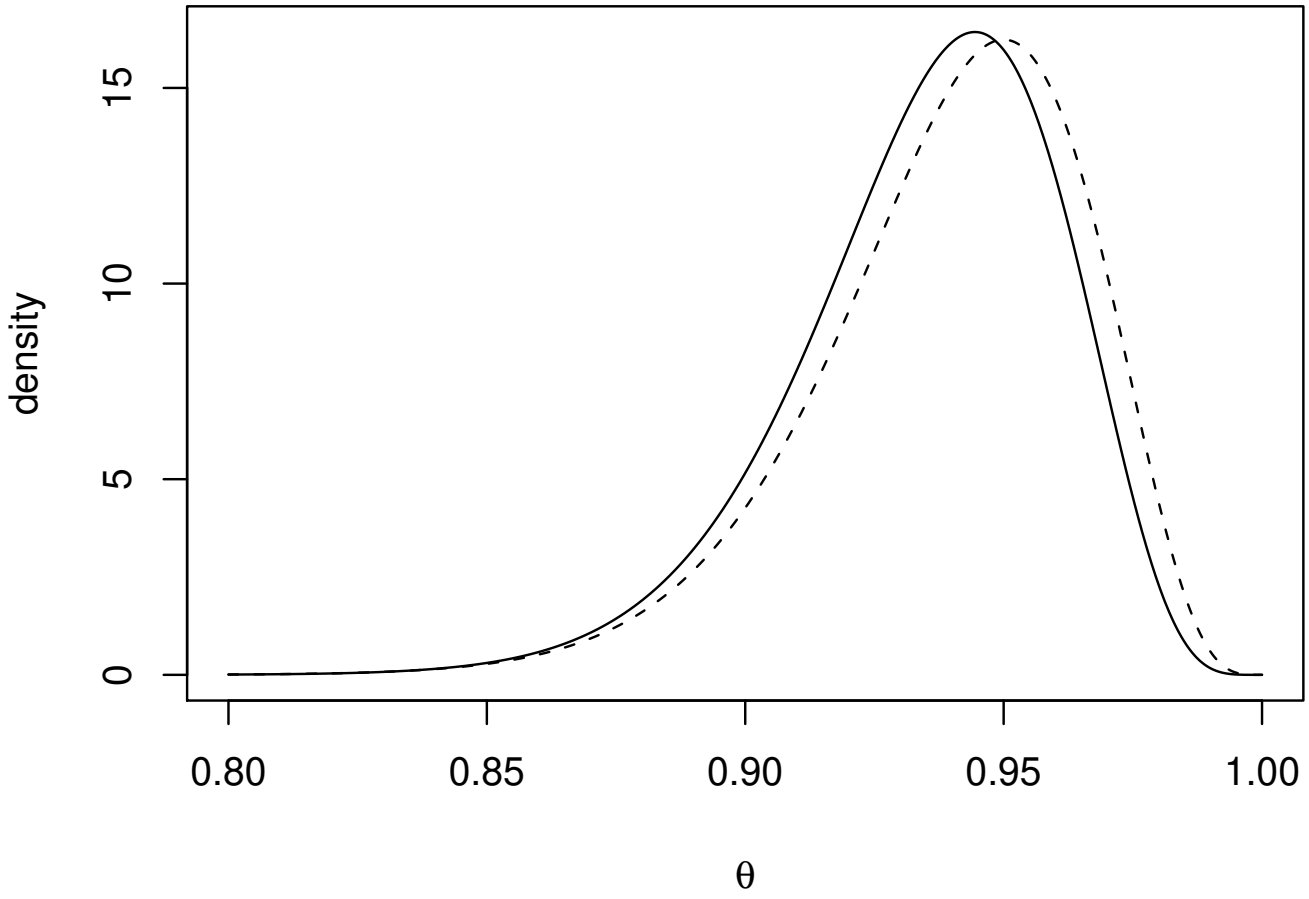

Example 2.2: Summary

	Prior (2.7)	Likelihood (2.8)	Posterior (2.9)
$Mode(\theta)$	0.950	0.900	0.944
$E(\theta)$	0.939	–	0.935
$SD(\theta)$	0.0263	–	0.0256

Notice that, having observed only a 90% success rate in the experiment, the posterior mode and mean are smaller than their prior values.

Also, the experiment has largely confirmed our ideas about θ, with the uncertainty about θ being only very slightly reduced.

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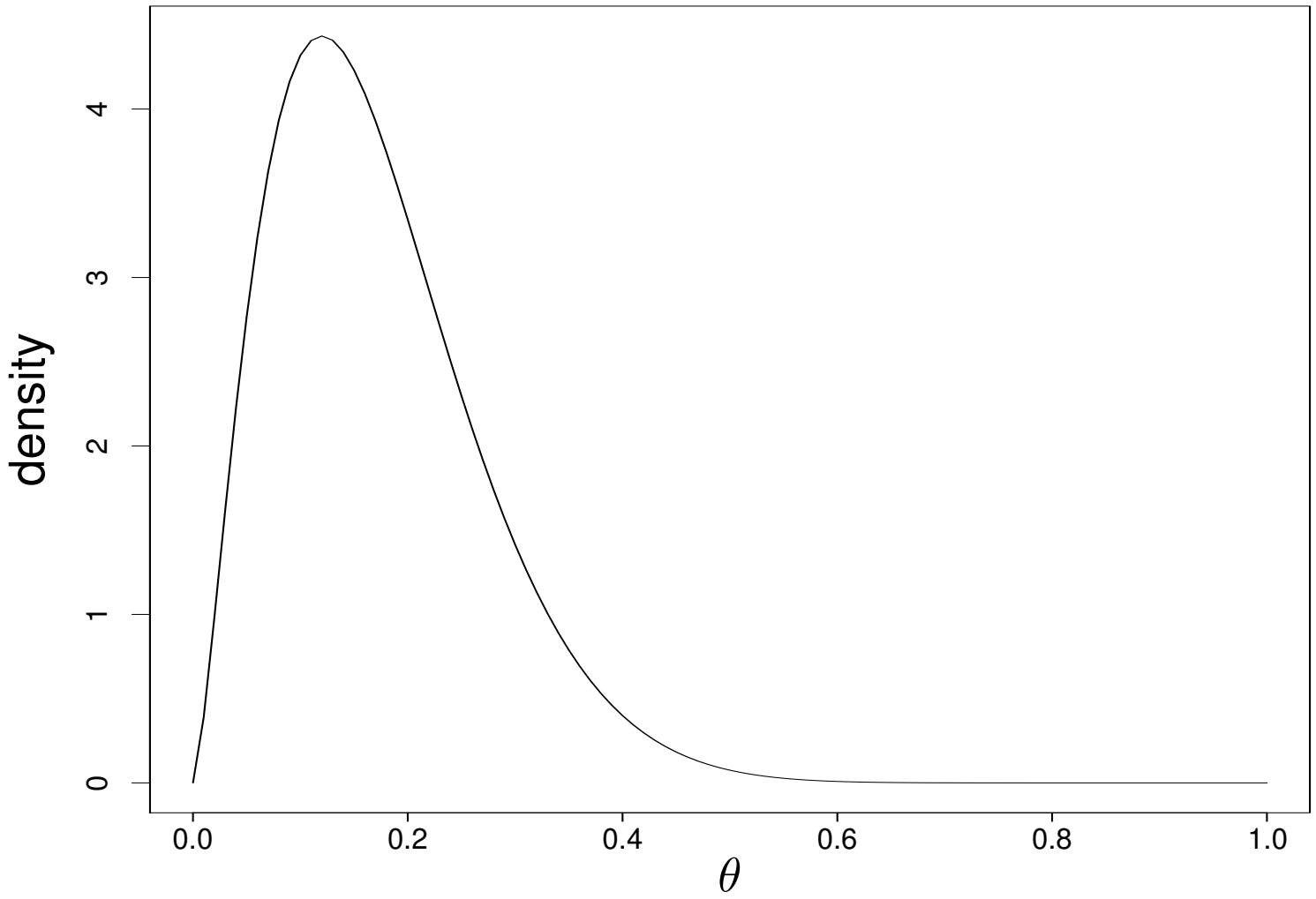
Example 2.3

