

MAS2903

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 2 2016/2017

MAS2903

An Introduction to Bayesian Methods

Time allowed: 1 hour 30 minutes

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FOUR questions in Section A and THREE questions in Section B.

Answers to questions should be entered directly on this question paper in the spaces provided. Rough work should be done on the blank sides of the pages. The rough work will not be marked. This question paper must be handed in, attached inside an anonymised cover sheet, at the end of the examination.

Calculators may be used.

Binomial distribution: If $X \sim \text{Bin}(k, \theta)$ then it has probability function

$$f(x|\theta) = \binom{k}{x} \theta^x (1 - \theta)^{k-x}, \quad x = 0, 1, \dots, \quad 0 < \theta < 1.$$

Also, $E(X) = k\theta$ and $\text{Var}(X) = k\theta(1 - \theta)$.

Poisson distribution: If $X \sim \text{Po}(\theta)$ then it has probability function

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \dots, \quad \lambda > 0.$$

Also, $E(X) = \theta$ and $\text{Var}(X) = \theta$.

Beta distribution: If $X \sim \text{Be}(\alpha, \beta)$ then it has density

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad \alpha > 0, \beta > 0.$$

Also, $E(X) = \alpha/(\alpha + \beta)$, $\text{Var}(X) = \alpha\beta/\{(\alpha + \beta)^2(\alpha + \beta + 1)\}$, and $\text{Mode}(X) = (\alpha - 1)/(\alpha + \beta - 2)$.

Exponential distribution: If $X \sim \text{Exp}(\lambda)$ then it has density

$$f(x|\lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Also, $E(X) = 1/\lambda$ and $\text{Var}(X) = 1/\lambda^2$.

Gamma distribution: If $X \sim \text{Ga}(\alpha, \lambda)$ then it has density

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha > 0, \lambda > 0.$$

Also, $E(X) = \alpha/\lambda$, $\text{Var}(X) = \alpha/\lambda^2$ and $\text{Mode}(X) = (\alpha - 1)/\lambda$.

Normal distribution: If $X \sim N(\mu, 1/\tau)$ then it has density

$$f(x|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x - \mu)^2\right\}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \tau > 0.$$

Also, $E(X) = \mu$ and $Var(X) = 1/\tau$.

Finally, if $X_1, X_2, \dots, X_n$ are independent $N(\mu, 1/\tau)$ random variables and μ has a prior distribution $N(b, 1/d)$ then the posterior distribution is $N(B, 1/D)$ where

$$B = \frac{db + n\tau\bar{x}}{d + n\tau}, \quad D = d + n\tau.$$

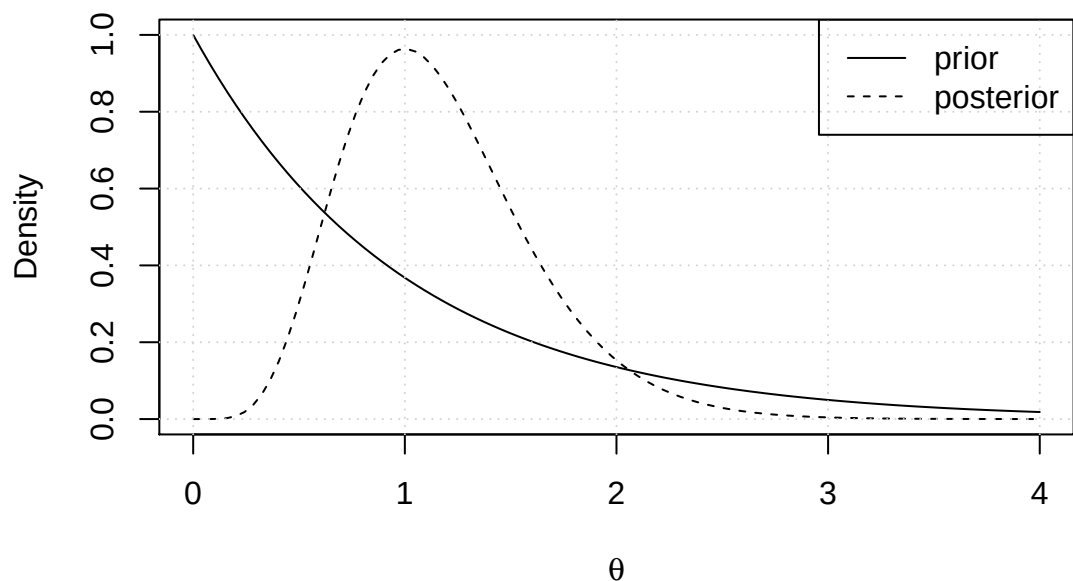
Beta-Binomial distribution: If $X \sim BetaBin(n, \alpha, \beta)$ then it has density

$$f(x|n, \alpha, \beta) = \binom{n}{x} \frac{B(x + \alpha, \beta + n - x)}{B(\alpha, \beta)}, \quad x = 0, 1, \dots, n \quad \alpha > 0, \beta > 0.$$

