

MAS2317 (3317): Specimen paper

NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS & STATISTICS

SEMESTER 2 2012/2013

MAS2317 (3317): Specimen paper

Introduction to Bayesian Statistics

Time allowed: 1 hour 30 minutes

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks indicate the relative weight of individual questions, they do not correspond directly to marks on the University scale.

There are FOUR questions in Section A and THREE (FOUR) questions in Section B.

Answers to questions should be entered directly on this question paper in the spaces provided. Rough work should be done on the blank sides of the pages. The rough work will not be marked. This question paper must be handed in, attached inside an anonymised cover sheet, at the end of the examination.

Beta distribution: If $X \sim Be(\alpha, \beta)$ then it has density

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \quad \alpha > 0, \beta > 0.$$

Also, $E(X) = \alpha/(\alpha + \beta)$ and $Var(X) = \alpha\beta/\{(\alpha + \beta)^2(\alpha + \beta + 1)\}$.

Exponential distribution: If $X \sim Exp(\lambda)$ then it has density

$$f(x|\lambda) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0.$$

Also, $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$.

Gamma distribution: If $X \sim Ga(\alpha, \lambda)$ then it has density

$$f(x|\alpha, \lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0, \quad \alpha > 0, \lambda > 0.$$

Also, $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$.

Normal distribution: If $X \sim N(\mu, 1/\tau)$ then it has density

$$f(x|\mu, \tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x - \mu)^2\right\}, \quad -\infty < x < \infty, \quad -\infty < \mu < \infty, \tau > 0.$$

Also, $E(X) = \mu$ and $Var(X) = 1/\tau$.

Note: If $X \sim N(0, 1)$ then the following quantiles apply:

x	1.2816	1.6449	1.9600	2.3263	2.5758
$\Pr(X < x)$	0.9	0.95	0.975	0.99	0.995

Poisson distribution: If $X \sim Po(\theta)$ then it has probability function

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \dots, \quad \lambda > 0.$$

Also, $E(X) = \theta$ and $Var(X) = \theta$.

SECTION A

A1. Briefly describe the frequentist, classical and subjective interpretations of probability. Give one drawback with each interpretation.

FREQUENTIST

- *Description*

- *Drawback*

CLASSICAL

- *Description*

- *Drawback*

SUBJECTIVE

- *Description*

- *Drawback*

[Total A1: 9 marks]

- A2.** (a) Suppose $E_1, E_2, \dots, E_n$ form a partition of the sample space. State and prove Bayes' Theorem. You may assume (and need not prove) the Law of Total Probability.
[5 (4) marks]

Answer:

- (b) A lecturer has noted from past experience that students who do the homework assignments have a 0.90 probability of passing her course. On the other hand, students who do not do the homework assignments for the course have a 0.25 probability of passing. The lecturer estimates that 75% of the students taking the course do their homework. Given a student passes the course, what is the probability that they did their homework?

[4 marks]

Answer:

[Total A2: 9 (8) marks]

A3. Suppose $X_1, X_2, \dots, X_n$ are a random sample from a Poisson $Po(\lambda)$ distribution.

(a) Write down the likelihood function for λ .

[1 mark]

Answer:

$$f(\mathbf{x}|\lambda) =$$

(b) Show that the log-likelihood function for λ is

$$\log f(\mathbf{x}|\lambda) = n\bar{x} \log \lambda - n\lambda + \text{constant}$$

where $\bar{x}$ is the sample mean.

[2 (1) marks]

Answer:

(c) The prior distribution proposed by Sir Harold Jeffreys can be written as $\pi(\lambda) \propto I(\lambda)^{1/2}$, where

$$I(\lambda) = E_{\mathbf{X}|\lambda} \left[-\frac{\partial^2}{\partial \lambda^2} \log f(\mathbf{X}|\lambda) \right].$$

Show that, for this model, Jeffreys' prior is

$$\pi(\lambda) \propto \frac{1}{\sqrt{\lambda}}, \quad \lambda > 0.$$

[6 (5) marks]

Answer:

[Part of this page has been left blank for your solution to the last question]

- (d) Suppose we model the annual number of road traffic accidents at a particular location in Newcastle according to a Poisson $Po(\lambda)$ distribution. In 2010 and 2011 we observe 4 and 6 road traffic accidents at this location. Show that the posterior distribution for λ , assuming Jeffreys' prior, is a gamma distribution; specifically,

$$\lambda|\mathbf{x} \sim Ga(10.5, 2).$$

[6 (4) marks]

Answer:

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- (e) Show that, for large n , the posterior distribution for λ is approximately a Normal distribution with mean and variance $\bar{x}$ and $\bar{x}/n$, respectively, where $\bar{x}$ is the sample mean.