Max, a video game pirate, is trying to identify the proportion of potential customers θ who might be interested in buying *Call of Duty: Elite* next month.

Based on the proportion of customers who have bought similarly violent games from him in the past, he assumes that $\theta \sim Be(2.5, 12)$ (see Example 3.3, Chapter 3).

Example 2.3

A plot of this prior density is shown in Figure 2.6.



Example 2.3

Max asks five potential customers if they would buy *Call of Duty: Elite* from him, and four say they would.

Using this information, what is Max's posterior distribution for θ?

Solution to Example 2.3 (1/2)

We have been told that the prior for θ is a $Be(2.5, 12)$ distribution – this has density given by

$$\pi(\theta) = \frac{\theta^{2.5-1}(1-\theta)^{12-1}}{B(2.5, 12)}$$

$$= 435.1867\theta^{1.5}(1-\theta)^{11}. \tag{2.10}$$

We have an observation on the random variable $X|\theta \sim Bin(5, \theta)$. This gives a likelihood function of

$$f(x = 4|\theta) = {}^{5}C_{4}\theta^{4}(1-\theta)^{1}$$

$$= 5\theta^{4}(1-\theta), \tag{2.11}$$

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Solution to Example 2.3 (2/2)

We combine our prior information (2.10) with the data (2.11) – to obtain our posterior distribution – using Bayes' Theorem.

The posterior density function is

$$\pi(\theta|x=4) \propto \pi(\theta)f(x=4|\theta)$$

$$\propto 435.1867\theta^{1.5}(1-\theta)^{11} \times 5\theta^{4}(1-\theta), \qquad 0<\theta<1,$$

$$= k\theta^{5.5}(1-\theta)^{12}, \qquad 0<\theta<1. \tag{2.12}$$

You should recognise this density function as one from the beta family. In fact, we have a $Be(6.5, 13)$, i.e. $\theta|x=4 \sim Be(6.5, 13)$.

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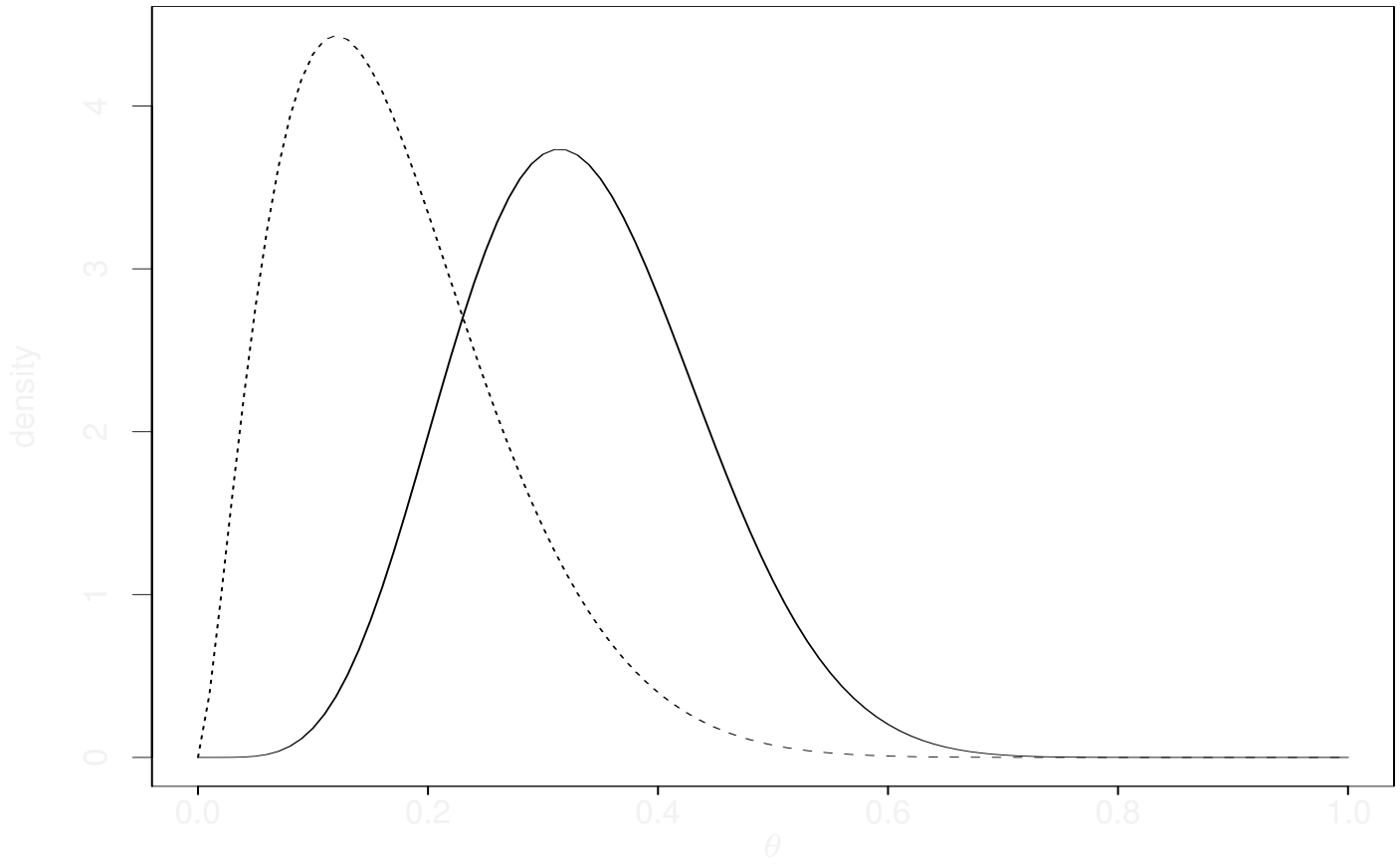

