



MAS1403

Quantitative Methods for Business Management

Semester 1, 2017–2018

Module leader: Dr. Lee Fawcett

Additional lecturers: Dr. Dave Walshaw and Dr. Ged Cowburn

Announcements: Written assignment (mini project)

You should be working on this now; worth 10% of the module;
deadline for submission: **4pm, Thursday 14th December**

Hints:

- Graphs: when comparing two or more groups, use the same scales (e.g. percentage rel. freq. histograms or polygons, x-axes etc.)...
- ...and where appropriate overlay graphs on the same panel
- 'Produce *appropriate* graphical / numerical summaries...':
Numerical – one measure of average + one measure of spread per dataset; Graphical – one or two at most per dataset
- Comments: Average? Where does the graph 'peak'?
Spread / dispersion? Outliers? Symmetric / asymmetric distribution? Normal distribution?

Announcements: Written assignment (mini project)

Submission:

- Hard-copy, posted through the **Stage 1 homework submission letterbox** on the 3rd floor of the **Herschel Building**
- Must have a personalised NESS cover sheet attached
- Personalised datasets for question 2!
- Marks for presentation: Type up solutions in WORD?

Announcements

- CBA3: Will go live at 00:01 this coming Saturday, 2nd December in both practice and assessed modes
- Deadline: **23:59 Friday 15th December**

Lecture 9

CONTINUOUS PROBABILITY MODELS

9. Continuous probability models

We have seen how **discrete** random variables can be modelled by discrete probability distributions such as the **binomial** and **Poisson** distributions.

We now consider how to model **continuous** random variables.

9. Continuous probability models

A variable is **discrete** if it takes a **countable** number of values.

For example,

- the number of **blue** cars that I count in a 5 minute period
- the number of **heads** observed when I flip a coin ten times
- Shoe sizes: $1, \dots, 12, 13, 1, 2, \dots$
- $r = 0, 0.1, 0.2, \dots, 0.9, 1.0$

In contrast, the values which a **continuous variable** can take form a **continuous scale**, with no “jumps”.

For example,

- Height
- Weight
- Temperature

An example

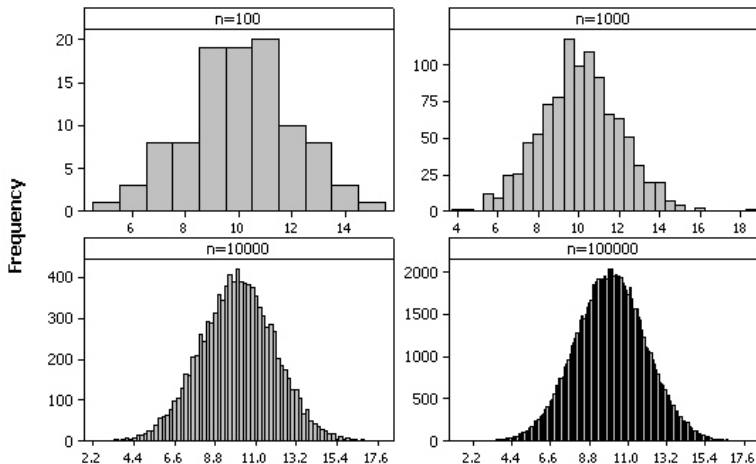
Think about **height**.

- In practice, we might only record height to the **nearest cm**
- If we could measure height *exactly* we'd find that everyone had a **different height**
- This is the essential difference between discrete and continuous variables
- If there are n people on the planet, the probability that someone's height is x would be $\frac{1}{n}$
- As n gets bigger and bigger, this probability tends to zero!!

An example

Consider taking a sample of values from the continuous random variable X .

