

MAS2317

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 2 2015/2016

MAS2317

Introduction to Bayesian Statistics

Time allowed: 1 hour 30 minutes

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FOUR questions in Section A and THREE questions in Section B.

Answers to questions should be entered directly on this question paper in the spaces provided. Rough work should be done on the blank sides of the pages. The rough work will not be marked. This question paper must be handed in, attached inside an anonymised cover sheet, at the end of the examination.

Poisson distribution: If $X \sim Po(\theta)$ then it has probability function

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \dots, \quad \lambda > 0.$$

Also, $E(X) = \theta$ and $Var(X) = \theta$.

Beta distribution: If $X \sim Be(\alpha, \beta)$ then it has density

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad \alpha > 0, \beta > 0.$$

Also, $E(X) = \alpha/(\alpha + \beta)$, $Var(X) = \alpha\beta/\{(\alpha + \beta)^2(\alpha + \beta + 1)\}$, and $Mode(X) = (\alpha - 1)/(\alpha + \beta - 2)$.

Exponential distribution: If $X \sim Exp(\lambda)$ then it has density

$$f(x|\lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Also, $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$.

Gamma distribution: If $X \sim Ga(\alpha, \lambda)$ then it has density

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha > 0, \lambda > 0.$$

Also, $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$.

Normal distribution: If $X \sim N(\mu, 1/\tau)$ then it has density

$$f(x|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x - \mu)^2\right\}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \tau > 0.$$

Also, $E(X) = \mu$, $Var(X) = 1/\tau$ and the following table gives some probabilities for $X \sim N(0, 1)$:

x	0.34	0.44	0.54	0.64	0.74
$\Pr(X < x)$	0.633	0.670	0.705	0.739	0.770

Finally, if $X_1, X_2, \dots, X_n$ are independent $N(\mu, 1/\tau)$ random variables and μ has a prior distribution $N(b, 1/d)$ then the posterior distribution is $N(B, 1/D)$ where

$$B = \frac{db + n\tau\bar{x}}{d + n\tau}, \quad D = d + n\tau.$$

SECTION A

A1. (a) (i) State Bayes theorem for a continuous parameter θ and a vector of data $\mathbf{x}$.

Answer:

(ii) Define the likelihood function in one sentence, supposing that the data is discrete.

Answer:

(iii) Describe what beliefs the prior distribution represents in one sentence.

Answer:

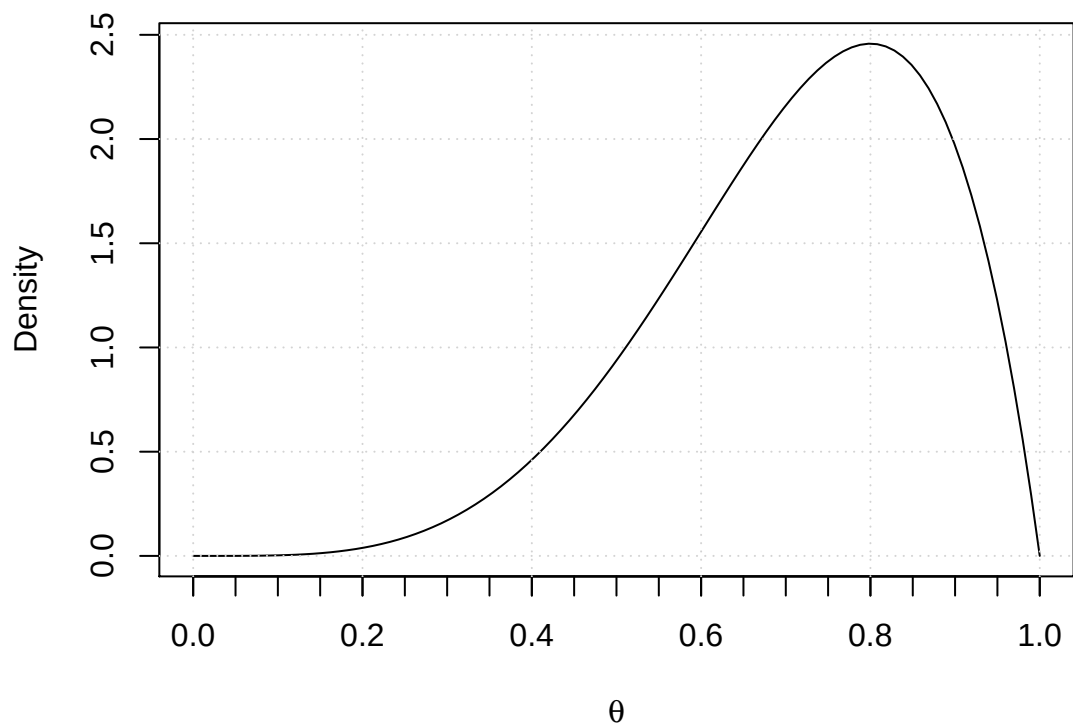
(iv) Describe what beliefs the posterior distribution represents in one sentence.