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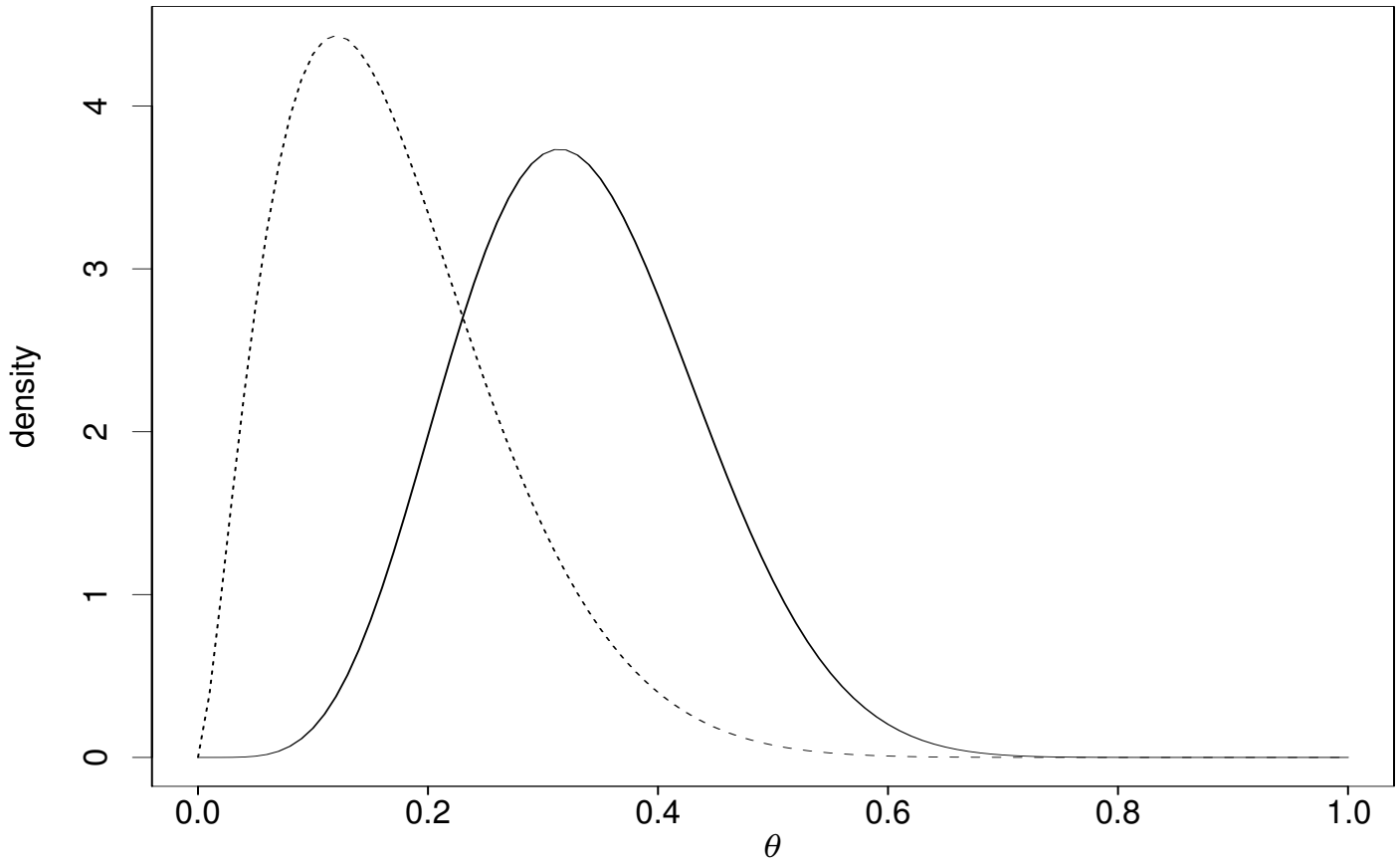

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$E(\theta)$	0.172	–	0.333
$SD(\theta)$	0.096	–	0.104

Notice how the posterior has been "pulled" from the prior towards the observed value: the mode has moved up from 0.12 to 0.314, and the mean has moved up from 0.172 to 0.333.

Having just one observation in the likelihood, we see that there is hardly any change in the standard deviation from prior to posterior: we would expect to see a decrease in standard deviation with the addition of more data values.

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Example 2.4

Table 2.3 shows some data on the times between serious earthquakes.

An earthquake is included if its magnitude is at least 7.5 on the Richter scale or if over 1000 people were killed.

Recording starts on 16 December 1902 (4500 killed in Turkistan).

The table includes data on 21 earthquakes, that is, 20 “waiting times” between earthquakes.

840	157	145	44	33	121	150	280	434	736
584	887	263	1901	695	294	562	721	76	710

Example 2.4

It is believed that earthquakes happen in a random haphazard kind of way and that times between earthquakes can be described by an exponential distribution.

Data over a much longer period suggest that this exponential assumption is plausible.

Therefore, we will assume that these data are a random sample from an exponential distribution with rate θ (and mean $1/\theta$). The parameter θ describes the rate at which earthquakes occur.

