

MAS2903

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS, STATISTICS & PHYSICS

SEMESTER 2 2018–2019

MAS2903

An Introduction to Bayesian Methods: Mid–semester test

Time allowed: 40 minutes

Candidates should attempt all questions. Marks for each question are indicated.

There are FIVE questions on this paper.

*Answers to questions should be entered directly on this question paper in the spaces provided.
This question paper must be handed in at the end of the test.*

Name:.....

Beta distribution: If $X \sim Be(\alpha, \beta)$ then it has density

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad \alpha > 0, \beta > 0.$$

Also, $E(X) = \alpha/(\alpha + \beta)$, $Var(X) = \alpha\beta/\{(\alpha + \beta)^2(\alpha + \beta + 1)\}$, and $Mode(X) = (\alpha - 1)/(\alpha + \beta - 2)$.

Binomial distribution: If $X \sim Bin(n, p)$ then it has probability function

$$f(x|n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n, \quad 0 \leq p \leq 1.$$

Also, $E(X) = np$ and $Var(X) = np(1-p)$.

Exponential distribution: If $X \sim Exp(\lambda)$ then it has density

$$f(x|\lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Also, $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$.

Gamma distribution: If $X \sim Ga(\alpha, \lambda)$ then it has density

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha > 0, \lambda > 0.$$

Also, $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$.

Geometric distribution: If $X \sim Geo(\theta)$ then it has probability function

$$f(x|\theta) = (1-\theta)^{x-1} \theta, \quad x = 1, 2, \dots, \quad 0 \leq \theta \leq 1.$$

Also, $E(X) = 1/\theta$ and $Var(X) = (1-\theta)/\theta^2$.

Normal distribution: If $X \sim N(\mu, 1/\tau)$ then it has density

$$f(x|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x-\mu)^2\right\}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \tau > 0.$$

Also, $E(X) = \mu$ and $Var(X) = 1/\tau$.

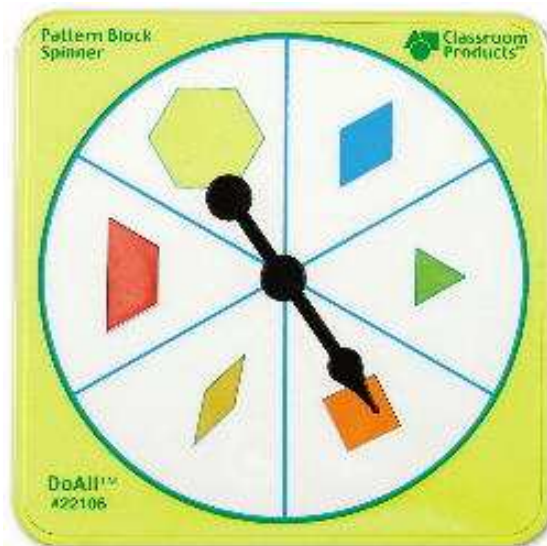
Poisson distribution: If $X \sim Po(\theta)$ then it has probability function

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \dots, \quad \lambda > 0.$$

Also, $E(X) = \theta$ and $Var(X) = \theta$.

1. Consider the scenarios below. State whether a classical, frequency or subjective interpretation of probability is being used to estimate θ_1 , θ_2 and θ_3 .

Scenario	Classical? Frequency? Subjective?
A jockey is about to take part in the Grand National horse race. From ten racehorses available, the jockey's manager will randomly select a horse for the jockey to ride. You assess that $\theta_1 = \Pr(\text{The horse } \textit{Bayesian Beauty} \text{ is chosen}) = 0.1$.	
You are going on a diving trip in the Great Barrier reef. You liaise with a biologist in an attempt to evaluate $\theta_2 = \Pr(\text{You see a loggerhead turtle})$.	
You are playing a board game with your friends, and need to spin a hexagon on the spinner shown below to "get out of jail". Since the start of the game, you have noticed that the hexagon has hardly ever been spun; thus, you suggest that $\theta_3 = \Pr(\text{spin a hexagon}) = 0.0001$.	



[Total Q1: 3 marks]

2. Road safety experts are trying to reduce vehicle speeds at 20 known safety “hotspots” in Newcastle. They test one of three road safety features at each “hotspot” according to the table below.

Safety feature	Number of “hotspots”
Speed camera	4
Speed bumps	6
“Speed kills” signs	10

Past studies show that a significant reduction in speed will be observed with probability 0.8, 0.7 and 0.2 if speed cameras, speed bumps and “Speed kills” signs are used (respectively).

For a randomly chosen “hotspot”, a significant reduction in speed was observed. Given this observation, find the posterior probability for each of the safety features used. Express your answers in the form $a/47$, $b/47$ and $c/47$, where a , b and c are to be found.

Answer:

[Total Q2: 8 marks]

3. Researchers are interested in the crime rate, θ per day, for towns in the Brazilian state of Paranhá.

(a) Discussions with officials from the state police force led to the screenshot shown overleaf, taken from the *MATCH* online elicitation tool.

(i) Name the elicitation procedure that has been used to fit a distribution to the beliefs of the Paranhá state police about the crime rate θ .

Answer:

(ii) Write down the fitted distribution for θ , and give the associated prior mean and standard deviation. Write down the parameters of your distribution to one decimal place.