Histograms of continuous r.v. for increasing sample sizes



An example

- As the sample size gets bigger, the **interval widths get smaller**
- the jagged profile of the histogram **smooths out** to become a curve
- When the sample size is infinitely large, this curve is known as the **probability density function** (pdf)

Features of the probability density function

The key features of pdfs are:

- 1 the area under a pdf is one: $P(-\infty < X < \infty) = 1$
- 2 areas under the curve correspond to probabilities
- 3 $P(X \leq x) = P(X < x)$ since $P(X = x) = 0$.

9. Continuous probability models

Over the next two weeks we will consider some particular probability distributions that are often used to describe continuous random variables.

We start with the most **important**, most **widely-used** statistical distribution of all time...

...wait for it...

The Normal Distribution

9.1 The Normal distribution

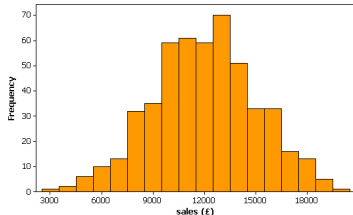
The **Normal distribution** is without doubt the most widely-used statistical distribution in many practical applications:

- Normality arises **naturally** in many physical, biological and social measurement situations
- Normality is important in **Statistical inference** (see *Semester 2 material*)
- The normal distribution has many guises:
 - Gaussian distribution
 - Laplacean distribution
 - “bell-shaped curve”

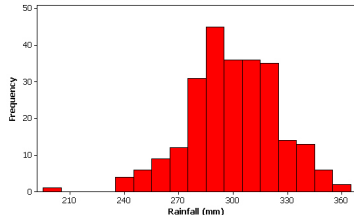


Some real-life examples

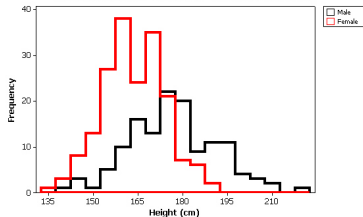
Histogram of weekly sales at 500 branches of HMV



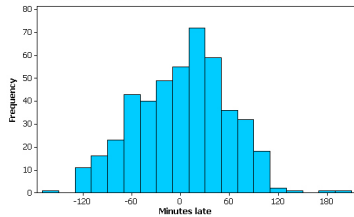
Histogram of monthly rainfall totals at 250 locations in the USA



Histograms to show the heights of MAS1403 students



Histogram to show arrival times of flights at Heathrow



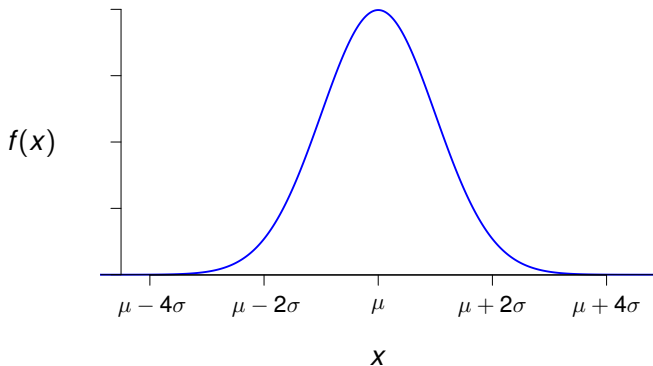
9.1 The Normal distribution

Recall the “parameters” of the binomial and Poisson distributions:

- The **binomial distribution** has two parameters, n and p
- the **Poisson distribution** has one parameter λ
- The **Normal distribution** has two parameters: the mean, μ , and the standard deviation, σ

9.1 The Normal distribution

The probability density function (pdf) of the Normal distribution has a “**bell-shaped**” profile:



9.1 The Normal distribution

We can think of the pdf as a smoothed percentage relative frequency histogram: **the area under the curve is 1**.

The (rather nasty!) formula for this pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}.$$

Unlike the binomial and Poisson distributions, there is **no simple formula** for calculating probabilities.

Don't worry though, probabilities from the Normal distribution can be determined using **statistical tables** (see page 51) or statistical packages such as `Minitab`.

