

4 Introduction to Probability

4.1 Definitions

An **experiment** is an activity where we do not know for certain what will happen, but we will *observe* what happens. An **outcome** is one of the possible things that can happen. The **sample space** is the set of all possible outcomes. An **event** is a set of outcomes.

All probabilities are measured on a scale ranging from zero to one, and can be expressed as fractions, decimal numbers or percentages. 

Notation: $P(A)$ represents the probability of the event A , e.g. $P(\text{Rain tomorrow})$. $P(\bar{A})$ is the probability that A does not occur (“not A ”).

The collection of all possible outcomes, that is the **sample space**, has a probability of 1. Two events are said to be **mutually exclusive** if both cannot occur simultaneously. Two events are said to be **independent** if the occurrence of one does not affect the probability of the other occurring.

Example 1

Do you think the following pairs of events are independent? 

- A : Molly plays table tennis, and B : Molly is good at maths
- C : Henry gets over 60 in MAS1403, and D : Henry gets under 40 in MAS1403

4.2 Measuring probability

1. Classical interpretation

Used when all possible outcomes are “equally likely”. In general, calculations follow from the formula

$$P(\text{Event}) = \frac{\text{Total number of outcomes in which event occurs}}{\text{Total number of possible outcomes}}.$$

2. Frequentist interpretation

When the outcomes of an experiment are not equally likely, we can perform the same experiment a large number of times and observe the outcome. The probability of an event can be estimated using the following formula:

$$P(\text{Event}) = \frac{\text{Number of times an event occurs}}{\text{Total number of times experiment performed}}.$$

3. Subjective interpretation

Probabilities are formulated subjectively using an individual's (sometimes expert) opinion. (Useful when the experiment can't be repeated.) For example, when we board an aeroplane, we judge the probability of it crashing to be sufficiently small that we are happy to undertake the journey.

4.3 Examples

1. *Chicken King* is a fast-food chain with 700 outlets in the UK. The geographic location of its restaurants is tabulated below:

		Region				Total
		NE	SE	SW	NW	
Population	Under 10,000	35	42	21	70	168
	10,000–100,000	70	105	84	35	294
	Over 100,000	175	28	35	0	238
Total		280	175	140	105	700

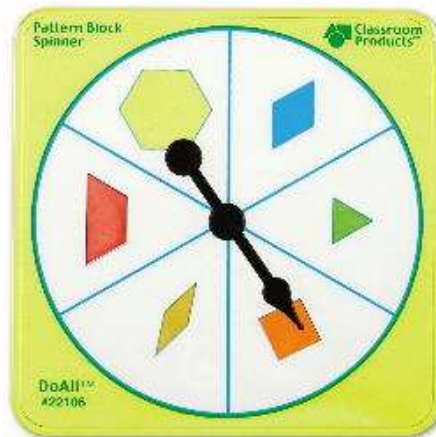
A health and safety organisation selects a restaurant at random for a hygiene inspection. Assuming that each restaurant is equally likely to be selected, calculate the following probabilities.

- (a) $P(\text{NE restaurant chosen})$,
- (b) $P(\text{Restaurant chosen from a city with a population over 100,000})$,
- (c) $P(\text{SW and city with a population under 10,000})$.

Solution



2. The spinner shown below is spun once.



Assuming each sector on the board is the same size, calculate the following probabilities.



(a) $P(\text{lands on a red shape}) =$

(b) $P(\text{lands on a triangle}) =$

(c) $P(\text{lands on a 4-sided shape}) =$

3. On the probability scale, how likely do you think it is that Newcastle United will be promoted this season? Which approach to probability would you use to estimate this?



4.4 The addition rule

The *addition rule* describes the probability of any of two or more events occurring. The addition rule for two events A and B is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

This describes the probability of *either* event A *or* event B happening.

Example 2

Prospective interns at internet startup *BlueFox* face two aptitude tests. If 35 percent of applicants pass the first test, 25 percent pass the second test, and 15 percent pass both tests, what percentage of applicants pass at least one test?

Solution



Note: if events A and B are mutually exclusive then $P(A \text{ and } B) = 0$ since A and B can't occur together. Therefore,

$$P(A \text{ or } B) = P(A) + P(B).$$

4.5 Exercises

1. Do you think the following pairs of events are independent or dependent? Explain.
 - (a) E : An individual has a high IQ
 F : An individual is accepted for a University place
 - (b) E_1 : An individual has a large outstanding credit card debt
 E_2 : An individual is allowed to extend his bank overdraft
2. The following data refer to a class of 18 students. Suppose that we will choose one student at random from this class.

