NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS, STATISTICS & PHYSICS

SEMESTER 2 2018–2019

MAS2903

An Introduction to Bayesian Methods: Mid–semester test

Time allowed: 40 minutes

Candidates should attempt all questions. Marks for each question are indicated.

There are FIVE questions on this paper.

Answers to questions should be entered directly on this question paper in the spaces provided. This question paper must be handed in at the end of the test.

Name:.....

Beta distribution: If $X \sim Be(\alpha, \beta)$ then it has density

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad \alpha > 0, \beta > 0.$$

Also, $E(X) = \alpha/(\alpha+\beta)$, $Var(X) = \alpha\beta/\{(\alpha+\beta)^2(\alpha+\beta+1)\}$,
and $Mode(X) = (\alpha-1)/(\alpha+\beta-2)$.

Binomial distribution: If $X \sim Bin(n, p)$ then it has probability function

$$f(x|n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n, \quad 0 \leq p \leq 1.$$

Also, $E(X) = np$ and $Var(X) = np(1-p)$.

Exponential distribution: If $X \sim Exp(\lambda)$ then it has density

$$f(x|\lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Also, $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$.

Gamma distribution: If $X \sim Ga(\alpha, \lambda)$ then it has density

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha > 0, \lambda > 0.$$

Also, $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$.

Geometric distribution: If $X \sim Geo(\theta)$ then it has probability function

$$f(x|\theta) = (1-\theta)^{x-1}\theta, \quad x = 1, 2, \ldots, \quad 0 \leq \theta \leq 1.$$

Also, $E(X) = 1/\theta$ and $Var(X) = (1-\theta)/\theta^2$.

Normal distribution: If $X \sim N(\mu, 1/\tau)$ then it has density

$$f(x|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x-\mu)^2\right\}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \tau > 0.$$

Also, $E(X) = \mu$ and $Var(X) = 1/\tau$.

Poisson distribution: If $X \sim Po(\theta)$ then it has probability function

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \ldots, \quad \lambda > 0.$$

Also, $E(X) = \theta$ and $Var(X) = \theta$.

1. Consider the scenarios below. State whether a classical, frequency or subjective interpretation of probability is being used to estimate θ_1, θ_2 and θ_3.

Scenario	**Classical? Frequency? Subjective?**
A jockey is about to take part in the Grand National horse race. From ten racehorses available, the jockey's manager will randomly select a horse for the jockey to ride. You assess that $\theta_1 = \Pr(\text{The horse } \textit{Bayesian Beauty} \text{ is chosen}) = 0.1$.	Classical **(1 mark)**
You are going on a diving trip in the Great Barrier reef. You liaise with a biologist in an attempt to evaluate $\theta_2 = \Pr(\text{You see a loggerhead turtle})$.	Subjective **(1 mark)**
You are playing a board game with your friends, and need to spin a hexagon on the spinner shown below to "get out of jail". Since the start of the game, you have noticed that the hexagon has hardly ever been spun; thus, you suggest that $\theta_3 = \Pr(\text{spin a hexagon}) = 0.0001$.	Subjective **(1 mark)**

[Total Q1: 3 marks]

2. Road safety experts are trying to reduce vehicle speeds at 20 known safety "hotspots" in Newcastle. They test one of three road safety features at each "hotspot" according to the table below.

Safety feature	Number of "hotspots"
Speed camera	4
Speed bumps	6
"Speed kills" signs	10

Past studies show that a significant reduction in speed will be observed with probability 0.8, 0.7 and 0.2 if speed cameras, speed bumps and "Speed kills" signs are used (respectively).

For a randomly chosen "hotspot", a significant reduction in speed was observed. Given this observation, find the posterior probability for each of the safety features used. Express your answers in the form $a/47$, $b/47$ and $c/47$, where a, b and c are to be found.

