

Univariate Extremes: The Impact of some Unfinished Business

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Structure of this talk

- Univariate return level estimation
 - ▶ The usual story...
 - ▶ Some unfinished business
- Extremes this decade
 - ▶ Spatial
 - ▶ Example: Wind speed analysis
- Attending to some unfinished business
 - ▶ Extremal dependence
 - ▶ Current aims and impact

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The usual story: Return level estimation



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Let $\{X_n\}$ denote a stationary sequence of random variables with common distribution function F , and let $M_n = \max\{X_1, \dots, X_n\}$.

It is typically the case that, as $n \rightarrow \infty$,

$$\Pr(M_n \leq x) \approx F^{n^\theta}(x), \quad (1)$$

where $\theta \in (0, 1)$ is known as the **extremal index**; see, for e.g., [Leadbetter & Rootzén \(1988\)](#).

As $\theta \rightarrow 0$ there is increasing dependence in the extremes of the process; for an independent process, $\theta = 1$.

The usual story: Return level estimation

The **Generalised Extreme Value** (GEV) distribution is the limiting model for F^n , with distribution function (d.f.)

$$\mathcal{G}(y; \mu, \varsigma, \xi) = \begin{cases} \exp \left[- (1 + \xi(y - \mu)/\varsigma)^{-1/\xi} \right], & \xi \neq 0 \\ \exp \left[-\exp \left(-(y - \mu)/\varsigma \right) \right] & \xi = 0, \end{cases}$$

defined on $\{y : 1 + \xi(y - \mu)/\varsigma > 0\}$, where $-\infty < \mu < \infty$, $\varsigma > 0$ and $-\infty < \xi < \infty$ are parameters of location, scale and shape.

- It can be shown that $F^{n\theta}$ is also GEV with d.f. $\mathcal{G}(y; \mu^*, \varsigma^*, \xi)$, provided long-range dependence is negligible¹
- In practice, the GEV is used to model maxima over some convenient calendar unit – usually years

¹Technically, this requires **Leadbetter's $D(u_n)$ condition** to hold; see **Leadbetter et al. (1983)**.

The usual story: Return level estimation

Pickands (1975) showed that for $\theta = 1$ and u large, $(X - u | X > u)$ follows a **Generalised Pareto** (GP) distribution with d.f.

$$\mathcal{H}(y) = \begin{cases} 1 - (1 + \xi y / \sigma)^{-1/\xi}, & \xi \neq 0 \\ 1 - \exp[-y/\sigma] & \xi = 0, \end{cases}$$

defined on $\{y : y > 0 \text{ and } (1 + \xi y / \sigma) > 0\}$, where $\sigma = \varsigma + \xi(u - \mu) > 0$ and ξ are the scale and shape.

- Extremal index θ can't be absorbed into the parameters of the GPD; careful consideration of **extremal dependence** required
(unfinished business #1)
- Graphical diagnostics available for **selection of threshold** u
(unfinished business #2)

The usual story: Return level estimation

The **r -year return level z_r** can then be obtained by inversion of $\mathcal{G}(z_r)$ or $\mathcal{H}^\theta(z_r)$.

For threshold excesses, on equating to $1 - r^{-1}$ this gives

$$z_r = \begin{cases} u + \sigma \xi^{-1} \left[(\lambda_u^{-1} w_r)^{-\xi} - 1 \right] & \xi \neq 0 \\ u - \sigma \log (\lambda_u^{-1} w_r) & \xi = 0, \end{cases}$$

where $w_r = 1 - [1 - (rn_y)^{-1}]^{1/\theta}$, λ_u is the rate of threshold excess and n_y is the (average) number of observations per year.

- **Seasonal variation** – single-season approach?

Smoothly-varying parameters? Seasonal piecewise... solve

$$\prod_{m=1}^M \mathcal{H}_m^{\theta_m}(z_r) = 1 - r^{-1}, \quad (2)$$

where m is the indicator of season (*unfinished business #3*)

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- Estimation of return levels is usually the **key aim** of any extreme value analysis
- **Examples:** *British Standards Institute* – 50 year wind speed; *Dutch Delta Commission* – 10,000 year sea-surge; *Office for Nuclear Regulation* – also 10,000 year safety specifications...
- Large / very large uncertainty on estimates:
 - ▶ Nidd river flows: $(270, 1250)\text{m}^3\text{s}^{-1}$ for z_{100}
(NERC Flood Studies Report, 1995)
 - ▶ Grand Isle wave-heights: $(10, 22.5)\text{m}$ for z_{105}
(Northrop & Jonathan, 2011)
 - ▶ Newlyn sea-surge: $(0.7, 6.5 \rightarrow \sim 30)\text{m}$ for z_{1000}
(Fawcett & Walshaw, 2012)
 - ▶ Carrington-like solar flare event: $(2, 600)\text{years}$
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Extremes this decade: Spatial!