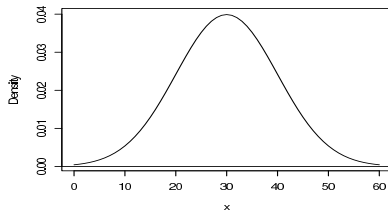
Characteristics of the Normal distribution

There are four important characteristics of the Normal distribution:

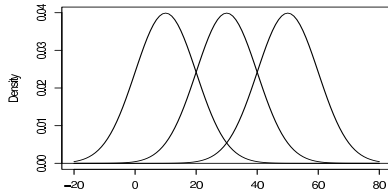
- 1 It is **symmetrical** about its mean, μ .
- 2 The mean, median and mode all **coincide**.
- 3 The area under the curve is equal to 1.
- 4 The curve extends in both directions to infinity (∞).

On the next slide are plots of the pdf for Normal distributions with different values of μ and σ .

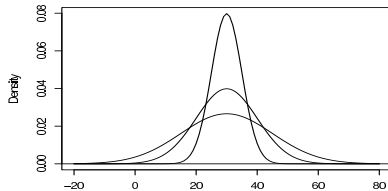
Normal pdf with mean 30 and sd 10



Normal pdfs with mean 10, 30, 50 and sd 10



Normal pdfs with mean 30 and sds 5, 10, 15



Notation

If a random variable X has a Normal distribution with mean μ and variance σ^2 , then we write

$$X \sim N(\mu, \sigma^2).$$

For example, a random variable X which follows a Normal distribution with mean **10** and variance **25** is written as

$$\begin{aligned} X &\sim N(10, 25) && \text{or} \\ X &\sim N(10, 5^2). \end{aligned}$$

It is important to note that the second parameter in this notation is the **variance** and not the **standard deviation**.

9.1.1 The *standard* Normal distribution

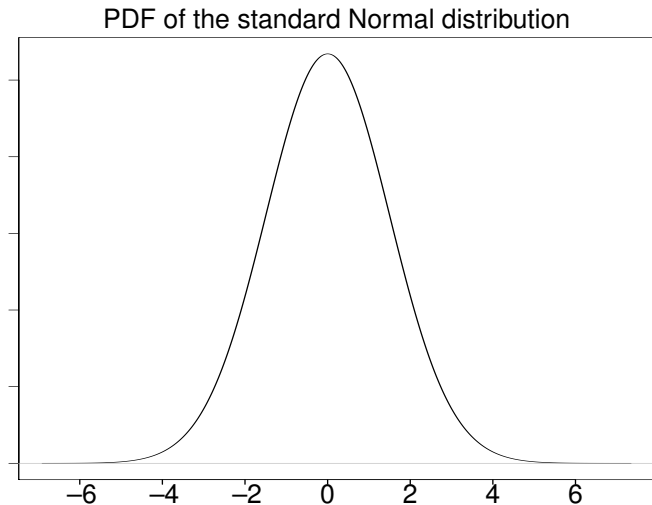
The **Standard** Normal distribution has a mean of **0** and a variance of **1**.

A random variable with this *standard* Normal distribution is usually given the letter Z , and so we say

$$Z \sim N(\mathbf{0}, \mathbf{1}).$$

If our random variable follows a **standard** Normal distribution, then we can obtain *cumulative probabilities* from statistical tables (see page 51 of the notes), which give “**less than or equal to**” probabilities.

Probability density function for Z



Example 1

For example, if $Z \sim N(0, 1)$:

- (a)** The probability that Z is less than or equal to -1.46 is $P(Z \leq -1.46)$. Therefore we look for the probability in tables corresponding to $z = -1.46$: row labelled -1.4 , column headed -0.06 .