Example 2.4

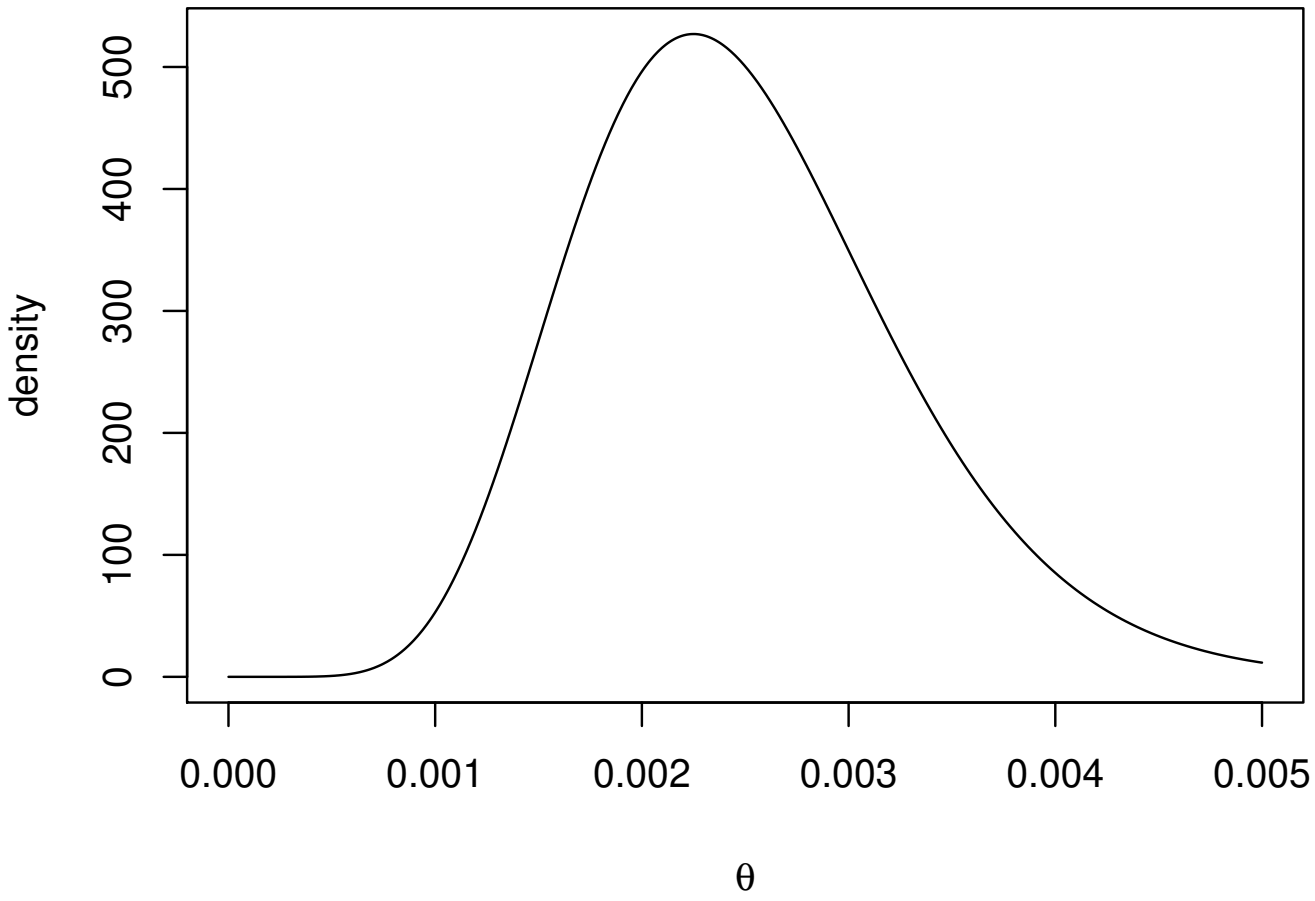
An expert on earthquakes has prior beliefs about the rate of earthquakes, θ, described by a $Ga(10, 4000)$ distribution (see Example 3.1, Chapter 3), which has density density

$$\pi(\theta) = \frac{4000^{10}\,\theta^{9} e^{-4000\theta}}{\Gamma(10)}, \quad \theta > 0, \tag{2.13}$$

and mean $E(\theta) = 0.0025$.

Example 2.4

A plot of this prior distribution can be found in Figure 2.8. As you might expect, the expert believes that only very small values of θ are likely, though larger values are not ruled out!



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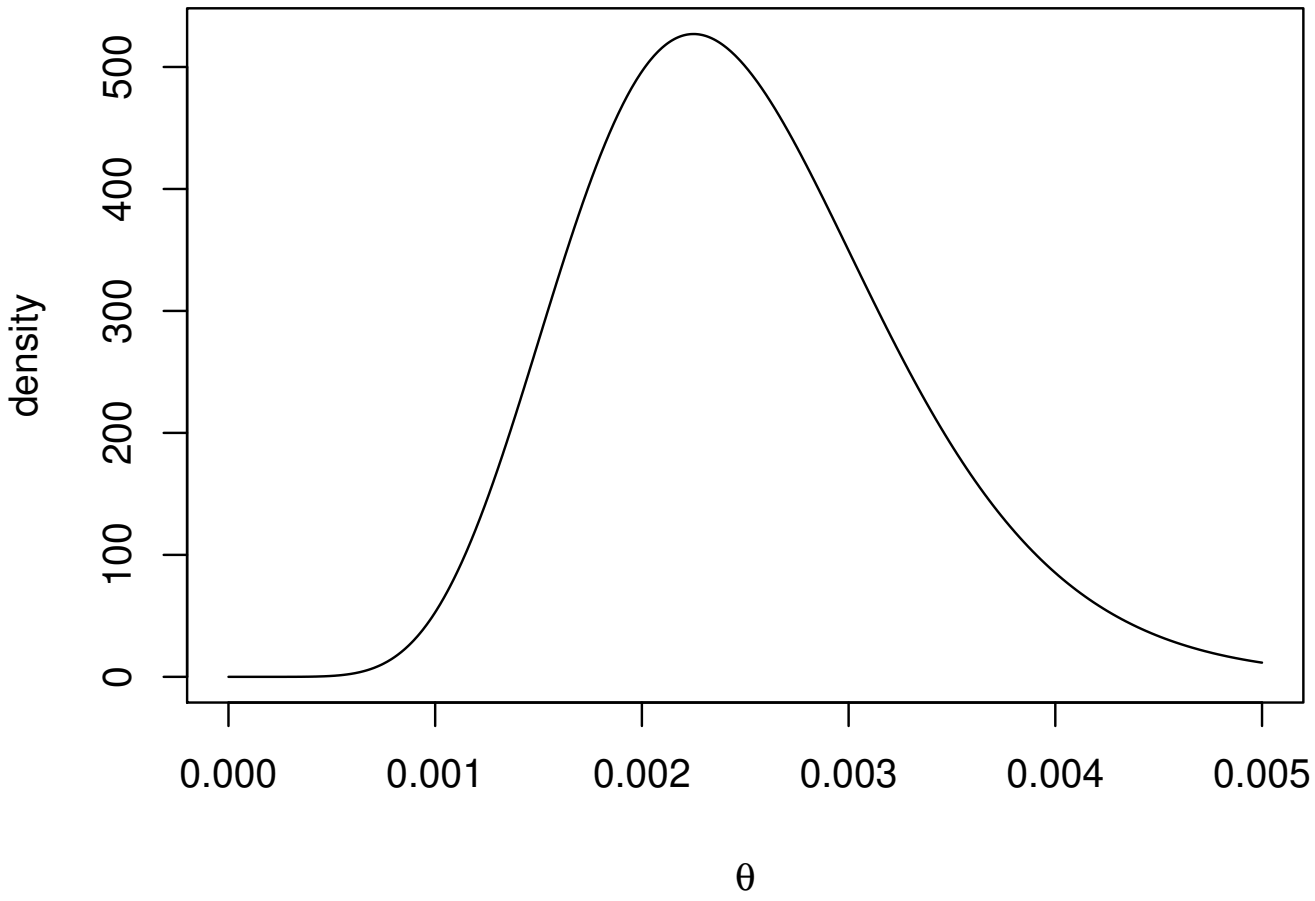

Solution to Example 2.4 (1/2)

The data are observations on $X_i|\theta \sim Exp(\theta)$, $i = 1, 2, \ldots, 20$ (independent).