- EVA conferences

Year	Location	% 'Applications' talks	% Spatial talks
2009 ²	Fort Collins	56	37
2011	Lyon	53	48
2013	Shanghai	48	65
2015	Ann Arbor	45	67
2017	Delft	38	74
2019	Zagreb	?	?

- Spatial extremes is...
 - ▶ Hard!
 - ▶ Problematic: Fawcett & Walshaw (2019): Assessing the assessment of spatial dependence
 - ▶ Inaccessible to practitioners: Bevilacqua & Padoan (2017)
 - ▶ Not threshold-based: for componentwise-maxima only

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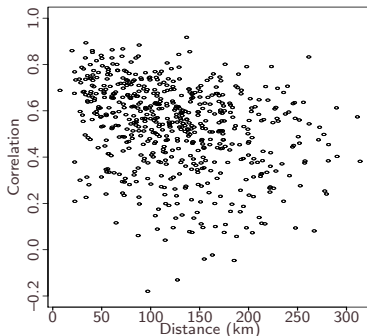
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Wind speed analysis

- 42 years of annual maximum wind speeds (km/h) at 35 (undisclosed!) locations near a proposed nuclear site in France
- New structures must be able to withstand wind speeds with an annual exceedance probability of 10^{-4}
- Decaying correlation between extremes with distance:



Wind speed analysis

- Annual maxima: GEV – location, scale and shape μ , ς and ξ
- Standard analysis:
 - ▶ Fit separate GEV to each site, obtaining $\Psi_d = (\hat{\mu}_d, \hat{\varsigma}_d, \hat{\xi}_d)$, $d = 1, \dots, 35$
 - ▶ Could obtain marginal estimates of return levels $\hat{z}_{r_d}$, or somehow combine the marginal Ψ_d to obtain a 'global' return level estimate $\hat{z}_r$
 - ▶ **Spatial dependence ignored**
- Ideas:
 - ▶ Bayesian hierarchical model for site effects – e.g. Sang & Gelfand (2010); **spatial structure ignored**
 - ▶ Latent variable models
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Wind speed analysis: Latent variable model

Suppose $\{Y(x)\}$ is a random process defined at all points x of a spatial domain $\mathcal{X}$, where $\mathcal{X}$ is a contiguous subset of $\mathbb{R}^2$.

We have annual maximum wind speeds at a finite subset $\mathcal{D} = \{x_1, \dots, x_{35}\}$ of sites of $\mathcal{X}$... how do we make inferences for the process elsewhere in $\mathcal{X}$?

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For $\{\mu(x), \varsigma(x), \xi(x)\}$ assume the following **trend surfaces**:

$$\begin{aligned}\mu(x) &= \beta_{0,\mu} + \beta_{1,\mu}\text{lon}(x) + \beta_{2,\mu}\text{lat}(x) + S_\mu(x; \alpha_\mu, \lambda_\mu) \\ \varsigma(x) &= \beta_{0,\varsigma} + \beta_{1,\varsigma}\text{lon}(x) + \beta_{2,\varsigma}\text{lat}(x) + S_\varsigma(x; \alpha_\varsigma, \lambda_\varsigma) \\ \xi(x) &= \beta_{0,\xi} + S_\xi(x; \alpha_\xi, \lambda_\xi),\end{aligned}$$

where S_* is a zero mean **Gaussian random field** with covariance function:

$$\alpha_* \exp\{-|h|/\lambda_*\}.$$

Then, conditional on the three Gaussian processes at sites in $\mathcal{D}$, we assume that

$$Y_i(x_d) | \{\mu(x_d), \varsigma(x_d), \xi(x_d)\} \sim \text{GEV}\{\mu(x_d), \varsigma(x_d), \xi(x_d)\},$$