Student Number	Sex	Height (m)	Weight (kg)	Shoe Size	Student Number	Sex	Height (m)	Weight (kg)	Shoe Size
1	M	1.91	70	11.0	10	M	1.78	76	8.5
2	F	1.73	89	6.5	11	M	1.88	64	9.0
3	M	1.73	73	7.0	12	M	1.88	83	9.0
4	M	1.63	54	8.0	13	M	1.70	55	8.0
5	F	1.73	58	6.5	14	M	1.76	57	8.0
6	M	1.70	60	8.0	15	M	1.78	60	8.0
7	M	1.82	76	10.0	16	F	1.52	45	3.5
8	M	1.67	54	7.5	17	M	1.80	67	7.5
9	F	1.55	47	4.0	18	M	1.92	83	12.0

Find the probabilities for the following events.

- (a) The student is female.
 - (b) The student's weight is greater than 70kg.
 - (c) The student's weight is greater than 70kg and the student's shoe-size is greater than 8.
 - (d) The student's weight is greater than 70kg or the student's shoe-size is greater than 8.
3. The regional manager of supermarket *Freshco* is interested in predicting sales patterns of breakfast cereal. If 85% of Freshco customers buy branded cereals (e.g. *Kellogg's* etc), 60% of customers buy Freshco's own-brand cereals, and 50% of customers buy both branded and Freshco's own-brand cereal, what percentage of Freshco customers do not buy breakfast cereal?

5 Conditional probability

5.1 The multiplication rule

The *multiplication rule* describes the probability of two (or more) events occurring. The probability of two events A and B both occurring is

$$P(A \text{ and } B) = P(A) \times P(B|A),$$

where $P(B|A)$ is the **conditional probability** of B given that A has already happened.

Example 1

A small company has 10 employees: 4 male and 6 female. You, as the manager, select two employees at random to attend a training session. What is the probability that you select two female employees?

Solution



Re-arranging the above expression for the **multiplication rule** gives a formula for calculating a conditional probability:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}.$$

Example 2

Recall that prospective interns at internet startup *BlueFox* face two aptitude tests. If 35 percent of applicants pass the first test, 25 percent pass the second test, and 15 percent pass both tests, what percentage of applicants pass the second test given that they passed the first test?

Solution



Independent events: two events A and B are independent if $P(B|A) = P(B)$, in which case

$$P(A \text{ and } B) = P(A) \times P(B).$$

Example 3

Are the outcomes of the two aptitude tests at internet startup *BlueFox* independent? Justify your answer.

Solution



Example 4

Employees at a Marketing firm are classified by age and sex as follows:

	under 30	30 to 50	over 50	
Male	0.275	0.125	0.025	
Female	0.325	0.175	0.075	

So, for example, 27.5% of employees are Male and under 30 years of age.

From this table, calculate

- (a) $P(\text{Male})$
- (b) $P(30 \text{ to } 50)$
- (c) $P(\text{Male}|30 \text{ to } 50)$
- (d) $P(30 \text{ to } 50|\text{Male})$
- (e) Are the events “Male” and “30 to 50” independent?

Solution

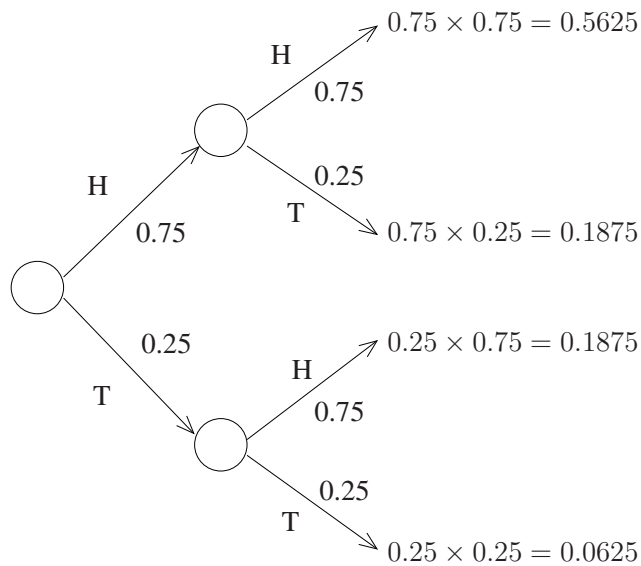


5.2 Tree diagrams

Tree diagrams (or probability trees) are simple, clear ways of presenting probabilistic information.

Example 5

Suppose we have a biased coin, with $P(\text{Head}) = 0.75$. Then the following tree diagram displays all outcomes, along with their associated probabilities, for two consecutive flips of the coin:



Important: multiply probabilities along branches (multiplication rule); the probabilities at the ends of the branches should add up to 1.

