

4 Bivariate distributions

In this section we consider two related topics: (i) how to simulate samples from continuous bivariate distributions, and (ii) transformations of bivariate random variables. This leads naturally to consideration of the bivariate normal distribution.

4.1 Sampling from bivariate distributions

Suppose we have a pair of continuous random variables (X, Y) with joint PDF $f_{X,Y}(x, y)$. Recall that

$$f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y | X = x)$$

where $f_X(x)$ is the **marginal** PDF of X and $f_{Y|X}(y | X = x)$ is the **conditional** PDF of Y given $X = x$. One way to simulate realizations of the pair (X, Y) is:

1. Simulate a realization x of X from the marginal distribution of X .
2. Simulate a realization y of Y from the conditional distribution of Y given $X = x$.

In the **trivariate case** we can add a third step: simulate Z given $Y = y$ and $X = x$.

Example 4.1: Suppose $X \sim U(0.1, 0.5)$ and

$$f_{Y|X}(Y | X = x) = \begin{cases} xe^{-xy}, & \text{when } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Write an R function to generate a $2 \times n$ **matrix** of samples from this distribution, and produce a scatter plot.

Note that, given that $X = x$, Y has an exponential distribution with parameter $\lambda = X$:

$$Y | X = x \sim \text{Exp}(x).$$

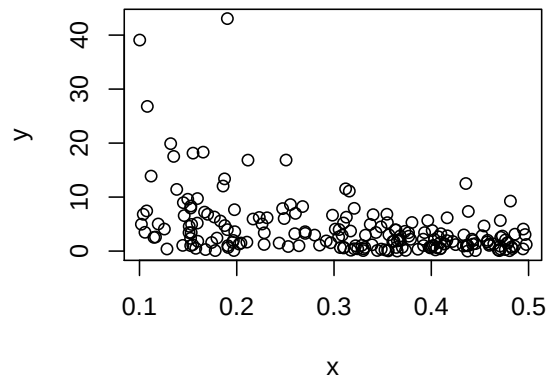
```

1 example4.1 = function(n) {
2   output = matrix(0, 2, n)
3   x = runif(n, 0.1, 0.5)
4   y = rexp(n, rate = x)
5   output[1, ] = x
6   output[2, ] = y
7   output
8 }
```

Make some notes about the R function.

To obtain a scatterplot:

```
1 test = example4.1(200)
2 plot(test[1,], test[2,], xlab = 'x', ylab = 'y')
```



Example 4.2: (Trivariate case.) Suppose that $X \sim U(-1, 1)$ and $Y \sim \text{Exp}(\lambda)$ independently of X and that

$$f_{Z|X,Y}(Z \mid X = x, Y = y) = \frac{1}{\sqrt{2\pi}y} e^{-\frac{1}{2}\left(\frac{z-x}{y}\right)^2}.$$

Write an R function to sample realizations (X, Y, Z) in the case $\lambda = 5$ and produce a scatterplot of (X, Z) . What would the plot look like if $\lambda = 1$?

Note that, given that $X = x, Y = y$ then Z has a normal distribution:

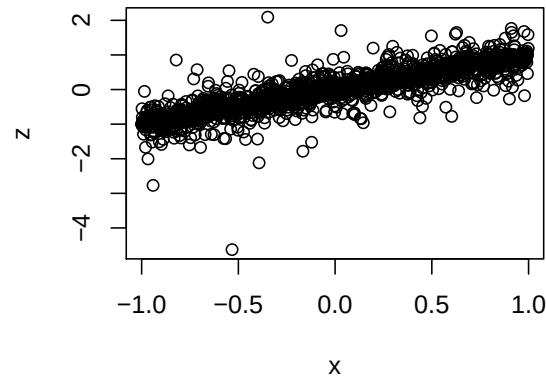
$$Z \mid X = x, Y = y \sim N(x, y^2).$$

```
1 example4.2 = function(n) {
2   output = matrix(0, 3, n)
3   x = runif(n, -1, 1)
4   y = rexp(n, rate = 5)
5   z = rnorm(n, mean = x, sd = y)
6   output[1, ] = x
7   output[2, ] = y
8   output[3, ] = z
9   output
10 }
```

Make some notes about the R function.