Therefore, the likelihood function for θ is

$$f(\boldsymbol{x}|\theta) = \prod_{i=1}^{20} \theta e^{-\theta x_i}, \qquad \theta > 0$$

$$= \theta^{20} \exp\left(-\theta \sum_{i=1}^{20} x_i\right), \qquad \theta > 0$$

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Solution to Example 2.4 (2/2)

We now apply Bayes Theorem to combine the expert opinion with the observed data. The posterior density function is

$$\pi(\theta|\boldsymbol{x}) \propto \pi(\theta) f(\boldsymbol{x}|\theta)$$

$$\propto \frac{4000^{10}\,\theta^{9} e^{-4000\theta}}{\Gamma(10)} \times \theta^{20} e^{-9633\theta}, \qquad \theta > 0$$

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The only continuous distribution which takes the form $k\theta^{g-1} e^{-h\theta}$, $\theta > 0$ is the $Ga(g, h)$ distribution.

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We now apply Bayes Theorem to combine the expert opinion with the observed data. The posterior density function is

$$\pi(\theta|\boldsymbol{x}) \propto \pi(\theta) f(\boldsymbol{x}|\theta)$$

$$\propto \frac{4000^{10}\,\theta^{9} e^{-4000\theta}}{\Gamma(10)} \times \theta^{20} e^{-9633\theta}, \qquad \theta > 0$$

$$= k\,\theta^{30-1} e^{-13633\theta}, \qquad \theta > 0. \tag{2.15}$$

The only continuous distribution which takes the form $k\theta^{g-1}e^{-h\theta}$, $\theta > 0$ is the $Ga(g, h)$ distribution.

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Example 2.4: Summary

The data have updated our beliefs about θ from a $Ga(10, 4000)$ distribution to a $Ga(30, 13633)$ distribution.

Plots of these distributions are given in Figure 2.9, and Table 2.4 gives a summary of the main changes induced by incorporating the data.

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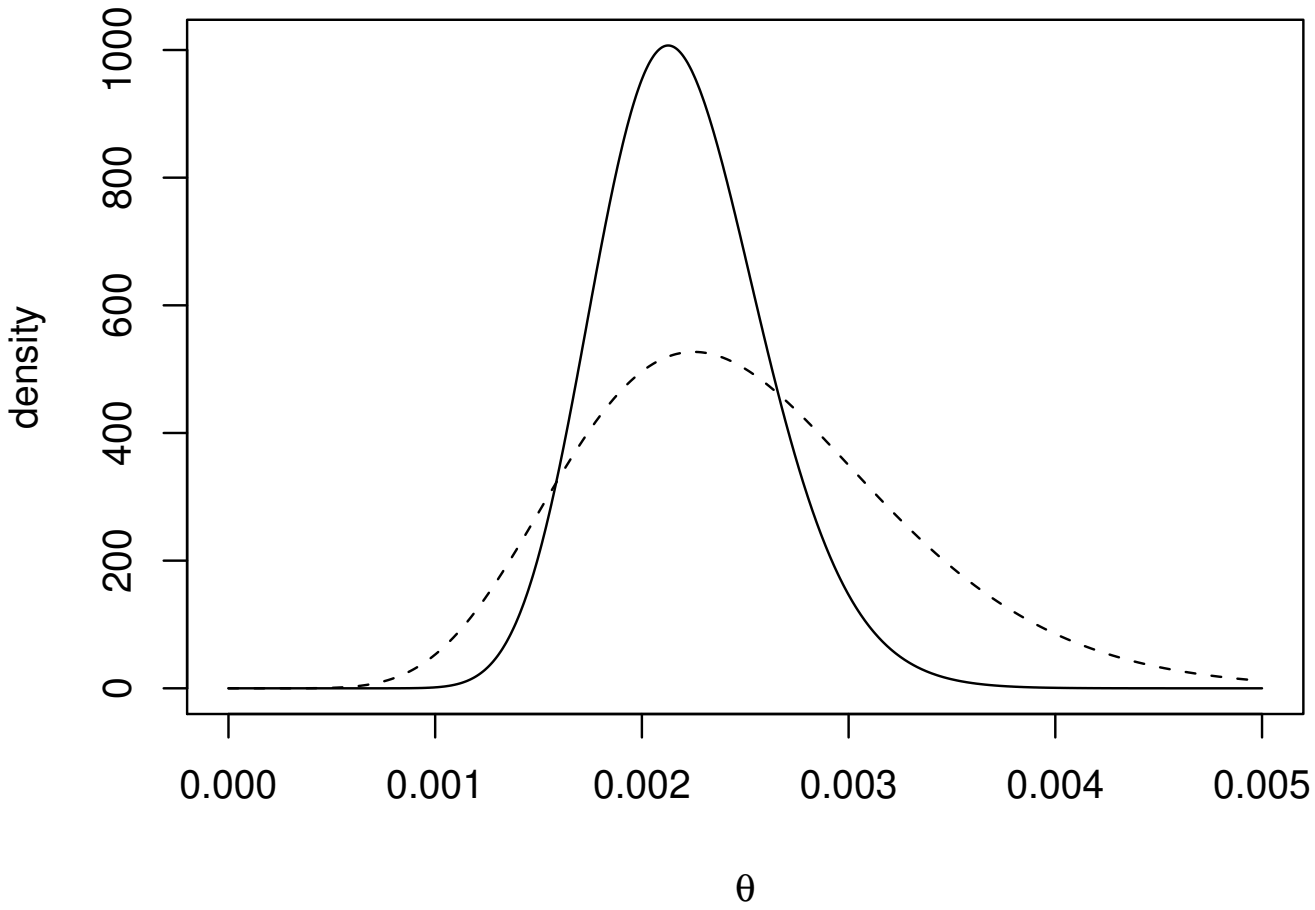
Plots of these distributions are given in Figure 2.9, and Table 2.4 gives a summary of the main changes induced by incorporating the data.

Example 2.4: Summary



Example 2.4: Summary

	Prior (2.13)	Likelihood (2.14)	Posterior (2.15)
$Mode(\theta)$	0.00225	0.00208	0.00213
$E(\theta)$	0.00250	–	0.00220
$SD(\theta)$	0.00079	–	0.00040

Notice that, as the mode of the likelihood function is close to that of the prior distribution, the information in the data is consistent with that in the prior distribution.

This results in a reduction in variability from the prior to the posterior distributions.

The similarity between the prior beliefs and the data has reduced the uncertainty we have about the likely earthquake rate θ.

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Example 2.5

We now consider the general case of the problem discussed in Example 2.4.

Suppose $X_i|\theta \sim Exp(\theta)$, $i = 1, 2, \ldots, n$ (independent) and our prior beliefs about θ are summarised by a $Ga(g, h)$ distribution (with g and h known), with density

$$\pi(\theta) = \frac{h^g \, \theta^{g-1} e^{-h\theta}}{\Gamma(g)}, \quad \theta > 0. \tag{2.16}$$

Determine the posterior distribution for θ.