[(5) marks]

Answer:

[Total A3: 15 (16) marks]

A4. (a) State the Factorisation Theorem.

[3 marks]

Answer:

- (b) Suppose the population follows a Geometric $Geo(\theta)$ distribution with probability density function

$$f(x|\theta) = \theta(1 - \theta)^{x-1}, \quad x = 1, 2, \dots$$

Show that $\sum_{i=1}^n X_i$ is sufficient for θ .

[4 marks]

Answer:

[Total A4: 7 marks]

SECTION B

B5. Marine biologists are concerned about the health of coral reefs in the Caribbean; anything more than a 5% reduction in live coral cover over the period of a year is seen as catastrophic. A team of researchers is interested in

$$\theta = \Pr(\text{more than a 5\% reduction in live coral cover})$$

for reefs in the Caribbean.

- (a) You are a statistician on the team and you attempt to elicit a prior distribution for θ , $\pi(\theta)$, via a consultation with one of the marine biologists. This results in the screenshot shown overleaf, taken from the *MATCH Uncertainty Elicitation Tool*. Use this to answer parts (i)–(iii) of this question.

- (i) Briefly explain the method of prior elicitation that the statistician has used. [3 marks]

Answer:

- (ii) Write down the elicited prior distribution for θ , rounding the parameters of this distribution to the nearest integer, and write down the corresponding density function $\pi(\theta)$. Also calculate the prior mean and prior standard deviation.

[5 (4) marks]

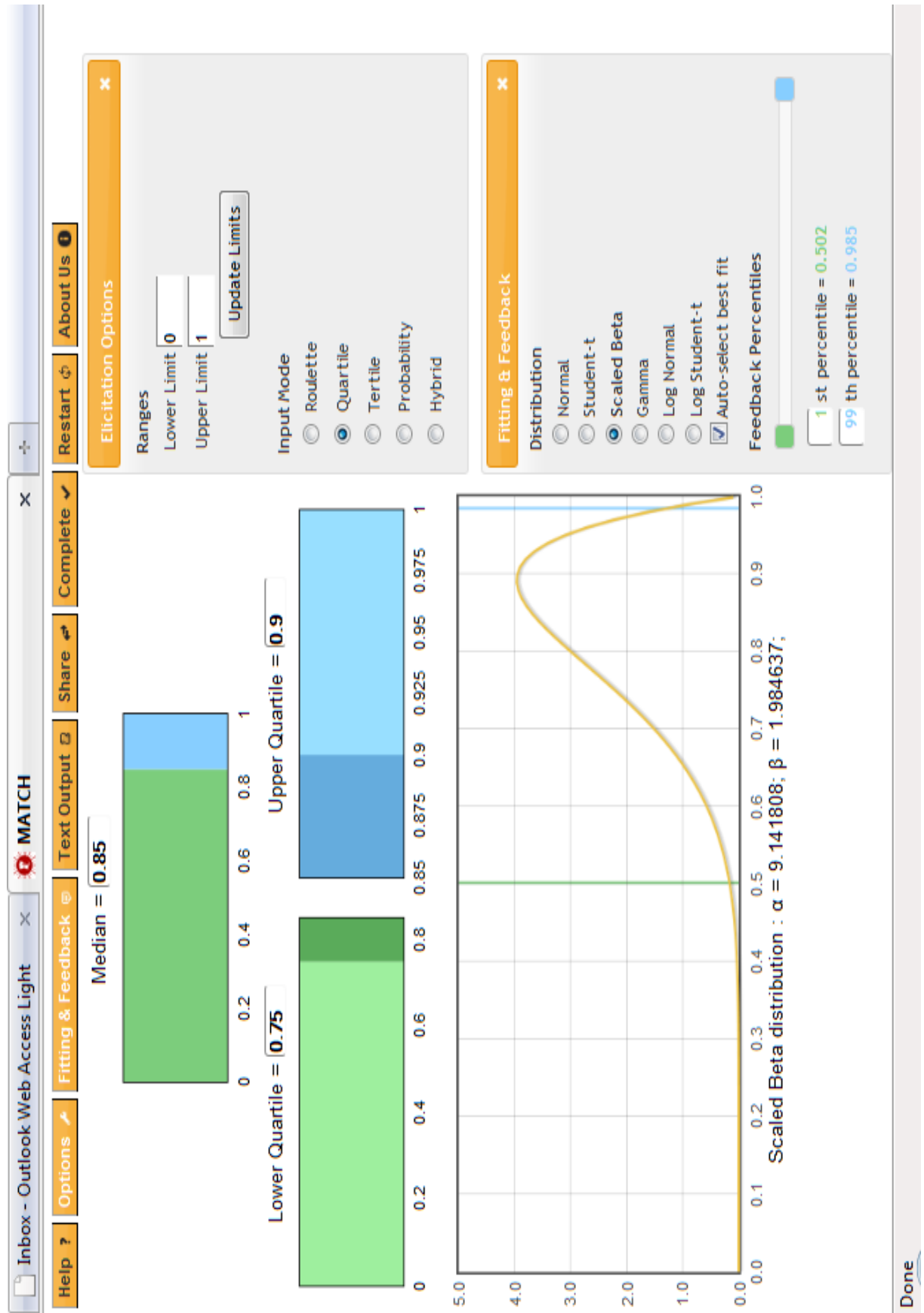
Answer:

$$\theta \sim$$

$$\pi(\theta) =$$

$$E(\theta) =$$

$$SD(\theta) =$$



- (iii) As part of the feedback stage of the elicitation process, phrase a question to the marine biologist in terms of the lower feedback percentile given in the *MATCH* output.

[3 (2) marks]

Answer:

- (b) Of the eight Caribbean reefs inspected by divers since this time last year, six have seen more than a 5% decline in live coral cover. Combine this with the expert's prior for θ to obtain the posterior distribution for θ , and write a short report for the marine biologists.

[9 (7) marks]

Answer:

[Part of this page has been left blank for your solution to the last question]

[Total B4: 20 (16) marks]

- B6.** (a) Describe the terms *substantial prior knowledge* and *vague prior knowledge*. Illustrate your answer with suitable diagrams.

[6 (5) marks]

Answer:

- (b) What is meant by the term *conjugate prior distribution*? Explain how conjugate prior distributions are used to represent vague prior knowledge.

[4 (3) marks]

Answer:

(c) Suppose that the lifetimes X of patients receiving a drug treatment follow a Gamma $Ga(4, \theta)$ distribution.

(i) Show that the likelihood function for θ , given a random sample of lifetimes $\mathbf{x}$, is

$$f(\mathbf{x}|\theta) = k\theta^{4n}e^{-n\bar{x}\theta}$$

where k is a positive constant (with respect to θ).

[3 marks]

Answer:

(ii) Suppose your prior beliefs about θ were described by a $Ga(g, h)$ distribution. Determine your posterior distribution for θ given the data $\mathbf{x}$.

[5 (4) marks]

Answer:

- (iii) Is the Gamma distribution a conjugate prior distribution here? Explain your answer.

[2 marks]

Answer:

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- (iv) Show that vague prior knowledge can be represented as $g \rightarrow 0$ and $h \rightarrow 0$.

[(3) marks]

Answer:

[Total B6: 20 marks]

B7. A random sample of size n is taken from a normal $N(\mu, 1/\tau)$ distribution (where τ is known), giving sample mean $\bar{x}$. Prior beliefs about μ follow a normal $N(b, 1/d)$ distribution. It can be shown that the likelihood function for μ is

$$f(\mathbf{x}|\mu) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left\{-\frac{n\tau}{2} [s^2 + (\bar{x} - \mu)^2]\right\},$$

where $s^2 = \sum_{i=1}^n (x_i - \bar{x})^2/n$.

(a) Obtain the posterior distribution $\pi(\mu|\mathbf{x})$.

[8 (6) marks]

Answer:

(b) Determine

- (i) C_B , a 95% Bayesian confidence interval for μ ;
- (ii) C_H , the 95% highest density interval (H.D.I.) for μ , and
- (iii) C_F , a 95% frequentist confidence interval for μ .

[9 marks]

Answer:

- (c) Explain any differences in interpretation between these confidence intervals.
[3 (2) marks]

Answer:

[Total B7: 20 (17) marks]

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B8. Suppose that the outcomes of an experiment form a random sample, each with probability function $f(x|\theta)$. Our current beliefs about θ are described by the posterior density function $\pi(\theta|\mathbf{x})$, obtained by combining experimental results $\mathbf{x}$ with a prior distribution for θ .

Suppose we are interested in the outcome Y of the next experiment.

By considering the distribution of (Y, θ) given the data $\mathbf{x}$, show that the predictive probability function for Y given the data $\mathbf{x}$ can be expressed as

$$f(y|\mathbf{x}) = \frac{f(y|\theta)\pi(\theta|\mathbf{x})}{\pi(\theta|\mathbf{x}, y)},$$

where $\pi(\theta|\mathbf{x}, y)$ is the posterior density of θ given both $\mathbf{x}$ and y .

Answer:

[Total B8: (7) marks]

THE END