Answer:

Let SC : Speed camera, SB : Speed bumps and SK : "Speed kills" signs. Also, let R : Significant reduction in speed. Then we have

$$P(SC) = 2/10; \quad P(SB) = 3/10; \quad P(SK) = 5/10,$$

and

$$P(R|SC) = 0.8; \quad P(R|SB) = 0.7; \quad P(R|SK) = 0.2.$$

According to the Law of Total Probability, we have

$$\begin{aligned} P(R) &= P(R|SC)P(SC) + P(R|SB)P(SB) + P(R|SK)P(SK) \\ &= \left(\frac{8}{10} \times \frac{2}{10}\right) + \left(\frac{7}{10} \times \frac{3}{10}\right) + \left(\frac{2}{10} \times \frac{5}{10}\right) = \frac{47}{100}. \qquad \textbf{(2 marks)} \end{aligned}$$

So

$$P(SC|R) = \frac{P(R|SC)P(SC)}{P(R)} = \frac{\frac{8}{10} \times \frac{2}{10}}{\frac{47}{100}} = \frac{16}{47}. \qquad \textbf{(2 marks)}$$

Similarly,

$$P(SB|R) = \frac{P(R|SB)P(SB)}{P(R)} = \frac{\frac{7}{10} \times \frac{3}{10}}{\frac{47}{100}} = \frac{21}{47}, \qquad \textbf{(2 marks)}$$

giving

$$P(SK|R) = 1 - \left(\frac{16}{47} + \frac{21}{47}\right) = \frac{10}{47}. \qquad \textbf{(2 marks)}$$

[Total Q2: 8 marks]

3. Researchers are interested in the crime rate, θ per day, for towns in the Brazilian state of Paranhá.

(a) Discussions with officials from the state police force led to the screenshot shown overleaf, taken from the *MATCH* online elicitation tool.

(i) Name the elicitation procedure that has been used to fit a distribution to the beliefs of the Paranhá state police about the crime rate θ.

Answer:

Trial roulette method **(1 mark)**

(ii) Write down the fitted distribution for θ, and give the associated prior mean and standard deviation. Write down the parameters of your distribution to one decimal place.

Answer:

$$\theta \sim Ga(3.7, 0.2) \qquad \textbf{(2 marks)}$$

$$E(\theta) = 3.7/0.2 = 18.5 \qquad \textbf{(1 mark)}$$

$$SD(\theta) = \sqrt{3.7/0.2^2} = 9.62 \qquad \textbf{(1 mark)}$$

(iii) Use the feedback percentiles as provided by *MATCH* to phrase a question to the police as part of the feedback and refinement process.

Answer:

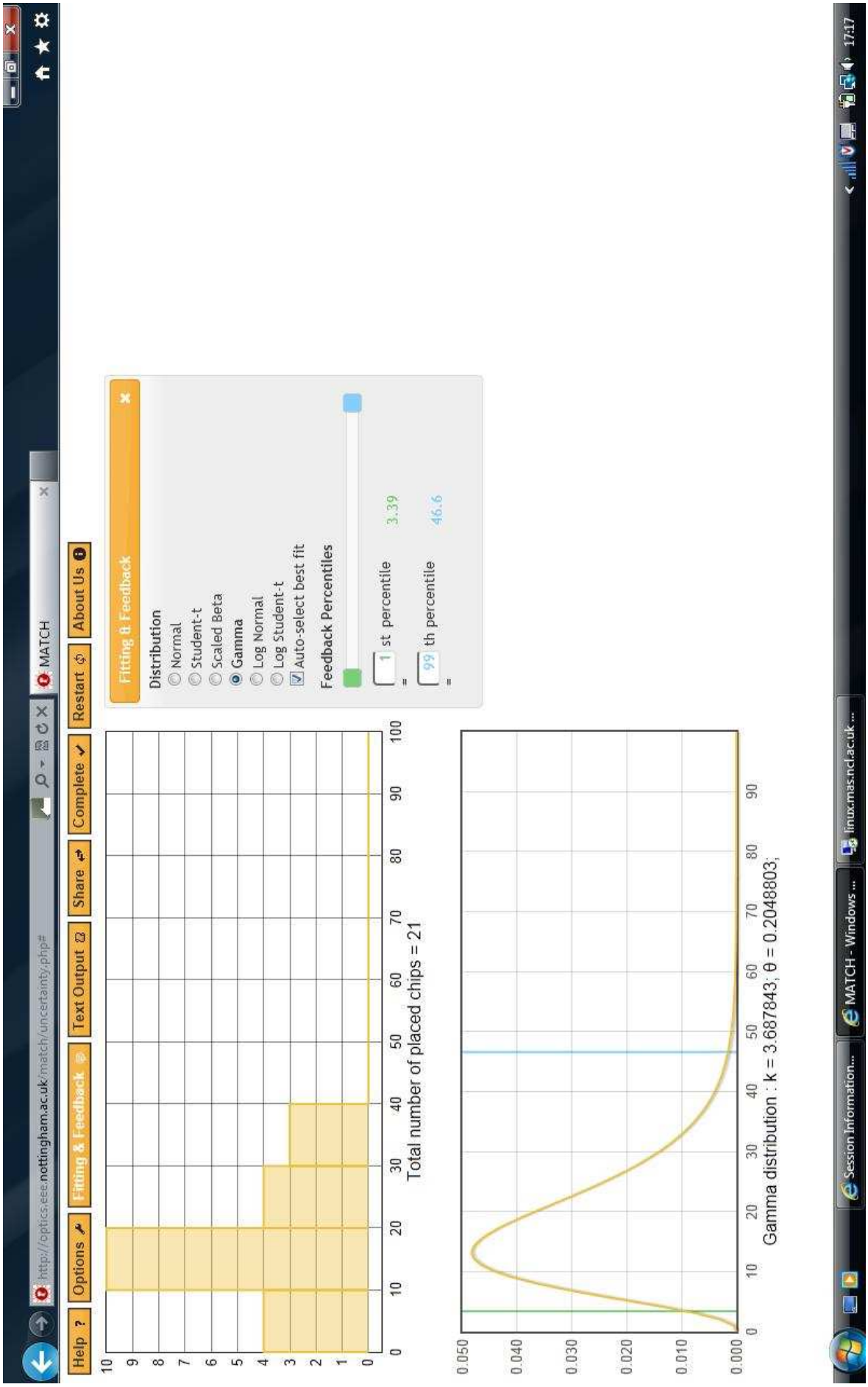
(3 marks) for something like:

"There is less than a one in a hundred chance that the daily crime rate for towns in Paranhá state will exceed 47. Does this seem reasonable?"

Or perhaps:

"Only once in a hundred days will the crime rate for towns in Paranhá state fall below about 3 or 4"

Question 3 continued on next page


http://optics.eee.nottingham.ac.uk/match/uncertainty.php#
MATCH
Help ?
Options
Fitting & Feedback
Text Output
Share
Complete
Restart
About Us
Fitting & Feedback
Distribution
Normal
Student-t
Scaled Beta
Gamma
Log Normal
Log Student-t
Auto-select best fit
Feedback Percentiles
1 st percentile 3.39
99 th percentile 46.6
Total number of placed chips = 21
Gamma distribution : k = 3.687843; θ = 0.2048803;

(b) The same team of researchers are now interested in X, the daily number of reported crimes in Coritiba (a town in Paranhá state). On Monday, Tuesday and Wednesday this week, they found that there were 80, 95 and 110 reported crimes in Coritiba (respectively).

Assuming a Poisson distribution for X with rate θ, show that

$$f(\boldsymbol{x}|\theta) \propto e^{-3\theta}\theta^{285}, \qquad \theta > 0.$$

Answer:

$$\begin{aligned} f(\boldsymbol{x}|\theta) &= \frac{e^{-\theta}\theta^{80}}{80!} \times \frac{e^{-\theta}\theta^{95}}{95!} \times \frac{e^{-\theta}\theta^{110}}{110!} \\ &\propto e^{-3\theta}\theta^{285}, \qquad \theta > 0. \qquad (\textbf{2 marks}) \end{aligned}$$

(c) Use your answers to parts (a) and (b) to obtain the posterior distribution for θ.

Answer:

We have

$$\pi(\theta|\boldsymbol{x}) \propto \theta^{2.7}e^{-0.2\theta} \times e^{-3\theta}\theta^{285} = \theta^{288.7-1}e^{-3.2\theta},$$

and so $\theta|\boldsymbol{x} \sim Ga(288.7, 3.2)$. **(4 marks)**
Must specify that the posterior is $Ga(288.7, 3.2)$ to get full marks - not enough just to state the density.

Question 3 continued on next page

(d) Use your answers to parts (a) and (c) to explain how beliefs about θ have been updated in light of the data.