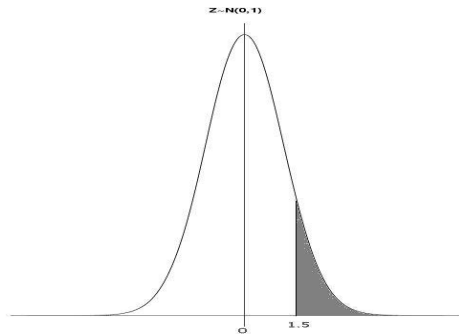
This gives $P(Z \leq -1.46) = \mathbf{0.0721}$.

- (b)** The probability that Z is less than or equal to 0.01 is $P(Z \leq 0.01)$. Therefore we look for the probability in tables corresponding to $z = 0.01$: row labelled 0.0 , column headed 0.01 .

This gives $P(Z \leq 0.01) = \mathbf{0.5040}$.

Example 1

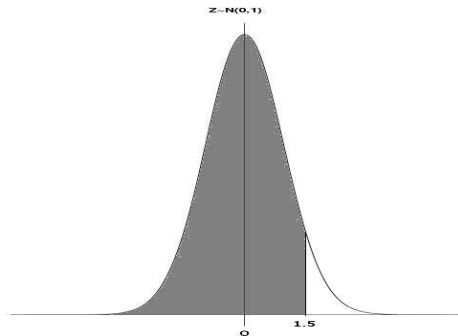
- (c) The probability that Z is greater than 1.5 is $P(Z > 1.5)$.
Now our tables give “less than” probabilities, and here we want a “greater than” probability.



So we find $P(Z < 1.5) = 0.9332$ and subtract this from 1 to give **0.0668**.

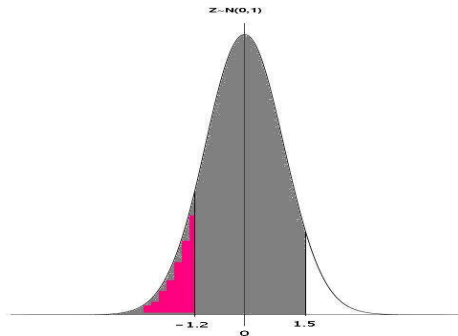
Example 1

- (d) What about the probability that Z lies between -1.2 and 1.5 ? It helps to think about this graphically.



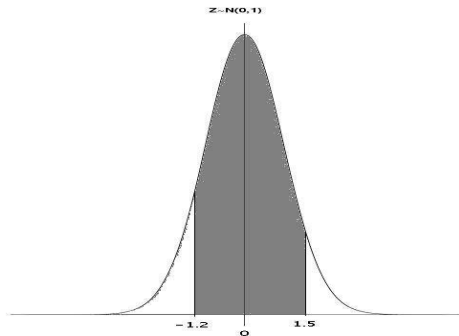
Example 1

- (d) What about the probability that Z lies between -1.2 and 1.5 ? It often helps to think about this graphically.



Example 1

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Doing so, gives

$$\begin{aligned} P(-1.2 < Z < 1.5) &= P(Z < 1.5) - P(Z \leq -1.2) \\ &= 0.9332 - 0.1151 \end{aligned}$$

Example 1

(e)

$$\begin{aligned}P(Z < 1.5) &= 1 - P(Z > 1.5) \\&= 1 - 0.0668 \quad \text{From part (c)} \\&= 0.9332.\end{aligned}$$

9.1.2 Probabilities from *any* Normal distribution

So how do we calculate probabilities for **any** Normal distribution, not just the **standard** Normal distribution (for which we have tables)?

Idea: “make” the Normal distribution that we have “look like” the standard Normal distribution, and then we can just use the tables as before!

But how? Use the **slide-squash** technique!

9.1.2 Probabilities from *any* Normal distribution

The formula which changes **any** Normal random variable X into the **standard** Normal random variable Z is given by

$$Z = \frac{X - \mu}{\sigma},$$

where

- μ is the mean
- σ is the standard deviation

This can be translated into probability statements:

$$P(X \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right),$$

which can be looked up in tables.

Example 2

If $X \sim N(10, 2^2)$, calculate $P(X \leq 8)$.