Solution to Example 2.5 (1/2)

The likelihood function for θ is

$$
\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta e^{-\theta x_i}, \qquad \theta > 0 \\
&= \theta^n e^{-\theta x_1 - \theta x_2 - \ldots - \theta x_n} \\
&= \theta^n e^{-\theta \sum x} \\
&= \theta^n e^{-n\bar{x}\theta}, \qquad \theta > 0. \qquad (2.17)
\end{aligned}
$$

Solution to Example 2.5 (2/2)

We now apply Bayes Theorem. The posterior density function is

$$\pi(\theta|\boldsymbol{x}) \propto \pi(\theta) f(\boldsymbol{x}|\theta)$$

$$\propto \frac{h^g \, \theta^{g-1} e^{-h\theta}}{\Gamma(g)} \times \theta^n e^{-n\bar{x}\theta}, \qquad \theta > 0.$$

$$\pi(\theta|\boldsymbol{x}) = k\theta^{g+n-1} e^{-(h+n\bar{x})\theta}, \qquad \theta > 0, \tag{2.18}$$

where k is a constant that does not depend on θ.

Therefore, the posterior distribution takes the form $k\theta^{g-1} e^{-h\theta}$, $\theta > 0$ and so must be a gamma distribution.

Thus we have $\theta|\boldsymbol{x} \sim Ga(g+n, h+n\bar{x})$.

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Example 2.5: Summary

If

- $X_i \sim Exp(\theta)$ and
- $\theta \sim Ga(g, h)$, then
- $\theta|\boldsymbol{x} \sim Ga(g + n, h + n\bar{x})$

The changes in our beliefs about θ are summarised in Table 2.5, taking $g \geq 1$.

	Prior (2.16)	Likelihood (2.17)	Posterior (2.18)
$Mode(\theta)$	$(g-1)/h$	$1/\bar{x}$	$(g+n-1)/(h+n\bar{x})$
$E(\theta)$	g/h	–	$(g+n)/(h+n\bar{x})$
$SD(\theta)$	$\sqrt{g}/h$	–	$\sqrt{g+n}/(h+n\bar{x})$

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Chapter 2 so far...

Model	Prior	Posterior
Binomial(k, θ)	$\theta \sim U(0, 1)$ $\theta \sim Beta(a, b)$	Beta Beta
Poisson(θ)	$\theta \sim Ga(g, h)$ $\theta \sim exp(g)$	Gamma ??
Exponential(θ)	$\theta \sim Ga(g, h)$ $\theta \sim exp(g)$	Gamma ??
Normal(μ, σ^2)	??	??

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Conjugacy

Definition (Conjugacy)

Suppose that data $\boldsymbol{x}$ are to be observed with distribution $f(\boldsymbol{x}|\theta)$.

A family $\mathfrak{F}$ of prior distributions for θ is said to be **conjugate** to $f(\boldsymbol{x}|\theta)$ if for every prior distribution $\pi(\theta) \in \mathfrak{F}$, the posterior distribution $\pi(\theta|\boldsymbol{x})$ is also in $\mathfrak{F}$.

Example 2.6

Suppose we have a random sample from a Normal distribution.

In Bayesian statistics, when dealing with the Normal distribution, the mathematics is more straightforward if we work with the **precision** ($= 1/\text{variance}$) of the distribution rather than the variance itself.

So we will assume that this population has unknown mean μ but known precision τ: $X_i|\mu \sim N(\mu, 1/\tau)$, $i = 1, 2, \ldots, n$ (independent), where τ is known.

Suppose our prior beliefs about μ can be summarised by a $N(b, 1/d)$ distribution, with probability density function

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \tag{2.19}$$

Determine the posterior distribution for μ.

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Determine the posterior distribution for μ.

Solution to Example 2.6 (1/5)

The likelihood function for μ is

$$
\begin{aligned}
f(\boldsymbol{x}|\mu) &= \prod_{i=1}^{n} \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x_i-\mu)^2\right\} \\
&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right\} \\
&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2}\sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu)^2\right\} \\
&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2}\sum_{i=1}^{n}([x_i-\bar{x}]+[\bar{x}-\mu])^2\right\}
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\end{aligned}
$$

Solution to Example 2.6 (2/5)

Giving

$$f(\boldsymbol{x}|\mu) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2}\left[\sum_{i=1}^{n}(x_i-\bar{x})^2 + 2(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\bar{x}) + \sum_{i=1}^{n}(\bar{x}-\mu)^2\right]\right\}$$

$$= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2}\left[\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2\right]\right\},$$

Let

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2;$$

Then

$$f(\boldsymbol{x}|\mu) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2}\left[s^2 + (\bar{x}-\mu)^2\right]\right\}. \qquad (2.20)$$

Solution to Example 2.6 (2/5)

Giving

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f(\boldsymbol{x}|\mu) &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2}\left[\sum_{i=1}^{n}(x_i-\bar{x})^2\right.\right. \\
&\qquad \left.\left. + 2(\bar{x}-\mu)\sum_{i=1}^{n}(x_i-\bar{x}) + \sum_{i=1}^{n}(\bar{x}-\mu)^2\right]\right\} \\
&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{\tau}{2}\left[\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2\right]\right\},
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Solution to Example 2.6 (3/5)

Applying Bayes Theorem, the posterior density function is

$$\pi(\mu|\boldsymbol{x}) \propto \pi(\mu) f(\boldsymbol{x}|\mu)$$

$$\propto \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\}$$

$$\times \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\}$$

$$= k_1 \exp\left\{-\frac{d}{2}(\mu - b)^2 - \frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\}$$

$$= k_1 \exp\left\{-\frac{1}{2}\left[d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2\right]\right\}$$

where k_1 is a constant that does not depend on μ.

Solution to Example 2.6 (3/5)

Applying Bayes Theorem, the posterior density function is

$$\pi(\mu|\boldsymbol{x}) \propto \pi(\mu) f(\boldsymbol{x}|\mu)$$

$$\propto \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\} \times \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\}$$

$$= k_1 \exp\left\{-\frac{d}{2}(\mu - b)^2 - \frac{n\tau}{2}\left[s^2 + (\bar{x} - \mu)^2\right]\right\}$$

$$= k_1 \exp\left\{-\frac{1}{2}\left[d(\mu - b)^2 + n\tau(\bar{x} - \mu)^2\right]\right\}$$

where k_1 is a constant that does not depend on μ.

Solution to Example 2.6 (3/5)

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Solution to Example 2.6 (4/5)

We can also simplify the exponent. We have

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d(\mu - b)^2 \quad + \quad n\tau(\bar{x} - \mu)^2 \\
&= d(\mu^2 - 2b\mu + b^2) + n\tau(\bar{x}^2 - 2\bar{x}\mu + \mu^2) \\
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Solution to Example 2.6 (5/5)

Let

$$B = \frac{db + n\tau\bar{x}}{d + n\tau} \qquad \text{and} \qquad D = d + n\tau. \tag{2.21}$$

Then

$$\pi(\mu|\boldsymbol{x}) = k_1 \exp\left\{-\frac{D}{2}(\mu - B)^2 - \frac{c}{2}\right\}$$

$$= k \exp\left\{-\frac{D}{2}(\mu - B)^2\right\}, \tag{2.22}$$

where k is a constant that does not depend on μ.

