

MAS2317/3317: Introduction to Bayesian Statistics

Problems sheet 1: Solutions to assessed questions

1. (12 marks in total)

Student answers will vary for this question. Full marks are available for anything that seems sensible – provided justification is given. Some example responses are shown below. Note that the question asks for the probabilities to be marked on a scale from $0 \rightarrow 1$; however, full marks should be awarded if the probabilities are simply stated in each part.

(a) (2 marks each for parts (i)–(iii))

- (i) Frequency interpretation: take a sample of students from the class (your friends, perhaps?) and estimate the probability as the proportion of students in your sample whose BMI is at least 22.5. Could also use a subjective assessment, just by “looking”, or by using information given in the lecture notes about the NU BMI study.
- (ii) Subjective assessment: For example, even if you don’t know much about football, you can easily check to see where Newcastle are in the league right now – currently 8th top. Without any further research, you could then specify a high probability, given that they are only two places from a top six position. After more research, and perhaps using your own knowledge about the Premier League, you might think a top six position at the end of the season is unlikely, given their up-and-coming games – thus specifying a much lower probability.
- (iii) Classical interpretation: because we “randomly select”, each number is equally likely. The probability is then $13/90$.

(b) (2 marks each for parts (i)–(iii))

- (i) We could repeat the experiment 100 times to get 100 estimates of the probability. Reading off the 2.5 and 97.5 percentiles could then give an estimate of the probability interval. No marks for confidence intervals, or even the word “confidence”! Alternatively, a subjective assessment could be valid.
- (ii) The estimate itself is subjective, so the range will be too. The range should reflect your degree of certainty – the more you know about football, and Newcastle United, the narrower your range!
- (iii) Since we have a fixed probability, the range will be $(13/90, 13/90)$.

5. (20 marks in total)

We have the following prior probabilities:

$$\Pr(\text{North}) = 0.8 \quad \text{and} \quad \Pr(\text{South}) = 0.2.$$

The table in the question gives $\Pr(x_j|\text{North})$ and $\Pr(x_j|\text{South})$, $j = 1, \dots, 4$.

- (a) We want $\Pr(\text{North}|x_j)$ and $\Pr(\text{South}|x_j)$, $j = 1, \dots, 4$. Since “North” and “South” form a partition, we can use Bayes’ Theorem. Thus,

$$\begin{aligned} \Pr(\text{North}|x_1) &= \frac{\Pr(x_1|\text{North})\Pr(\text{North})}{\Pr(x_1|\text{North})\Pr(\text{North}) + \Pr(x_1|\text{South})\Pr(\text{South})} \\ &= \frac{0.3 \times 0.8}{0.3 \times 0.8 + 0.05 \times 0.2} = \frac{24}{25} = 0.96. \end{aligned}$$

We can find $\Pr(\text{North}|x_j)$, $j = 2, \dots, 4$ in a similar fashion; we can then obtain

$$\Pr(\text{South}|x_j) = 1 - \Pr(\text{North}|x_j),$$

giving the following results:

(2 marks for each North| x_j ; 1 mark for each South| x_j)

	Identifier			
	x_1	x_2	x_3	x_4
North	24/25	4/11	2/3	40/41
South	1/25	7/11	1/3	1/41

Thus, our rules for differential diagnosis would be: **(2 marks)**

Fresh debris found $\longrightarrow$ Search North
Signal detected $\longrightarrow$ Search South
Fresh debris found *and* signal detected $\longrightarrow$ Search North
No identifiers $\longrightarrow$ Search North

(b)

$$\begin{aligned} \Pr(\text{Misdiagnosis}) &= \Pr(x_1, \text{South}) + \Pr(x_2, \text{North}) + \Pr(x_3, \text{South}) + \Pr(x_4, \text{South}) \\ &= (0.05 \times 0.2) + (0.1 \times 0.8) + (0.2 \times 0.2) + (0.05 \times 0.2) \\ &= 0.14. \end{aligned} \quad \textbf{(4 marks)}$$

(c) We have **(2 marks)**

Prior odds : $\Pr(\text{South}) : \Pr(\text{North}) = 0.2 : 0.8 = 1 : 4$ **against** the South
Posterior odds : $\Pr(\text{South}|x_2) : \Pr(\text{North}|x_2) = 7/11 : 4/11 = 7 : 4$ **in favour** of the South

9. **(8 marks in total)**

The likelihood function is given by

$$\begin{aligned} f(\mathbf{x}|\sigma) &= \prod_{i=1}^n \frac{x_i}{\sigma^2} \exp \left\{ -\frac{x_i^2}{2\sigma^2} \right\} \\ &= \prod_{i=1}^n x_i \times \sigma^{-2n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n x_i^2 \right\}, \quad \sigma > 0, \\ &= h(\mathbf{x}) \times g \left(\sum_{i=1}^n x_i^2, \sigma \right), \end{aligned}$$

where $h(\mathbf{x}) = \prod_{i=1}^n x_i$ and $g(t, \sigma) = \sigma^{-2n} \exp \left\{ -\frac{t}{2\sigma^2} \right\}$.

Therefore, by the Factorisation Theorem, $T = \sum_{i=1}^n X_i^2$ is sufficient for σ .