Translate X into Z using the slide-squash rule:

$$\begin{aligned} Z &= \frac{X - \mu}{\sigma} \\ &= \frac{8 - 10}{2} \\ &= -1. \end{aligned}$$

Then, from the table on page 51,

$$P(Z \leq -1) = 0.1587.$$

Example 3

Suppose X is the IQ of a randomly selected 18–19 year old and that X follows a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$. Thus, we have:

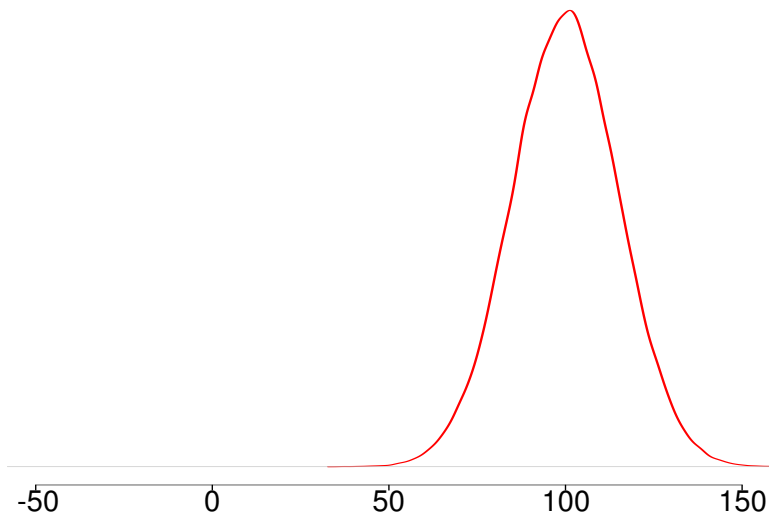
$$X \sim N(100, 15^2).$$

Find the following probabilities.

- (a) The probability that an 18–19 year old has an IQ less than 110.
- (b) The probability that an 18–19 year old has an IQ greater than 110.
- (c) The probability that an 18–19 year old has an IQ greater than 125.
- (d) The probability that an 18–19 year old has an IQ between 95 and 115.