Therefore, the posterior distribution takes the form $k \exp\{-D(\mu - B)^2/2\}$, $-\infty < \mu < \infty$ and so must be a normal distribution: we have $\mu|\boldsymbol{x} \sim N(B, 1/D)$.

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If we have a random sample from a $N(\mu, 1/\tau)$ distribution (with τ known) and our prior beliefs about μ follow a $N(b, 1/d)$ distribution then, after incorporating the data, our (posterior) beliefs about μ follow a $N(B, 1/D)$ distribution.

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The changes in our beliefs about μ are summarised in Table 2.6.

	Prior (2.19)	Likelihood (2.20)	Posterior (2.21)
$Mode(\mu)$	b	$\bar{x}$	$(db + n\tau\bar{x})/(d + n\tau)$
$E(\mu)$	b	–	$(db + n\tau\bar{x})/(d + n\tau)$
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Notice that the way prior information and observed data combine is through the parameters of the normal distribution:

$$b \to \frac{db + n\tau\bar{x}}{d + n\tau} \qquad \text{and} \qquad d \to d + n\tau.$$

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Notice also that the posterior variance (and precision) does not depend on the data, and the posterior mean is a convex combination of the prior and sample means, that is,

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\begin{aligned}
B &= \frac{db + n\tau\bar{x}}{d + n\tau} \\
&= \frac{db}{d + n\tau} + \frac{n\tau\bar{x}}{d + n\tau} \\
&= \frac{d}{d + n\tau} b + \frac{n\tau}{d + n\tau}\bar{x} \\
&= \left(\frac{d}{d + n\tau}\right) b + \left(\frac{d + n\tau - d}{d + n\tau}\right)\bar{x} \\
&= \alpha b + (1 - \alpha)\bar{x}, \qquad \alpha = \frac{d}{d + n\tau}.
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Example 2.6: Summary

This equation for the posterior mean, which can be rewritten as

$$E(\mu|\boldsymbol{x}) = \alpha E(\mu) + (1 - \alpha)\bar{x},$$

arises in other models and is known as the **Bayes linear rule** (see Problems Sheet 2, question 3).

Example 2.6: Summary

Notice that the posterior mean is greater than the prior mean if and only if the likelihood mode (sample mean) is greater than the prior mean, that is

$$E(\mu|\boldsymbol{x}) > E(\mu) \quad \Longleftrightarrow \quad Mode[f(\boldsymbol{x}|\mu)] > E(\mu).$$

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Example 2.7

The ages of *Ennerdale granophyre* rocks can be determined using the relative proportions of rubidium–87 and strontium–87 in the rock.

An expert in the field suggests that the ages of such rocks (in millions of years) $X|\mu \sim N(\mu, 8^2)$ and that a prior distribution $\mu \sim N(370, 20^2)$ is appropriate.

A rock is found whose chemical analysis yields $x = 421$. What is the posterior distribution for μ and what is the probability that the rock will be older than 400 million years?

Solution to Example 2.7 (1/2)

We have $n = 1$, $\bar{x} = x = 421$, $\tau = 1/64$, $b = 370$ and $d = 1/400$. Therefore, using the results in Example 2.6,

$$\begin{aligned} B &= \frac{db + n\tau\bar{x}}{d + n\tau} \\ &= \frac{370/400 + 421/64}{1/400 + 1/64} = 414.0 \qquad \text{and} \end{aligned}$$

$$\begin{aligned} D &= d + n\tau \\ &= 1/400 + 1/64 = 1/7.43^2. \end{aligned}$$

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Solution to Example 2.7 (2/2)

Thus, the posterior distribution is $\mu|x = 421 \sim N(414.0, 7.43^2)$.

The (posterior) probability that the rock will be older than 400 million years is

$$\Pr(\mu > 400|x = 421) = 0.9702$$

calculated using the R commands `1-pnorm(400,414,7.43)` or `1-pnorm(-1.884)` (or indeed by hand using tables).

Without the chemical analysis, the only basis for determining the age of the rock is via the prior distribution.

The (prior) probability that the rock will be older than 400 million years is $\Pr(\mu > 400) = 0.0668$ calculated using the R command `1-pnorm(400,370,20)`.

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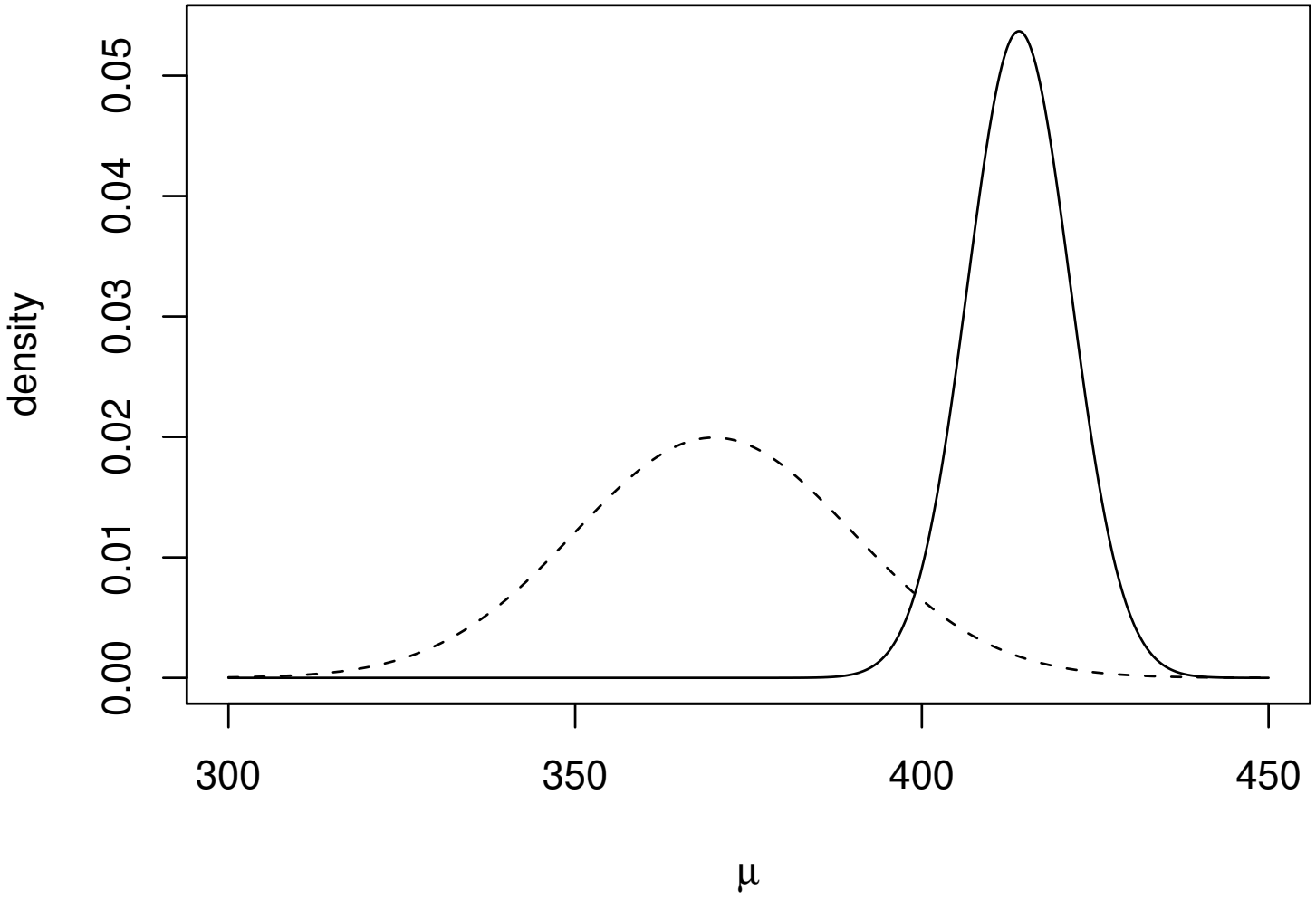
Example 2.7: Summary

