

MAS2317/3317: Problems Class 4

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Thursday 27th March, 4–5pm

Problems Class 4

The work from problems sheet 3 is being marked as we speak, and will be returned soon (check emails).

The work in this problems class will correspond to the (unstarred) questions on problems sheet 4.

This should supplement the assessed (starred) questions on problems sheet 4 that are due in by no later than **4pm, Tuesday 6th May**.

If you need extra help with any of the material in problems sheet 4, there is a drop-in session after Easter.

There is also an **optional** practical after Easter.

Questions 7 and 8 correspond to material in the very last part of the notes, and so we will come back to these in the last problems class of the course.

Question 1

Suppose that the random variable Y follows a $Ga(a, b)$ distribution.

Determine the probability density function for the random variable $X = 1/\sqrt{Y}$.

Solution to Question 1

Suppose $Y \sim \text{Ga}(a, b)$ and $X = 1/\sqrt{Y}$. The density of X is

$$\begin{aligned} f_X(x) &= f_Y(x^{-2}) \left| \frac{d}{dx} x^{-2} \right| \\ &= \frac{b^a (x^{-2})^{a-1} e^{-bx^{-2}}}{\Gamma(a)} \times 2x^{-3}, \quad x > 0 \\ &= \frac{2b^a x^{-2a-1} e^{-b/x^2}}{\Gamma(a)}, \quad x > 0. \end{aligned}$$

Question 2

Using a random sample from a $\text{Bin}(k, \theta)$ (with k known), determine the posterior distribution for θ assuming

- (i) vague prior knowledge;
- (ii) the Jeffreys prior distribution;
- (iii) a very large sample.

Solution to question 2

The **conjugate prior distribution** is a $Beta(g, h)$ distribution. Using this prior distribution, the posterior density is

$$\pi(\theta|\mathbf{x}) \propto \pi(\theta) f(\mathbf{x}|\theta)$$

$$\propto \theta^{g-1}(1-\theta)^{h-1} \times \prod_{i=1}^n \theta^{x_i}(1-\theta)^{k-x_i}, \quad 0 < \theta < 1$$

$$\propto \theta^{g+n\bar{x}-1}(1-\theta)^{h+nk-n\bar{x}-1}, \quad 0 < \theta < 1$$

$$\text{i.e. } \theta|\mathbf{x} \sim Beta(G = g + n\bar{x}, H = h + nk - n\bar{x}).$$

Solution to question 2(i)

We represent **vague prior information** by taking a conjugate prior distribution with large variance.

As the beta distribution restricts values to the range $(0,1)$, there is a finite upper limit to the variance.

Intuitively, the maximum variance is achieved when the probability density is pushed to the extremes of the range, that is, equal mass at $\theta = 0$ and $\theta = 1$ – this distribution is obtained in the limit $g \rightarrow 0$ and $h \rightarrow 0$.

Thus we will take this limit to represent vague prior information. Hence the posterior distribution under vague prior information is

$$\theta | \mathbf{x} \sim \text{Beta}(n\bar{x}, nk - n\bar{x}).$$

Solution to question 2(ii)

The **Jeffreys prior distribution** is

$$\pi(\theta) \propto \sqrt{I(\theta)}.$$

Now

$$\begin{aligned} f(\mathbf{x}|\theta) &\propto \prod_{i=1}^n \theta^{x_i} (1 - \theta)^{k - x_i} \\ &\propto \theta^{n\bar{x}} (1 - \theta)^{kn - n\bar{x}}. \end{aligned}$$

Solution to question 2(ii)

Therefore

$$\log f(\mathbf{x}|\theta) = \text{constant} + n\bar{x} \log \theta + n(k - \bar{x}) \log(1 - \theta)$$

$$\Rightarrow \frac{\partial}{\partial \theta} \log f(\mathbf{x}|\theta) = \frac{n\bar{x}}{\theta} - \frac{n(k - \bar{x})}{1 - \theta}$$

$$\Rightarrow \frac{\partial^2}{\partial \theta^2} \log f(\mathbf{x}|\theta) = -\frac{n\bar{x}}{\theta^2} - \frac{n(k - \bar{x})}{(1 - \theta)^2}$$

$$\begin{aligned} \Rightarrow I(\theta) &= E_{\mathbf{X}|\theta} \left[-\frac{\partial^2}{\partial \theta^2} \log f(\mathbf{X}|\theta) \right] \\ &= \frac{nE_{\mathbf{X}|\theta}(\bar{X})}{\theta^2} + \frac{n[k - E_{\mathbf{X}|\theta}(\bar{X})]}{(1 - \theta)^2}. \end{aligned}$$

Solution to question 2(ii)

Now $E_{X|\theta}(\bar{X}) = E_{X|\theta}(X) = k\theta$. Therefore

$$I(\theta) = \frac{nk}{\theta} + \frac{nk}{1-\theta} = \frac{nk}{\theta(1-\theta)}.$$

Hence the Jeffreys prior for this model is

$$\begin{aligned}\pi(\theta) &\propto \sqrt{I(\theta)} \\ &\propto \sqrt{\frac{nk}{\theta(1-\theta)}}, \quad 0 < \theta < 1 \\ &\propto \theta^{-1/2}(1-\theta)^{-1/2}, \quad 0 < \theta < 1.\end{aligned}$$

Solution to question 2(ii)

This is a $Beta(1/2, 1/2)$ prior distribution and so the resulting posterior distribution is

$$\theta|\mathbf{x} \sim Beta(1/2 + n\bar{x}, 1/2 + nk - n\bar{x}).$$

The **asymptotic posterior distribution** (as $n \rightarrow \infty$) is

$$\theta|\mathbf{x} \sim N\left(\hat{\theta}, J(\hat{\theta})^{-1}\right),$$

where

$$J(\theta) = -\frac{\partial^2}{\partial \theta^2} \log f(\mathbf{x}|\theta) = \frac{n\bar{x}}{\theta^2} + \frac{n(k - \bar{x})}{(1 - \theta)^2}.$$

Solution to question 2(iii)

Now

$$\begin{aligned}\frac{\partial}{\partial \theta} \log f(\mathbf{x}|\theta) = 0 &\implies \frac{n\bar{x}}{\hat{\theta}} - \frac{n(k - \bar{x})}{1 - \hat{\theta}} = 0 \\ &\implies \hat{\theta} = \frac{\bar{x}}{k} \\ &\implies J(\hat{\theta}) = \frac{nk^3}{\bar{x}(k - \bar{x})} \\ &\implies J(\hat{\theta})^{-1} = \frac{\bar{x}(k - \bar{x})}{nk^3}.\end{aligned}$$

Therefore, for large n , the posterior distribution for θ is

$$\theta|\mathbf{x} \sim N\left(\frac{\bar{x}}{k}, \frac{\bar{x}(k - \bar{x})}{nk^3}\right) \quad \text{approximately.}$$

Question 4

Use techniques from MAS2302 to obtain 95% frequentist confidence intervals for $\hat{\theta}_C$ and $\hat{\theta}_I$.

Solution to Question 4

From MAS2302, you should (but probably can't!) recall that, for the binomial proportion p ,

$$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right)$$

where n is the number of trials. The variance is estimated using $\hat{p}$, and recall that, for MLEs generally,

$$\text{Var}(\theta) = I(\theta)^{-1},$$

where

$$I(\theta) = E \left[-\frac{\partial^2}{\partial \theta^2} \ell(\theta) \right].$$

Thus, a 95% frequentists CI for p is given by

$$\hat{p} \pm 1.96 \times \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}.$$