To obtain a scatterplot:

```
1 output = example4.2(2000)
2 plot(output[1,], output[3,], xlab = 'x', ylab = 'z')
```



4.2 Transformations of bivariate random variables

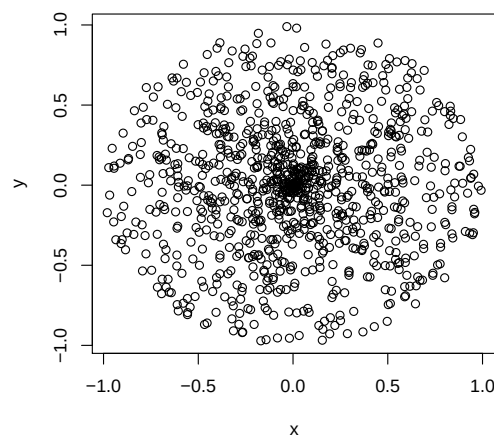
Motivating example. Consider distributions of points that lie on a disc of radius 1. Suppose (R, Θ) are the polar coordinates of a random point. Suppose that $R \sim U(0, 1)$ and $\Theta \sim U(0, 2\pi)$. What does this distribution look like? Is it uniform over the disc?

Here is some R code to sample from this distribution and to create a scatterplot:

```
1 disc = function(n) {
2   r = runif(n, 0, 1)
3   theta = runif(n, 0, 2*pi)
4   x = r*sin(theta)
5   y = r*cos(theta)
6   data.frame(x, y)
7 }
```

To obtain a scatterplot:

```
1 plot(disc(1000))
```



This distribution is **not** uniform on the disc: there seem to be more points in the centre than near the edges. So what should the distribution of (R, Θ) be?

The following result relates the PDF of a pair of random variables (X, Y) with the PDF of a pair (U, V) where U and V are **functions** of X and Y .

Let X and Y have joint PDF $f_{X,Y}(x, y)$, and let $A \subseteq \mathbb{R}^2$ be the set

$$A = \{(x, y) : f_{X,Y}(x, y) > 0\}.$$

Suppose $T : A \rightarrow B$ is an invertible map where $B \subseteq \mathbb{R}^2$ is the image of T . Then define random variables $(U, V) = T(X, Y)$. Then the joint PDF of U, V is

$$f_{U,V}(u, v) = |J| f_{X,Y}(T^{-1}(u, v)) \quad \text{when } (u, v) \in B$$

where J is the determinant

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

If $(u, v) \notin B$ then $f_{U,V}(u, v) = 0$.

The proof is omitted.

Example 4.3: If (X, Y) are uniformly distributed on the unit disc, find the joint PDF of (R, Θ) where R and Θ are the corresponding polar coordinates. Also find the marginal PDFs for R and Θ .

Here $A = \text{disk with unit radius}$ ~~$\{x, y\}$~~ $\{(x, y) : x^2 + y^2 \leq 1\}$.

$$B = \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$$

T is the transformation

$$(x, y) \mapsto ((x^2 + y^2)^{1/2}, \tan^{-1}(y/x)) = (r, \theta)$$

The inverse is

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta) = (x, y)$$

$$J = \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = r(\cos^2 \theta + \sin^2 \theta)$$

$$= r$$

If x, y are uniprm on A then

$$f_{x,y}(x,y) = \begin{cases} 1/\pi & (x,y) \in A \quad (\text{disk has area } \pi) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{so } f_{R,\Theta}(r,\theta) = \frac{1}{\pi} \times r$$

$\nearrow$ $f_{x,y}(T^{-1}(r,\theta))$ $\nwarrow$ $|J|$

$\Rightarrow \Theta \sim U(0, 2\pi)$ and $f_R(r) = 2r$ is marginal PDF of R .

Example 4.4: If X and Y are independent $\text{Exp}(1)$ random variables, and $U = X + Y$, $V = X/(X + Y)$, then find the distribution of U and V .

$$u = x + y, \quad v = \frac{x}{x+y} \quad \text{defines map } T.$$

$$\text{So } A = (0, \infty) \times (0, \infty) \quad (x, y \text{ both } +ve)$$

$$B = (0, \infty) \times (0, 1) \quad (V \in (0, 1)).$$

$$T^{-1}: (u, v) \mapsto (uv, u(1-v)) = (x, y).$$

$$J = \det \begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix} = -uv - u(1-v) = -u$$

$$f_{x,y}(x,y) = e^{-(x+y)} \quad (X, Y \sim \text{Exp}(1))$$

$$\begin{aligned} \text{So } f_{u,v}(u,v) &= |u| \times e^{-(uv + u - uv)} \\ &= u e^{-u} \quad \text{Since } u \geq 0. \end{aligned}$$

Hence U & V are independent $\therefore$ joint PDF factorizes.

$$f_u(u) = u e^{-u} \quad 0 \leq u < \infty$$

$$\& \quad V \sim U(0,1).$$