

MAS2317 Exam 2016: Solutions

Question 1

ai

$$\pi(\theta|\mathbf{x}) = \frac{\pi(\theta)f(\mathbf{x}|\theta)}{f(\mathbf{x})}$$

aii It is the probability of $\mathbf{x}$ given θ , viewed as a function of θ .

aiii It encapsulates beliefs about θ before observing any data.

aiv It encapsulates beliefs about θ after observing the data.

av

$$f(\mathbf{x}) = \int \pi(\theta)f(\mathbf{x}|\theta)d\theta$$

bi It's the value of θ maximising the density: 0.8.

bii Beta. (Shape and/or range of non-zero values are wrong for the others.)

bihi This is $\int_0^{0.6} \pi(\theta|\mathbf{x})d\theta$. From the graph this is clearly between 0 and 0.5.

The only matching answer is 0.23.

biv The distribution is skewed to the left so the mean is slightly less than the mode: 0.71.

bv Both endpoints of a HDI interval have the same density. Only the last two answers appear to satisfy this. Integrating the pdf over the interval should give 0.95. This is clearly not the case for (0.68, 0.89). Hence the answer is (0.41, 0.98).

Question 2

a The data is continuous but only positive values make sense as waiting times, so we want a distribution on non-negative reals. The exponential distribution puts positive density on all non-negative reals.

b The likelihood function is

$$\begin{aligned} f(\mathbf{x}|\lambda) &= \prod_{i=1}^7 [\lambda \exp(-\lambda x_i)] \\ &= \lambda^7 \exp\left(-\lambda \sum_{i=1}^7 x_i\right) \\ &= \lambda^7 \exp(-13\lambda) \end{aligned}$$

c The posterior distribution is

$$\begin{aligned}\pi(\lambda|\mathbf{x}) &\propto \pi(\lambda)f(\mathbf{x}|\lambda) \\ &= \lambda \exp(-2\lambda)\lambda^7 \exp(-13\lambda) \\ &= \lambda^8 \exp(-15\lambda)\end{aligned}$$

This is a $Ga(9, 15)$ distribution.

d This is an unsuitable prior as it places positive density on negative values of λ .

Question 3

a The prior is $\mu \sim N(b, 1/d)$ where $b = 85, d = 1/400$.

Observations follow a $X \sim N(\mu, 1/\tau)$ distribution with $\tau = 1/9$.

From the lecture notes, $\mu|\mathbf{x} \sim N(B, 1/D)$ where

$$D = d + n\tau = 1/400 + 10/9 = 1.14$$

$$B = (db + n\tau\bar{x})/D = 99.37$$

b

$$\begin{aligned}\Pr(\mu > 100|\mathbf{x}) &= \Pr(Z > (100 - 99.3)/0.95) \\ &= \Pr(Z > 0.74) = 0.23\end{aligned}$$

- c We want a posterior precision of $D = 1.5^{-2} = 4/9$. Solving $d+n\tau = 4/9$ for n gives $n = 3.97$. So n is the minimum sample size giving the required precision.

Question 4

- a Let X be the number of imperfections. The likelihood function is:

$$\Pr(X = 7|\theta) = \frac{\theta^7 e^{-\theta}}{7!}$$

This gives $\Pr(X = 7|\theta = 9) = 0.117$ and $\Pr(X = 7|\theta = 3) = 0.022$.

Applying Bayes theorem we have:

$$\begin{aligned}\Pr(\theta = 9|X = 7) &= \frac{\Pr(X = 7|\theta = 9) \times \Pr(\theta = 9)}{\Pr(X = 7|\theta = 9) \times \Pr(\theta = 9) + \Pr(X = 7|\theta = 3) \times \Pr(\theta = 3)} \\ &= \frac{0.117 \times 0.5}{0.117 \times 0.5 + 0.022 \times 0.5} \\ &= 0.844\end{aligned}$$

And so $\Pr \theta = 3|X = 7) = 1 - \Pr(\theta = 9|X = 7) = 0.156$.

- b The predictive probability function is:

$$\begin{aligned}f(y|x = 7) &= f(y|\theta = 9) \Pr(\theta = 9|x = 7) + f(y|\theta = 3) \Pr(\theta = 3|x = 7) \\ &= \frac{9^y e^{-9}}{y!} \times 0.844 + \frac{3^y e^{-3}}{y!} \times 0.156 \\ &= [9^y \times 0.0001041579 + 3^y \times 0.007766783]/y!\end{aligned}$$

Question 5

a The “trial roulette” method is being used. Advantage:

- Expert can construct distribution interactively.
- No need for the expert to think about quartiles/other summaries.

We have $\theta \sim Be(11, 2)$.

b This is not consistent, as the median for the distribution in part (a) is $11/13 = 0.85$.

The notes suggest using the bisection method. Next we’d elicit upper and lower quartiles, fit some parameteric distribution to this (e.g. using MATCH), and finally check the results seem reasonable to the expert.