Example 6

A small company has 10 employees: 4 male and 6 female. You, as the manager, select two employees at random to attend a training session. What is the probability that you select one male and one female employee?

Solution



Example 7

Joe has a Business Management exam on Thursday morning. On Wednesday night he is free to choose one (and only one) of the following activities: (a) go to the cinema, (b) go to the pub, (c) stay home and watch TV, (d) stay home and study. The probabilities that he elects these alternatives are 0.14, 0.45, 0.25 and 0.16, respectively. His conditional probabilities of passing the exam given (a), (b), (c) and (d) are 0.4, 0.05, 0.5 and 0.8 respectively. Find

- (i) the probability that Joe goes to the pub and passes his exam;
- (ii) the probability that Joe passes his exam;
- (iii) the probability that Joe went to the pub, given that he passed his exam.

Solution

Use the space provided below to construct a tree diagram for this example.



(i) $P(\text{Joe goes to Pub and passes exam}) =$

(ii) $P(\text{Joe passes exam}) =$

(iii) $P(\text{Joe went to Pub} \mid \text{Joe passed exam}) =$

5.3 Exercises

1. An on-line retailer conducts a survey of 200 customers and obtains the following results.

	Age		
	Under 30	30 to 45	Over 45
Male	60	20	40
Female	40	30	10

A customer is selected at random.

- What is the probability that the customer is male and aged 30 to 45?
 - Given that this customer is aged 30 to 45, what is the probability that they are male?
 - Given that this customer is female, what is the probability that they are 45 or under?
 - Now suppose that two customers are selected at random. What is the probability that both are Male?
2. If Vinny goes to the cinema, there is a 60% chance he will then also go to the bar afterwards. However, if he doesn't go to the cinema, this reduces to just 30%. On Friday night, Vinny decides to go to the cinema only if his friend Julia also goes. Vinny has no idea about Julia's intentions this Friday and so is just as likely to go to the cinema as he is to not go. Let C be the event that Vinny goes to the cinema, and B the event that Vinny goes to the bar, this Friday. Using a probability tree diagram, or otherwise, find
- $P(C)$
 - $P(\bar{C})$
 - $P(\bar{B}|C)$
 - $P(\bar{B}|\bar{C})$
 - $P(C \text{ and } B)$
 - $P(B)$

6 Decision-making using probability

6.1 Expected Monetary Value

The *Expected Monetary Value (EMV)* of a single event is simply the probability of that event multiplied by its monetary value.

Example 1

Suppose you win £5 if you pull an ace from a pack of cards, the *EMV* would be

$$EMV(Ace) = P(Ace) \times \text{MonetaryValue}(Ace) = \frac{4}{52} \times 5 = 0.38.$$

Your expected return would be 38 pence; if you repeated this bet a large number of times, you would come out, on average, 38 pence better off per bet. Therefore you would want to pay no more than 38p for such a bet.

In general, for more complicated problems involving several options,

$$EMV = \sum \{P(\text{Event}) \times \text{Monetary value of Event}\}$$

where the sum is over all possible events. We choose the option with the largest EMV.

Example 2

Synaptec is a small technology company with a new product that they wish to launch on to the market. It could go for

- a direct approach, launching onto the domestic market through traditional channels,
- it could launch only on the internet,
- or it could license the product to a larger company through the payment of a licence fee irrespective of the success of the product.

Initial market research suggests that demand for the product can be classed into three categories: high, medium or low, and these categories will occur with probabilities 0.2, 0.35 and 0.45. Likely profits (in £K) to be earned under each option are

	High	Medium	Low
Direct	100	55	-25
Internet	46	25	15
Licence	20	20	20

How should the company launch the product?