This highlights the benefit of taking the chemical measurements.

Note that the large difference between these probabilities is not necessarily due to the expert's prior distribution being inaccurate, *per se*.

It is probably due to the large prior uncertainty about rock ages, as shown in Figure 2.10.

Example 2.7: Summary



Posterior distributions and sufficient statistics

We have already met the concept of sufficient statistics. Not surprisingly they also play a role in Bayesian Inference.

Suppose that we have data $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T$ available and we want to make inferences about the parameter θ in the statistical model $f(\boldsymbol{x}|\theta)$.

If T is a sufficient statistic then by the Factorisation Theorem

$$f(\boldsymbol{x}|\theta) = h(\boldsymbol{x})\, g(t, \theta).$$

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Posterior distributions and sufficient statistics

Therefore, using Bayes Theorem,

$$\begin{aligned}\pi(\theta|\boldsymbol{x}) &\propto \pi(\theta)\, f(\boldsymbol{x}|\theta) \\ &\propto \pi(\theta)\, h(\boldsymbol{x})\, g(t,\theta) \\ &\propto \pi(\theta)\, g(t,\theta).\end{aligned}$$

It can be shown that, up to a constant not depending on θ, $g(t,\theta)$ is equal to the probability (density) function of T; that is

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Posterior distributions and sufficient statistics

However, applying Bayes Theorem to the data t gives

$$\pi(\theta|t) \quad \propto \quad \pi(\theta)\, f_T(t|\theta)$$

and so, since

$$\pi(\theta|\boldsymbol{x}) \quad \propto \quad \pi(\theta|t)$$

and both are probability (density) functions, we have

$$\pi(\theta|\boldsymbol{x}) \quad = \quad \pi(\theta|t).$$

Posterior distributions and sufficient statistics

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This is what we would expect if all the information about θ in the data were contained in the sufficient statistic.

Example 2.8

Suppose we have a random sample from an exponential distribution with a gamma prior distribution, that is, $X_i|\theta \sim Exp(\theta)$, $i = 1, 2, \ldots, n$ (independent) and $\theta \sim Ga(g, h)$.

Determine a sufficient statistic T for θ and verify that $\pi(\theta|\boldsymbol{x}) = \pi(\theta|t)$.

Solution to Example 2.8 (1/2)

The density of the data is

$$
\begin{aligned}
f_{\boldsymbol{X}}(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta e^{-\theta x_i} \\
&= \theta^n \exp\left(-\theta \sum_{i=1}^{n} x_i\right) \\
&= 1 \times \theta^n \exp\left(-\theta \sum_{i=1}^{n} x_i\right) \\
&= h(\boldsymbol{x})\, g(\Sigma x_i, \theta)
\end{aligned}
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and therefore, by the Factorisation Theorem, $T = \sum_{i=1}^{n} X_i$ is sufficient for θ.

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Solution to Example 2.8 (2/2)

Now $T|\theta \sim Ga(n, \theta)$ and so

$$f_T(t|\theta) = \frac{\theta^n t^{n-1} e^{-\theta t}}{\Gamma(n)}, \quad \theta > 0.$$

Also

$$\pi(\theta) = \frac{h^g \, \theta^{g-1} e^{-h\theta}}{\Gamma(g)}, \quad \theta > 0.$$

Therefore, by Bayes Theorem,

$$\begin{aligned}
\pi(\theta|t) &\propto \pi(\theta) f(t|\theta) \\
&\propto \frac{h^g \, \theta^{g-1} e^{-h\theta}}{\Gamma(g)} \times \frac{\theta^n t^{n-1} e^{-\theta t}}{\Gamma(n)}, \qquad \theta > 0 \\
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and so the posterior distribution is $\theta|t \sim Ga(g+n, h+t)$. This is the same as the result we obtained previously for $\theta|\boldsymbol{x}$.

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and so the posterior distribution is $\theta|t \sim Ga(g+n, h+t)$. This is the same as the result we obtained previously for $\theta|\boldsymbol{x}$.

Example 2.9

Suppose we have a random sample from a normal distribution with known variance and a normal prior distribution for the mean parameter, that is, $X_i|\mu \sim N(\mu, 1/\tau)$, $i = 1, 2, \ldots, n$ (independent) and $\mu \sim N(b, 1/d)$.

Determine a sufficient statistic T for μ and verify that $\pi(\mu|\boldsymbol{x}) = \pi(\mu|t)$.

Solution to Example 2.9 (1/3)

Recall from Equation (2.20) that

$$
\begin{aligned}
f_{\boldsymbol{X}}(\boldsymbol{x}|\mu) &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2}\left[s^2 + (\bar{x}-\mu)^2\right]\right\} \\
&= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau s^2}{2}\right\} \times \exp\left\{-\frac{n\tau}{2}(\bar{x}-\mu)^2\right\} \\
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Solution to Example 2.9 (2/3)

Now $T|\mu \sim N(\mu, 1/(n\tau))$ (by the Central Limit Theorem) and so

$$f_T(t|\mu) = \left(\frac{n\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{n\tau}{2}(t-\mu)^2\right\}.$$

Also

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu-b)^2\right\}.$$

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Solution to Example 2.9 (3/3)

Therefore, by Bayes' Theorem,

$$\begin{aligned}
\pi(\mu|t) &\propto \pi(\mu)f(t|\mu) \\
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&\qquad \times \left(\frac{n\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{n\tau}{2}(t-\mu)^2\right\} \\
&\propto \exp\left\{-\frac{d}{2}(\mu-b)^2 - \frac{n\tau}{2}(t-\mu)^2\right\} \\
&\vdots \\
&\propto \exp\left\{-\frac{D}{2}(\mu-B)^2\right\}
\end{aligned}$$

where B and D are as in Equation (2.21), with t replacing $\bar{x}$; that is, $\mu|t \sim N(B, 1/D)$, the same distribution as $\mu|\boldsymbol{x}$.

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