Answer:

(v) Give an equation for the normalising constant $f(\mathbf{x})$.

Answer:

(b) The following plot shows a posterior density $\pi(\theta|\mathbf{x})$.



Use the plot to complete each of the statements below. Circle a single answer for each statement.

(i) The posterior mode is:

2.46 0.25 0.8 0.71

(ii) The posterior distribution is:

Uniform Beta Normal Gamma

(iii) The value of $\Pr(\theta < 0.6|\mathbf{x})$ is:

0.60 0 1.56 0.23

(iv) The posterior mean is:

2.46 0.25 0.8 0.71

(v) The 95% HDI interval estimate is:

(0.33, 0.95) (0.36, 1) (0.41, 0.98) (0.68, 0.89)

[10 marks]

A2. You are analysing data from a recent meteor shower. The following times (in minutes) between meteors were observed: 5.2, 2.0, 2.9, 0.7, 1.7, 0.2, 0.3.

- (a) It is assumed that the data are independent realisations of a random variable $X \sim \text{Exp}(\lambda)$. Explain why $\text{Exp}(\lambda)$ is a reasonable distribution to use. [2 marks]

Answer:

- (b) Derive the likelihood function for this model. [2 marks]

Answer:

- (c) A $Ga(2, 2)$ prior distribution is used. Derive the posterior distribution. [5 marks]

Answer:

- (d) You also want to investigate using a less informative prior. A colleague suggests a $N(1, 1)$ distribution. Why would this be a bad choice? [1 mark]

Answer:

[10 marks]

A3. A medical researcher takes blood pressure measurements from 10 patients in a clinical trial. The mean of the observations is 99.4 (in units of mm Hg). She assumes that each observation is independently distributed as $N(\mu, 3^2)$. A prior study suggests that $\mu \sim N(85, 20^2)$.

Use results and tables from this paper's formula sheet to answer the following questions.

- (a) Show that the posterior distribution for μ is $N(99.4, 0.95^2)$.
[4 marks]

Answer:

- (b) Find the posterior probability that μ is above 100. [3 marks]

Answer:

- (c) The researcher wonders if she could have used a smaller sample size. Find the minimum sample size required to reduce the posterior standard deviation to 1.5. [3 marks]

Answer:

[10 marks]

A4. In steel manufacture some imperfections are inevitable. The number of imperfections visible in a slice of a particular size indicates the quality of a block of steel. This number is assumed to follow a Poisson distribution with mean θ . In factory A, $\theta = 9$ and in factory B, $\theta = 3$.

- (a) A slice of steel is investigated to determine which of the two factories it came from. It contains 7 imperfections. Calculate the posterior distribution for θ , under the prior distribution $\Pr(\theta = 9) = \Pr(\theta = 3) = 0.5$. [5 marks]

Answer:

- (b) Another slice from a different part of the same block is made. Compute the predictive probability function of Y , the number of imperfections found this time. [5 marks]