Also, $E(X) = n\alpha/(\alpha + \beta)$ and $Var(X) = n\alpha\beta \frac{(\alpha + \beta + n)}{(\alpha + \beta)^2(\alpha + \beta + 1)}$.

SECTION A

- A1. (a) The following plot shows a prior density elicited from an expert and a posterior density.



Use the plot to complete each of the statements below. Circle a single answer for each statement.

- (i) The prior distribution is:

Normal Exponential Uniform Poisson

- (ii) The posterior distribution is:

Normal Exponential Gamma Beta

- (iii) A 95% highest density interval (HDI) based on the expert's beliefs alone is:

(0, 3.00) (0.55, 2.17) (0.05, 3.69) (0, 1.00)

- (iv) Compared to the prior, the posterior has:

Higher variance Lower variance Equal variance

- (v) The posterior mean is:

0 1 1.16 2.17

- (vi) The prior mode is:

0 1 1.16 2.17

(b) For each of the following events, state which probability interpretation you would use and give a short explanation as to why.

- (i) The **bayesian.org** website is unavailable due to technical problems for at least 1 hour today.

Answer:

- (ii) You pick 5 playing cards from a shuffled deck and they all have the same suit.

Answer:

[10 marks]

A2. A passport is suspected of being a forgery. Three scenarios are considered: the passport is real, it's a forgery made by Fred the forger or it's a forgery made by Carol the counterfeiter. Each scenario is initially judged to be equally likely. Then an expert is asked to examine the passport and determine if it is real or a forgery. The probability that an expert says the passport is real is 99% for a real passport, 80% for a forgery from Fred or 40% for a forgery from Carol.

- (a) The expert believes the passport is real. Given this, what is the probability of each scenario?

Answer:

- (b) A second expert is consulted who believes the passport is a forgery. What is the probability of each scenario now, given the answers of both experts? You may assume the two experts give independent answers.

Answer:

[10 marks]

A3. Sapelo Island is a barrier island off the southeast coast of the USA, and is on the migratory path of the rare black pelican. The random variable X_i represents the number of birds that are seen on day i at an observation spot on the island. It is assumed that $X_i \sim Po(\theta)$ and the observations are independent. In the first week of April 2014, the following counts were made: 2 1 3 0 1 2 3.

(a) Show that the likelihood function is proportional to $e^{-7\theta}\theta^{12}$.

Answer:

(b) Prove that the maximum likelihood estimate for θ is 12/7.

Answer:

- (c) Demonstrate that under a $Ga(2, b)$ prior for θ the posterior distribution is $Ga(14, b + 7)$.

Answer:

- (d) For what range of b values is the posterior mode for θ greater than the maximum likelihood estimate?

Answer:

[10 marks]

A4. Let X be the observed speed (in miles per hour) of a vehicle at a traffic sensor location. It is assumed that X has a $N(\mu, 1/\tau)$ distribution with $\tau = 0.1$. Here μ is the true speed and τ is the precision of the measurement.

(a) A local traffic expert tells you that a typical value of μ is 30. Also she expects that 95 times out of 100 the value would fall in the range 10 to 50. Based on this, a $N(30, 10^2)$ prior has been selected.

(i) Explain why this prior is a good choice to capture the expert's beliefs.

Answer:

(ii) Explain why $Exp(1/30)$ would be a bad choice of prior distribution.

Answer:

(b) Under the prior, $\Pr(\mu > 60) = 0.0013$. Use this to come up with a prior refinement question to ask the expert.

Answer:

- (c) An observation $x = 8$ is made. Use the results on the formula sheet to show that the posterior is $N(10, 100/11)$.

Answer:

- (d) It is decided to change the prior to incorporate the constraint $\mu > 5$. Write down the density of the resulting posterior up to proportionality.