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$$\begin{aligned}\mu(x) &= \beta_{0,\mu} + \beta_{1,\mu}\text{lon}(x) + \beta_{2,\mu}\text{lat}(x) + S_\mu(x; \alpha_\mu, \lambda_\mu) \\ \varsigma(x) &= \beta_{0,\varsigma} + \beta_{1,\varsigma}\text{lon}(x) + \beta_{2,\varsigma}\text{lat}(x) + S_\varsigma(x; \alpha_\varsigma, \lambda_\varsigma) \\ \xi(x) &= \beta_{0,\xi} + S_\xi(x; \alpha_\xi, \lambda_\xi),\end{aligned}$$

where S_* is a zero mean **Gaussian random field** with covariance function:

$$\alpha_* \exp\{-|h|/\lambda_*\}.$$

Then, conditional on the three Gaussian processes at sites in $\mathcal{D}$, we assume that

$$Y_i(x_d) | \{\mu(x_d), \varsigma(x_d), \xi(x_d)\} \sim \text{GEV}\{\mu(x_d), \varsigma(x_d), \xi(x_d)\},$$

$$d = 1, \dots, 35.$$

Wind speed analysis: Prior specification

- A joint prior density π must be specified for the parameters $(\alpha_\mu, \alpha_\varsigma, \alpha_\xi, \lambda_\mu, \lambda_\varsigma, \lambda_\xi, \boldsymbol{\beta}_\mu, \boldsymbol{\beta}_\varsigma, \beta_{0,\xi})$.
- To reduce computational burden, we use **conjugate priors** wherever possible. For example, for $\mu(x)$:
 - ▶ Assume α_μ , $\boldsymbol{\beta}_\mu$ and λ_μ are independent
 - ▶ $\alpha_\mu \sim \text{InvGa}(a_\mu, b_\mu)$; $\boldsymbol{\beta}_\mu \sim N_3(\mathbf{m}_\mu, V_\mu)$
 - ▶ No conjugate prior exists for λ_μ , for which we assume a $\text{Ga}(c_\mu, d_\mu)$ distribution
- Similar specifications are made for $\varsigma(x)$ and $\xi(x)$
- We assign the following values to the hyper-parameters:

	α_*		λ_*		$\boldsymbol{\beta}_*$	
	a_*	b_*	c_*	d_*	$\mathbf{m}_*$	V_*
$\mu(x)$	1	12	5	3	0	$v_{i,j} = 100, i = j; 0 \text{ otherwise}$
$\varsigma(x)$	1	1	5	3	0	$v_{i,j} = 100, i = j; 0 \text{ otherwise}$
$\xi(x)$	1	0.04	5	3	0	100

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Wind speed analysis: MCMC sampling

Given a current value of the Markov chain:

$$\Psi^t = (\boldsymbol{\mu}^t, \boldsymbol{\varsigma}^t, \boldsymbol{\xi}^t, \alpha_\mu^t, \lambda_\mu^t, \alpha_\varsigma^t, \lambda_\varsigma^t, \alpha_\xi^t, \lambda_\xi^t, \boldsymbol{\beta}_\mu^t, \boldsymbol{\beta}_\varsigma^t, \beta_{0,\xi}^t),$$

the next state Ψ^{t+1} is obtained as follows:

1: Updating the GEV parameters at each site

Each component of $\boldsymbol{\mu}^t = \{\mu^t(x_1), \dots, \mu^t(x_{35})\}$ is updated singly:

- Generate a proposal $\mu^*(x_d)$ from a symmetric random walk and compute

$$A\{\mu^t(x_d), \mu^*(x_d)\} = \min \left\{ 1, \frac{\pi\{y_d | \mu^*(x_d), \varsigma^t(x_d), \xi^t(x_d)\} \times \pi(\boldsymbol{\mu}^* | \alpha_\mu, \lambda_\mu, \boldsymbol{\beta}_\mu)}{\pi\{y_d | \mu^t(x_d), \varsigma^t(x_d), \xi^t(x_d)\} \times \pi(\boldsymbol{\mu}^t | \alpha_\mu, \lambda_\mu, \boldsymbol{\beta}_\mu)} \right\}$$

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2: Updating the regression parameters

Due to the use of conjugate priors, β_μ is drawn directly from a MVN having covariance matrix and mean vector

$$\Sigma_\mu = \left\{ V_\mu^{-1} + \mathbf{X}_\mu^\top (V_\mu^t)^{-1} \mathbf{X}_\mu \right\}^{-1} \quad \text{and}$$
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The regression parameters for the GEV scale and shape are updated similarly.