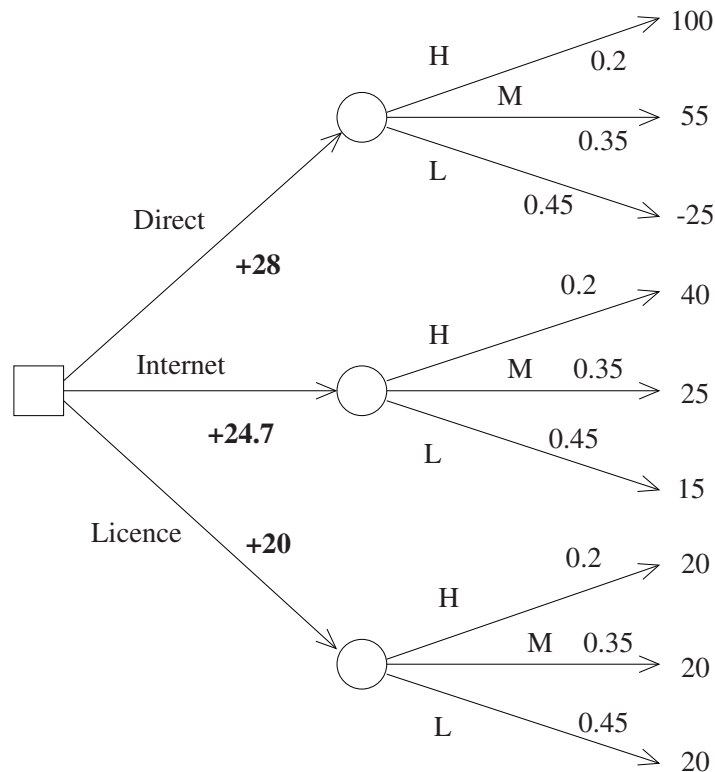


6.2 Decision trees

When we include a decision in a tree diagram (see Chapter 5) we use a rectangular node, called a *decision node* to represent the decision. The diagram is then called a *decision tree*.

Example 3

The decision tree for the last example (Example 2) would look like this:



Key points:

- There are no probabilities at a decision node but we evaluate the expected monetary values of the options.
- In a decision tree the first node (on the left) is always a decision node.
- There may also be other decision nodes.
- If there is another decision node then we evaluate the options there and choose the best one (based on EMV), and the expected monetary value of this option becomes the expected monetary value of the branch leading to the decision node.
- We work “backwards” through the tree (from right to left), evaluating EMVs and making decisions at each decision node.

Example 4

Charlotte Watson, the manager of a small sales company, has the opportunity to buy a fixed quantity of a new type of Android tablet which she can then offer for sale to clients.

The decision to buy the product and offer it for sale would involve a fixed cost of £200,000. The number of tablets that will be sold is uncertain, but Charlotte judges that:

- Sales will be “poor” with probability 0.2; this will result in an income of £100,000.
- Sales will be “moderate” with probability 0.5; this will result in an income of £220,000.
- Sales will be “good” with probability 0.3; this will result in an income of £350,000.

For an additional fixed cost of £30,000, market research can be conducted to aid the decision-making process. The outcome of the market research can be either positive or negative, with probabilities 0.58 and 0.42, respectively. Knowing the outcome of the market research changes the probabilities for the main sales project as follows:

Market research	Main sales probabilities		
	Poor	Moderate	Good
Positive	0.15	0.45	0.4
Negative	0.6	0.35	0.05

Charlotte has various options:

- Buy the tablets, without market research.
- Pay for the market research.
- Do nothing.

If she pays for the market research then, depending on the outcome, she can:

- Buy the tablets.
- Do nothing.

(a) Draw a decision tree for this problem.

(b) Use expected monetary value to determine the optimal course of action for Charlotte.

The following page is left blank for your solution to this question





6.3 Exercises

1. *Picoplex Technologies* have developed a new manufacturing process which they believe will revolutionise the smartphone industry. They are, however, uncertain how they should go about exploiting this advance.

Initial indications of the likely success of marketing the process are 55%, 30%, 15% for “high success”, “medium success” and “probable failure”, respectively. The company has three options; they can go ahead and develop the technology themselves, licence it or sell the rights to it. The financial outcomes (in £ millions) for each option are given in the table below.

	“high success”	“medium success”	“failure”
Develop	80	40	–100
Licence	40	30	0
Sell	25	25	25

- (a) Draw a decision tree to represent the company’s problem.
 - (b) Calculate the *Expected Monetary Value* for all possible decisions the company may take and hence determine the optimal decision for the company.
2. The manager of a small business has the opportunity to buy a fixed quantity of a new product and offer it for sale for a limited time.

The decision to buy the product and offer it for sale would involve a fixed cost of £150,000. The amount that would be sold is uncertain but the manager judges that:

- There is a probability of 0.3 that sales will be “poor” with an income of £80,000.
- There is a probability of 0.5 that sales will be “medium” with an income of £160,000.
- There is a probability of 0.2 that sales will be “good” with an income of £240,000.

For an additional fixed cost of £20,000, the product can be sold for a trial period before a final decision is made. No income is made from this trial. The result of the trial will be “poor” with probability 0.33, “medium” with probability 0.40 or “good” with probability 0.27. Knowing the outcome of the trial changes the probabilities for the main sales project:

Trial outcome	Main sales probabilities		
	Poor	Medium	Good
Poor	0.7	0.2	0.1
Medium	0.2	0.6	0.2
Good	0.1	0.2	0.7

The manager also has the option to do nothing.

- (a) Draw a decision tree for this problem.
- (b) Use expected monetary value to determine the optimal course of action for this business.