Answer:

Comparing prior with posterior gives:

	Mean	St. dev.
Prior	18.5	9.62
Posterior	90.2	5.31

Thus, in light of the data:

- Our beliefs about the daily crime rate have soared from around 18/19, on average, to about 90... this might cast doubt on the credibility of the police information (or perhaps the data? Probably not, though...!) **(2 marks)**
- We are now much more certain in our beliefs about the daily crime rate, the standard deviation having reduced from 9.6 to 5.3 **(2 marks)**

[**Total Q3: 18 marks**]

4. The *Weibull distribution*, with probability density function

$$f(x|\lambda) = \frac{3x^2}{\lambda}\exp\left\{-\frac{x^3}{\lambda}\right\}, \qquad x > 0, \quad \lambda > 0,$$

is often used to model river discharges.

(a) Suppose we observe a series of river discharges $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ (in thousand m^3/s). Assuming we have a sample of independent observations from a Weibull distribution with rate λ, obtain the likelihood function $f(\boldsymbol{x}|\lambda)$.

Answer:

$$\begin{aligned} f(\boldsymbol{x}|\lambda) &= \prod_{i=1}^{n} \frac{3x_i^2}{\lambda}\exp\left\{-\frac{x_i^3}{\lambda}\right\} \\ &= \left(3^n \prod_{i=1}^{n} x_i^2\right)\lambda^{-n}\exp\left\{-\frac{1}{\lambda}\sum_{i=1}^{n} x_i^3\right\}, \qquad \lambda > 0. \qquad \textbf{(4 marks)} \end{aligned}$$

(b) Use the Factorisation Theorem to find a sufficient statistic T for λ.

Answer:

We have

$$f(\boldsymbol{x}|\lambda) = h(\boldsymbol{x}) \times g(\lambda, t),$$

where

$$h(\boldsymbol{x}) = 3^n \prod_{i=1}^{n} x_i^2 \quad \text{and} \quad g(\lambda, t) = \lambda^{-n}\exp\{-t/\lambda\},$$

where $t = \sum_{i=1}^{n} x_i^3$.

Thus, by the Factorisation Theorem, $T = \sum_{i=1}^{n} X_i^3$ is sufficient for λ. **(3 marks)**

Question 4 continued on next page

(c) An *inverse gamma distribution*, $InvGa(a, b)$, is used as a prior for the rate of river discharges, with density

$$\pi(\lambda) = \frac{b^{(a-1)} e^{-b/\lambda}}{\Gamma(a-1)\lambda^a}, \quad a > 0, \quad b > 0.$$

Assuming the Weibull likelihood obtained in part (a), find the posterior distribution for λ. Clearly specify the parameters in this posterior, expressed in terms of your sufficient statistic $T = t$ where appropriate.

Answer:

We have

$$\begin{aligned}
\pi(\lambda|\boldsymbol{x}) &\propto \pi(\lambda) \times f(\boldsymbol{x}|\lambda) \\
&\propto \exp\{-b/\lambda\}\, \lambda^{-a} \times \lambda^{-n} \exp\{-t/\lambda\} \\
&\propto \exp\left\{-\frac{1}{\lambda}(b+t)\right\} \lambda^{-a-n} \\
&\propto \frac{e^{-(b+t)/\lambda}}{\lambda^{(a+n)}},
\end{aligned}$$

and so $\lambda|\boldsymbol{x} \sim InvGa(b+t, a+n)$, where $t = \sum_{i=1}^{n} x_i^3$. **(7 marks)**

For full marks, students must have the density in terms of t, not x_i^3, as the question asks (deduct **(1 mark)** if they don't do this). Also, for **(2 marks)** they must fully identify the posterior as $InvGa(b+t, a+n)$.

[Total Q4: 14 marks]

5. A random sample of size n is to be taken from a Normal distribution with mean μ and variance 9. The prior for μ is $N(b, 1)$.

Find a lower bound for n such that the posterior standard deviation is no more than 0.5.

Answer:

From the lecture notes, we know that the posterior standard deviation is $1/(\sqrt{d + n\tau})$ where d and τ are the precisions of the prior distribution, and the data, respectively.

Thus, we need

$$\begin{aligned} \frac{1}{\sqrt{1 + n/9}} &\leq 0.5 \\ 1 + \frac{n}{9} &\geq 4 \\ n &\geq (4 - 1) \times 9 = 27. \qquad \textbf{(7 marks)} \end{aligned}$$

[**Total Q5: 7 marks**]

THE END