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Due to the use of conjugate priors, α_μ is drawn directly from an inverse Gamma with shape and rate

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4: Updating the range parameters of the covariance function

Here the situation is non-conjugate and requires a Metropolis step.

- Generate a proposal $\lambda_\mu^* \sim U(\lambda_\mu^t - \epsilon_\lambda, \lambda_\mu^t + \epsilon_\lambda)$ and compute the acceptance probability:

$$A(\lambda_\mu^t, \lambda_\mu^*) = \min \left\{ 1, \frac{\pi(\boldsymbol{\mu}^t | \alpha_\mu^t, \lambda_\mu^*, \boldsymbol{\beta}_\mu^*)}{\pi(\boldsymbol{\mu}^t | \alpha_\mu^t, \lambda_\mu^t, \boldsymbol{\beta}_\mu^t)} \times \left(\frac{\lambda_\mu^*}{\lambda_\mu^t} \right)^{c_\mu - 1} \exp \left(\frac{\lambda_\mu^t - \lambda_\mu^*}{d_\mu} \right) \right\}$$

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Wind speed analysis: Results

- MCMC scheme run for 1M iterations, after a burn-in of 10,000
- After around 17 hours (!), the following results were obtained
- Convergence was checked by using multiple start points and eyeballing the chains, nothing more...

	β_0 (kmh ⁻¹)	β_1 (kmh ⁻¹ /km lon)	β_2 (kmh ⁻¹ /km lat)
$\mu(x)$	-0.75(-40.72, 38.05)	-10.20(-32.70, -6.74)	6.46(3.19, 9.77)
$\varsigma(x)$	0.21(-38.34, 38.23)	-2.95(-9.34, 3.45)	0.94(-0.22, 1.93)
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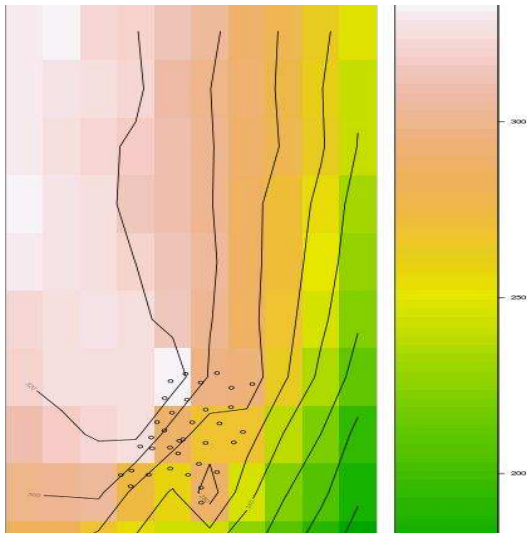
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Wind speed analysis: Return level inference



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		z_{10}	z_{50}	z_{1000}
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	$\kappa = 20$	190 (172, 219)	209 (188, 248)	394 (275, 512)
All excesses	(Arrival times)	182 (169, 197)	201 (188, 216)	302 (234, 382)
	(Markov chain)	184 (172, 198)	204 (190, 220)	308 (248, 396)
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- Previous seminars: discussed the merits of using all threshold excesses versus cluster peak excesses
- Need to think carefully about dependence, and in particular estimation of the extremal index θ
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- Ledford & Tawn (1997) show that, for $x_j, x_{j+1} > u$,

$$F(x_j, x_{j+1}) = F(x_j) + F(x_{j+1}) + \overline{F}(x_j, x_{j+1}) - 1, \quad \text{with}$$

$$\begin{aligned} \overline{F}(x_j, x_{j+1}) &= \mathcal{L} \{ -1/\log F(x_j), -1/\log F(x_{j+1}) \} \\ &\quad \times \{ -\log F(x_j) \}^{1/2\eta} \{ -\log F(x_{j+1}) \}^{1/2\eta} \end{aligned}$$

and $F(\cdot)$ assumed GP with (σ, ξ, λ_u)