Answer:

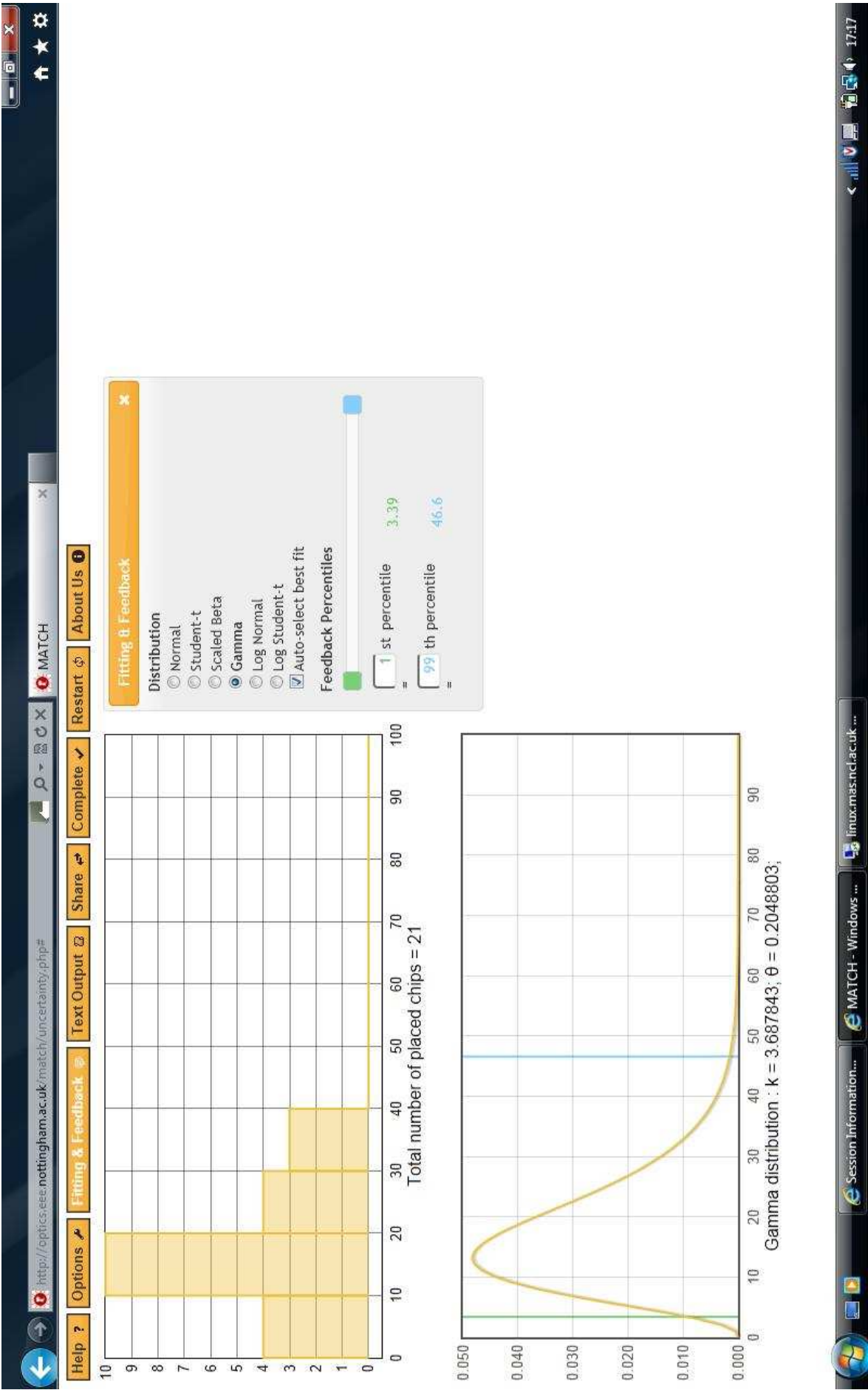
$$\theta \sim$$

$$E(\theta) =$$

$$SD(\theta) =$$

(iii) Use the feedback percentiles as provided by *MATCH* to phrase a question to the police as part of the feedback and refinement process.

Answer:



- (b) The same team of researchers are now interested in X , the daily number of reported crimes in Curitiba (a town in Paraná state). On Monday, Tuesday and Wednesday this week, they found that there were 80, 95 and 110 reported crimes in Curitiba (respectively).

Assuming a Poisson distribution for X with rate θ , show that

$$f(\mathbf{x}|\theta) \propto e^{-3\theta} \theta^{285}, \quad \theta > 0.$$

Answer:

- (c) Use your answers to parts (a) and (b) to obtain the posterior distribution for θ .

Answer:

- (d) Use your answers to parts (a) and (c) to explain how beliefs about θ have been updated in light of the data.

Answer:

[Total Q3: 18 marks]

4. The *Weibull distribution*, with probability density function

$$f(x|\lambda) = \frac{3x^2}{\lambda} \exp \left\{ -\frac{x^3}{\lambda} \right\}, \quad x > 0, \quad \lambda > 0,$$

is often used to model river discharges.

- (a) Suppose we observe a series of river discharges $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ (in thousand m³/s). Assuming we have a sample of independent observations from a Weibull distribution with rate λ , obtain the likelihood function $f(\mathbf{x}|\lambda)$.

Answer:

- (b) Use the Factorisation Theorem to find a sufficient statistic T for λ .

Answer:

- (c) An *inverse gamma distribution*, $InvGa(a, b)$, is used as a prior for the rate of river discharges, with density

$$\pi(\lambda) = \frac{b^{(a-1)}e^{-b/\lambda}}{\Gamma(a-1)\lambda^a}, \quad a > 0, \quad b > 0.$$

Assuming the Weibull likelihood obtained in part (a), find the posterior distribution for λ . Clearly specify the parameters in this posterior, expressed in terms of your sufficient statistic $T = t$ where appropriate.

Answer:

[Total Q4: 14 marks]

5. A random sample of size n is to be taken from a Normal distribution with mean μ and variance 9. The prior for μ is $N(b, 1)$.

Find a lower bound for n such that the posterior standard deviation is no more than 0.5.

Answer:

[Total Q5: 7 marks]

THE END