Other reasonable answers would also be acceptable e.g. eliciting the prior standard deviation instead of quartiles.

c The posterior is

$$\begin{aligned}\pi(\theta|\mathbf{x}) &\propto \pi(\theta)f(\mathbf{x}|\theta) \\ &\propto \theta^{10}(1-\theta)\theta^{21}(1-\theta)^9 \\ &\propto \theta^{31}(1-\theta)^{10}\end{aligned}$$

This is a $Be(32, 11)$ distribution.

d The likelihood is

$$f(\mathbf{x}|\theta) = \binom{n}{\phi} \theta^{n\phi} (1-\theta)^{n(1-\phi)}$$

Hence the log likelihood is

$$\ell(\theta) = \log \binom{n}{\phi} + n\phi \log \theta + n(1-\phi) \log(1-\theta)$$

Its first derivative is

$$\ell'(\theta) = n\phi/\theta - n(1-\phi)/(1-\theta)$$

Solving this equal to zero gives MLE $\hat{\theta} = \phi$. The second derivative is

$$\ell''(\theta) = -n\phi\theta^{-2} - n(1-\phi)(1-\theta)^{-2}$$

Hence the observed information is

$$J(\hat{\theta}) = -\ell''(\theta) = n\phi^{-1} + n(1-\phi)^{-1} = n\phi^{-1}(1-\phi)^{-1}$$

The asymptotic posterior is

$$\theta|\mathbf{x} \sim N(\hat{\theta}, J(\hat{\theta})^{-1})$$

i.e. $N(\phi, \phi(1-\phi)/n)$.

Question 6

- a The statistic $T(X)$ is sufficient for θ if $f(\mathbf{x}|T(X) = t)$ does not depend on θ .
- b The statistic $T(X)$ is sufficient for θ if and only if $f(\mathbf{x}|\theta) = h(\mathbf{x})g(t(\mathbf{x}), \theta)$.
- c The likelihood is

$$\begin{aligned} f(\mathbf{x}|\lambda) &= \prod_{i=1}^n \lambda e^{-\lambda x_i} \\ &= \lambda^n \exp(-\lambda \sum_{i=1}^n x_i) \end{aligned}$$

The factorisation theorem applies with $h(\mathbf{x}) = 1$ and $g(t(x), \lambda) = \lambda^n \exp(-\lambda t)$.

- d It is sufficient. Note $T' = \log(\sum_{i=1}^n X_i) = \log T$. The factorisation theorem applies with $h(\mathbf{x}) = 1$ and $g(t(x), \lambda) = \lambda^n \exp(-\lambda \exp(t'))$.
- e Using Bayes theorem

$$\begin{aligned} \pi(\lambda|\mathbf{x}) &\propto \pi(\lambda)f(\mathbf{x}|\lambda) \\ &\propto \lambda^{g-1} \exp(-h\lambda) \lambda^n \exp(-\lambda \sum_{i=1}^n x_i) \\ &= \lambda^{g+n-1} \exp(-\lambda[h + \sum_{i=1}^n x_i]) \end{aligned}$$

So the posterior is a $Ga(g + n, h + t(x))$ distribution.

The likelihood for T is

$$f(t|\lambda) \propto \lambda^n t^{n-1} \exp(-t\lambda)$$

So using Bayes theorem

$$\begin{aligned}\pi(\lambda|t) &\propto \pi(\lambda)f(t|\lambda) \\ &\propto \lambda^{g-1} \exp(-h\lambda) \lambda^n t^{n-1} \exp(-t\lambda) \\ &\propto \lambda^{g+n-1} \exp(-\lambda[h+t])\end{aligned}$$

This is also a $Ga(g+n, h+t(x))$ distribution.

Question 7

a The likelihood is

$$\begin{aligned}f(\mathbf{x}|\tau) &= \prod_{i=1}^n \frac{\tau^{1/2}}{(2\pi)^{1/2}} \exp(-\frac{\tau}{2}(x_i - 1)^2) \\ &\propto \tau^{n/2} \exp\left(-\frac{\tau}{2} \sum_{i=1}^n (x_i - 1)^2\right)\end{aligned}$$

Hence the posterior is

$$\begin{aligned}
\pi(\tau|\mathbf{x}) &\propto \pi(\tau)f(\mathbf{x}|\tau) \\
&\propto \tau^{a-1} \exp(-\lambda\tau) \tau^{n/2} \exp\left(-\frac{\tau}{2} \sum_{i=1}^n (x_i - 1)^2\right) \\
&\propto \tau^{n/2+a-1} \exp\left(-\tau\left[\lambda + \frac{1}{2} \sum_{i=1}^n (x_i - 1)^2\right]\right)
\end{aligned}$$

This is a $Ga(n/2 + a, \lambda + \frac{1}{2} \sum_{i=1}^n (x_i - 1)^2)$ distribution.

b Consider a Gamma distribution with mean μ and variance σ^2 . Then $\mu = \alpha/\lambda$ and $\sigma^2 = \alpha/\lambda^2$. So $\alpha = \mu^2/\sigma^2$ and $\lambda = \mu/\sigma^2$. The posterior in the limit $\sigma^2 \rightarrow \infty$ is therefore $Ga(n/2, \frac{1}{2} \sum_{i=1}^n (x_i - 1)^2)$.

ci It is invariant to reparameterisation.

cii The likelihood is

$$\begin{aligned}
f(\mathbf{x}|\tau) &= \prod_{i=1}^n \frac{\tau^{1/2}}{(2\pi)^{1/2}} \exp(-\frac{\tau}{2}(x_i - 1)^2) \\
&\propto \tau^{n/2} \exp\left(-\frac{\tau}{2} \sum_{i=1}^n (x_i - 1)^2\right)
\end{aligned}$$

Hence the posterior is

$$\begin{aligned}
\pi(\tau|\mathbf{x}) &\propto \pi(\tau)f(\mathbf{x}|\tau) \\
&\propto \tau^{n/2-1} \exp\left(-\frac{\tau}{2} \sum_{i=1}^n (x_i - 1)^2\right)
\end{aligned}$$

This is again a $Ga(n/2, \frac{1}{2} \sum_{i=1}^n (x_i - 1)^2)$ distribution.

ciii Yes since

$$\int_{\tau=0}^{\infty} \tau^{-1} d\tau = [\log \tau]_{\tau=0}^{\infty} = \infty.$$