- Bortot (1998) proposes the bivariate slowly varying function:

$$\mathcal{L}(x, y) = a + \{w(1-w)\}^{-1/2} [1 - V \{ (1-w)^{-1}, w^{-1} \}],$$

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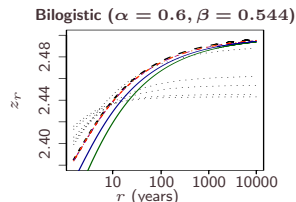
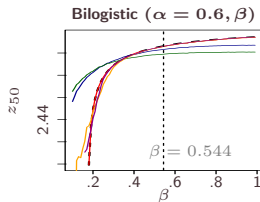
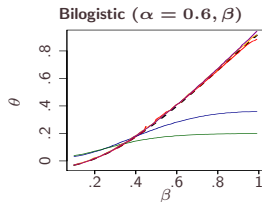
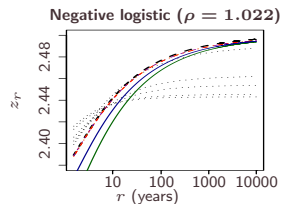
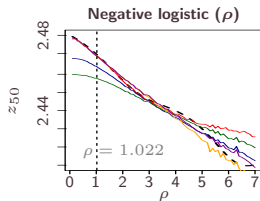
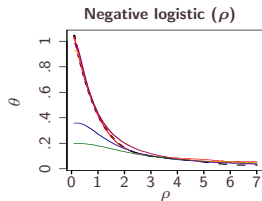
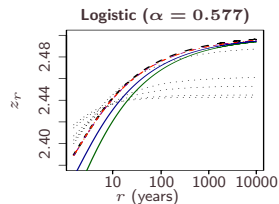
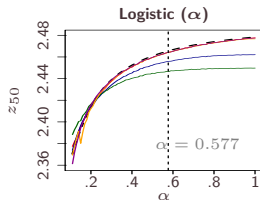
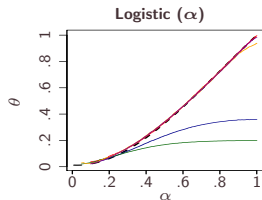
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Logistic Extreme Value Markov Process with $\beta = 0.5$ ($\theta = 0.328$)

Estimator	$u = 95\%$ quantile	$u = 99\%$ quantile
Arrival times	0.36 (0.056)	0.39 (0.070)
EV Markov chain	0.33 (0.020)	0.33 (0.025)
New	0.33 (0.035)	0.35 (0.065)
Clusters	0.15 (0.077)	0.18 (0.114)
Blocks	0.21 (0.060)	0.25 (0.102)

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- Incorporate unfinished business #2 and #3
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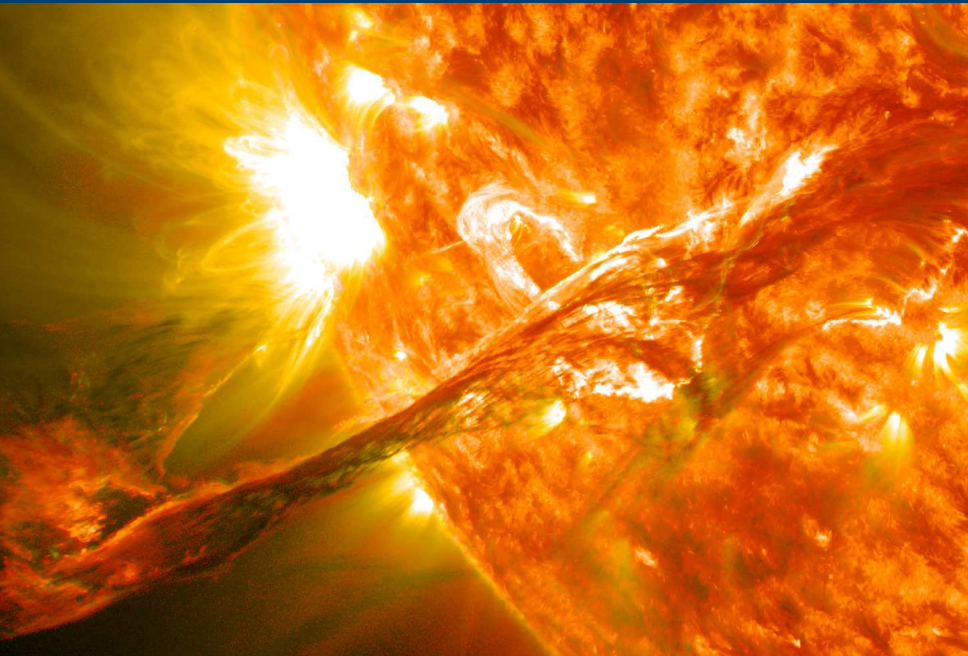
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