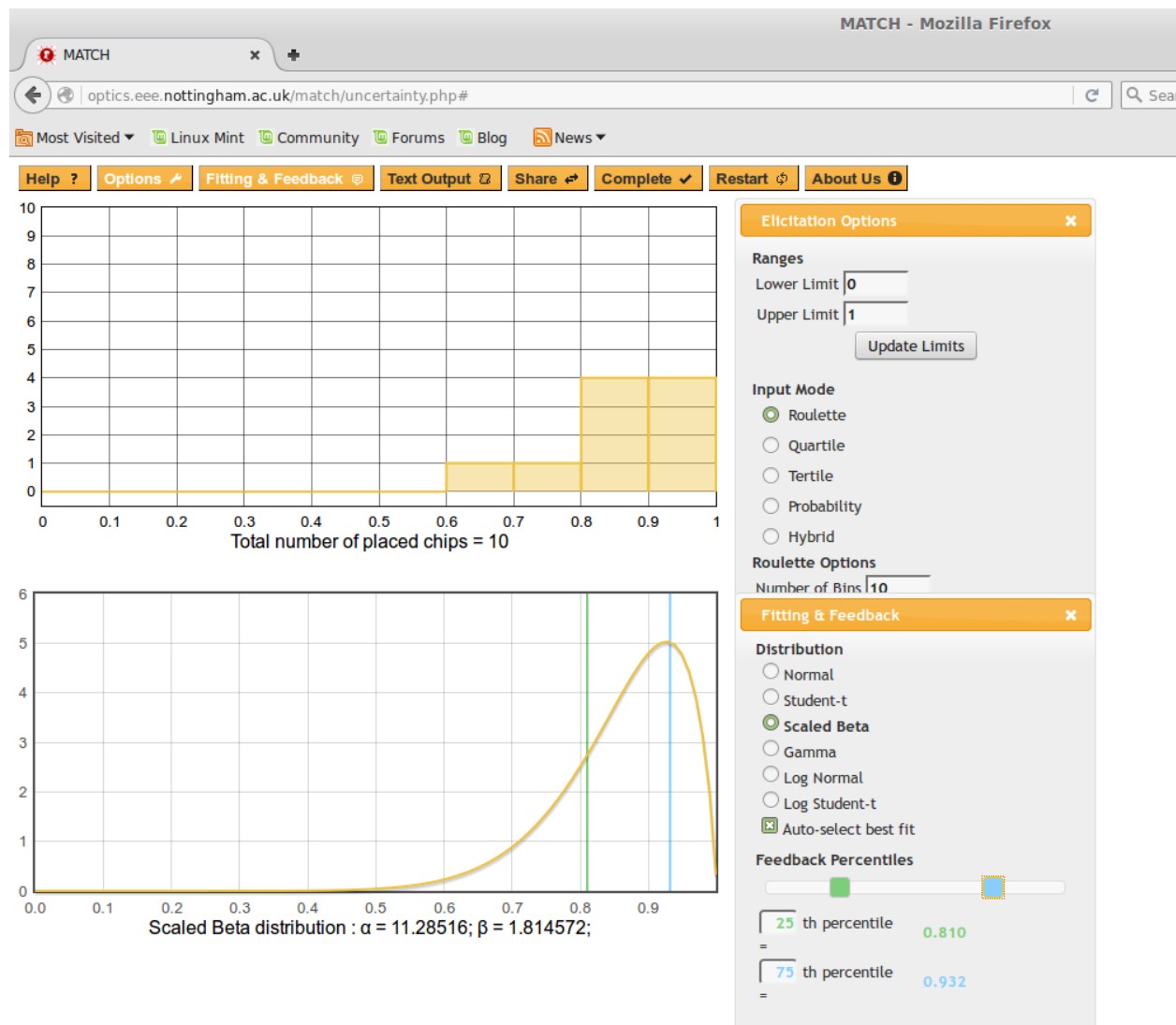
Answer:

[10 marks]

SECTION B

B5. You are analysing opinion poll data from n people for an election with candidates Mia and Edward. You assume that the number supporting Mia can be modelled as a $\text{Bin}(n, \theta)$ distribution.

- (a) A discussion with an expert results in the screenshot below taken from the *MATCH Uncertainty Elicitation Tool*. Describe the elicitation procedure which is being used and two of its advantages. Write down the prior distribution for θ (round your parameters to the nearest integer.) [5 marks]



Answer:

- (b) A second expert tells you that he thinks the value of θ is equally likely to be above or below 0.8. Explain why this is not consistent with the prior from part (a), and describe what further steps you would take to elicit a prior distribution from this second expert. [4 marks]

Answer:

- (c) Suppose 21 people out of 30 in the opinion poll support Mia. Calculate the posterior distribution for θ under the prior from part (a). [3 marks]

Answer:

- (d) Derive the asymptotic posterior in the case where a large number of people, n , were questioned and the proportion supporting Mia was ϕ . [8 marks]

Answer:

[20 marks]

B6. (a) Define a sufficient statistic. [3 marks]

Answer:

(b) State the factorisation theorem for proving sufficiency. [3 marks]

Answer:

- (c) Suppose $X_1, X_2, \dots, X_n$ are independent $Exp(\lambda)$ random variables. Show that $T = \sum_{i=1}^n X_i$ is a sufficient statistic. [3 marks]

Answer:

- (d) State and prove whether $T' = \log \sum_{i=1}^n X_i$ is also a sufficient statistic. [3 marks]

Answer:

- (e) Let $t(\mathbf{x}) = \sum_{i=1}^n x_i$. Show that $\pi(\lambda|t(\mathbf{x}))$ is the same as $\pi(\lambda|\mathbf{x})$ when a $Ga(g, h)$ prior is used. You may assume the property that $T|\lambda \sim Ga(n, \lambda)$. [8 marks]

Answer:

[20 marks]

B7. Suppose $X_1, X_2, \dots, X_n$ are independent and identically distributed as $N(1, 1/\tau)$ i.e. normal with mean 1 and unknown precision τ .

- (a) Suppose the prior distribution is $\tau \sim Ga(\alpha, \lambda)$. Show that the posterior distribution is $Ga(\alpha + n/2, \lambda + \frac{1}{2} \sum_{i=1}^n (x_i - 1)^2)$. [7 marks]

Answer:

- (b) Use the result of part (a) to find the posterior under a vague Gamma prior with mean μ . [6 marks]

Answer:

- (c) The Jeffreys prior for this model is $\pi(\tau) \propto \tau^{-1}$.

- (i) Describe a general benefit of using Jeffreys prior. [2 marks]

Answer:

- (ii) Compare the posterior using the Jeffreys prior to the results of part (b). [3 marks]

Answer:

- (iii) Prove whether or not the Jeffreys prior is an improper distribution. [2 marks]

Answer:

[20 marks]

THE END