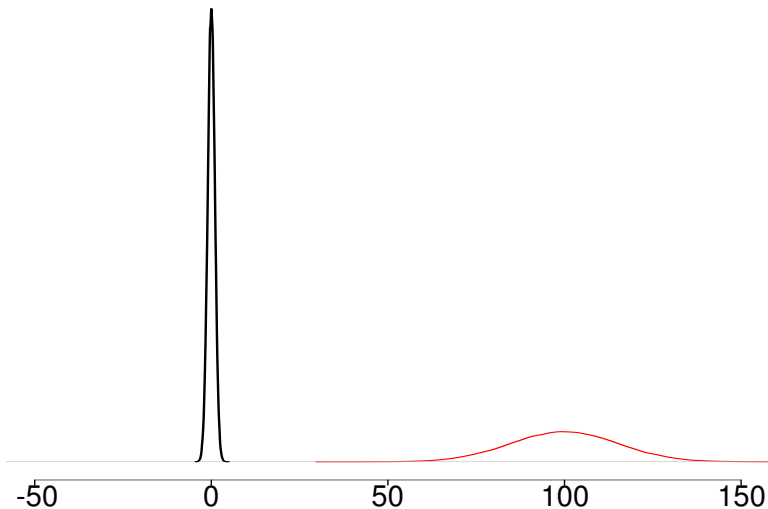
Example 3

Distribution of IQs



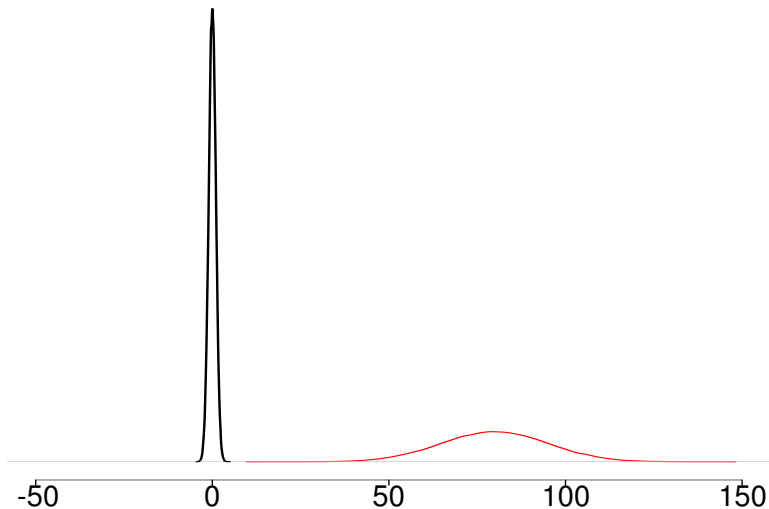
Example 3

Slide-squash



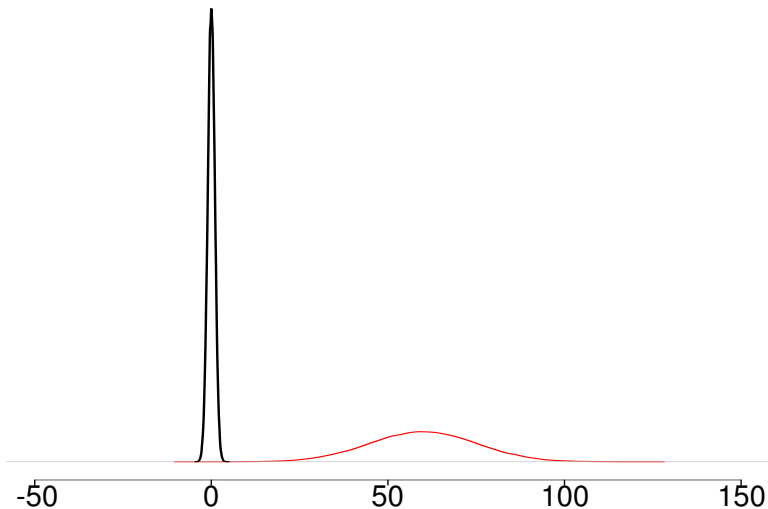
Example 3

Slide-squash



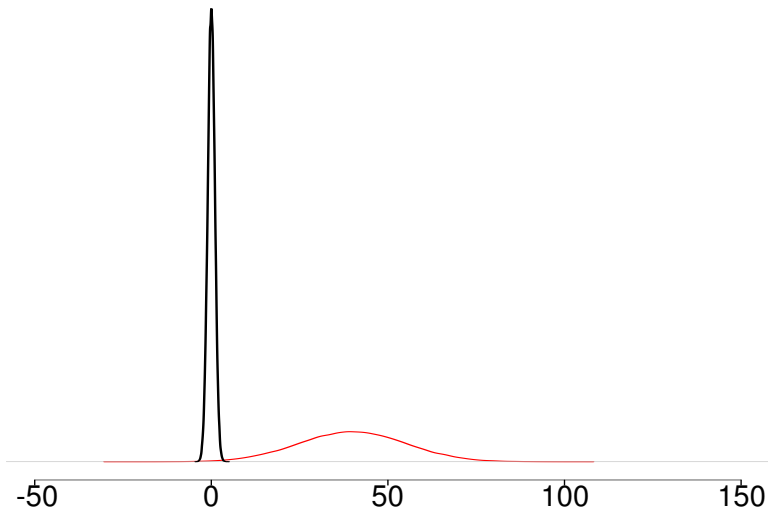