Answer:

[10 marks]

SECTION B

B5. Suppose $X_i|\theta \sim \text{Exp}(\theta)$ for $i = 1, 2, \dots, n$ are independent random variables. Under a $Ga(g, h)$ prior, the posterior for θ is $Ga(g', h')$ where $g' = g + n$, $h' = h + n\bar{x}$ and $\bar{x}$ is the mean observation.

- (a) Define a conjugate prior. State whether the $Ga(g, h)$ prior in this question is conjugate and explain why. [2 marks]

Answer:

- (b) Suppose (for this part of the question only) that $g' = 6$ and $h' = 3$.

- (i) Show that a 95% posterior credible interval $[a, b]$ must satisfy the following equation. [3 marks]

$$\int_a^b \theta^5 e^{-3\theta} d\theta = \frac{38}{243}$$

Answer:

- (ii) State what other condition is needed to get a highest density interval (HDI). [2 marks]

Answer:

- (c) Suppose $Y_j|\theta \sim \text{Exp}(\theta)$ for $j = 1, 2, \dots, m$ are also observed. Derive the posterior for θ given both sets of observations. You may assume that the X_i and Y_j variables are all independent. [3 marks]

Answer:

- (d) Derive the asymptotic posterior for θ given $X_1, X_2, \dots, X_n$. [10 marks]

Answer:

[20 marks]

B6. Suppose $X_1, X_2, \dots, X_n$ are independent and identically distributed $\text{Bin}(k, \theta)$ random variables, where k is some fixed value, and θ is an unknown parameter.

(a) Show that the likelihood function is

$$f(\mathbf{x}|\theta) = c(\mathbf{x})\theta^{n\bar{x}}(1 - \theta)^{n(k-\bar{x})}, \quad \text{for } 0 < \theta < 1,$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ is the mean of the observed values, and $c(\mathbf{x})$ is a function of $\mathbf{x}$ which you should derive. [2 marks]

Answer:

(b) Find a sufficient statistic for θ . [2 marks]

Answer:

- (c) Show that $E(\bar{X}|\theta) = k\theta$ where $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ is the mean of the random variables. [2 marks]

Answer:

- (d) Show that the Jeffreys prior for this model is $\pi(\theta) \propto \theta^{-1/2}(1-\theta)^{-1/2}$ for $0 < \theta < 1$. [10 marks]

Answer:

- (e) Explain whether or not the prior from part (d) is improper. [1 mark]

Answer:

- (f) Suppose $k = 3$, $n = 5$ and $\bar{x} = 0.5$. Find the posterior distribution under the Jeffreys prior. [3 marks]

Answer:

[20 marks]

- B7.** (a) Suppose that X is the number of tropical storms occurring over the Atlantic Ocean in July 2017, and let θ be the monthly rate of occurrence of such storms. A meteorologist assumes that $X|\theta \sim Po(\theta)$, with the following prior for θ :

Level of activity	θ	$\pi(\theta)$
Busy	8	0.2
Normal	3	0.8

- (i) Suppose that six tropical storms are observed over the Atlantic Ocean in July 2017. Show that given this the probability of busy activity is 0.377 and the probability of normal activity is 0.623. [4 marks]

Answer:

- (ii) Suppose that we now want to predict Y , the number of tropical storms over the Atlantic Ocean during August 2017. Determine the predictive probability function for Y , $f(y|x = 6)$. Assume that tropical storms occur at the same rate in August as they do in July. [3 marks]

Answer:

- (iii) The table below shows the posterior predictive distribution for Y . Use this table to find an approximate 95% prediction interval for Y . [3 marks]

y	0	1	2	3	4	5	6
$f(y x = 6)$	0.0437	0.1314	0.1980	0.2002	0.1545	0.0997	0.0592
y	7	8	9	10	11	12	13
$f(y x = 6)$	0.0360	0.0242	0.0175	0.0128	0.0090	0.0059	0.0036

Answer:

- (b) Let ψ be the probability that a tropical storm develops into a hurricane. After a discussion with the meteorologist, we are able to elicit the following prior distribution for ψ : $\psi \sim Be(2, 23)$, $0 < \psi < 1$. One of the last five tropical storms over the Atlantic Ocean developed into a hurricane. For the next five tropical storms observed in the future, show that the predictive probability distribution for the number that will develop into a hurricane is Beta-Binomial $BetaBin(5, 3, 27)$ (see formula sheet for definition). [10 marks]

Answer:

[20 marks]

THE END