Example 3

Slide-squash



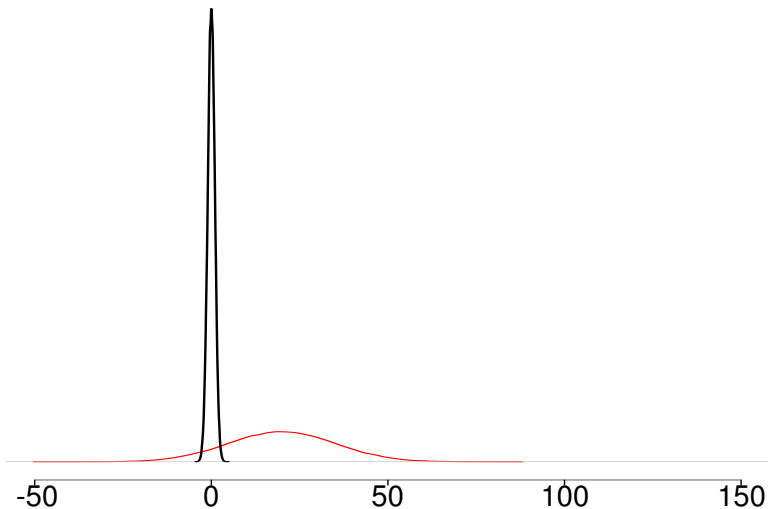
Example 3

Slide-squash



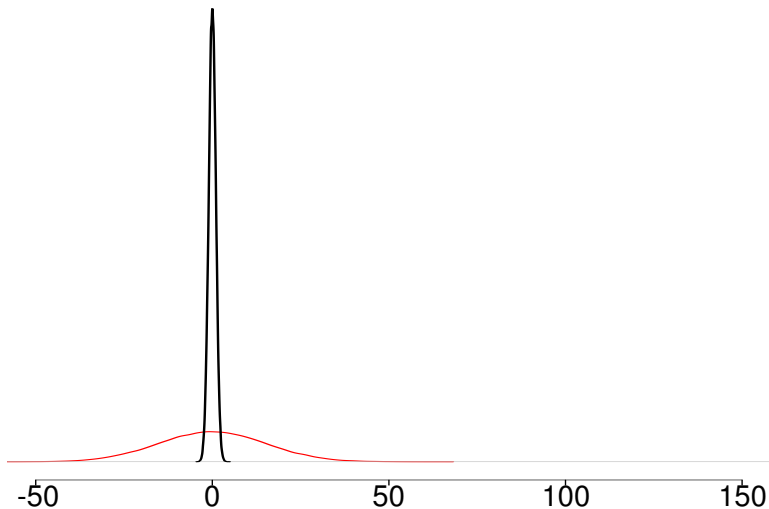
Example 3

Slide-squash



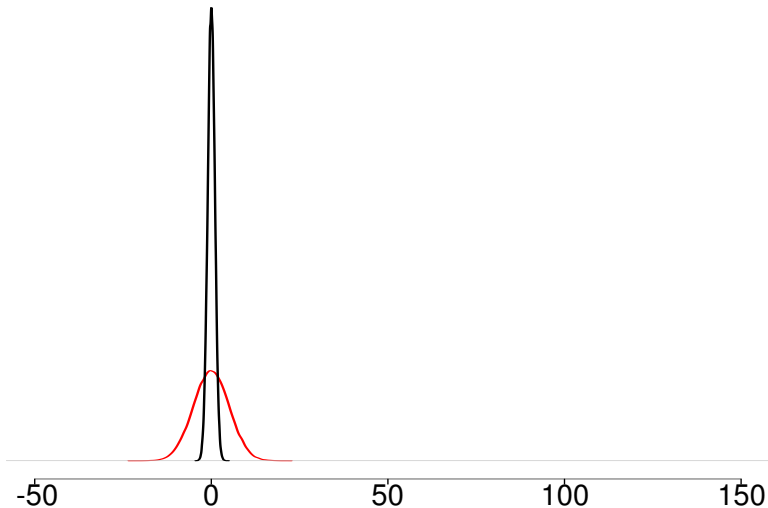
Example 3

Slide-squash



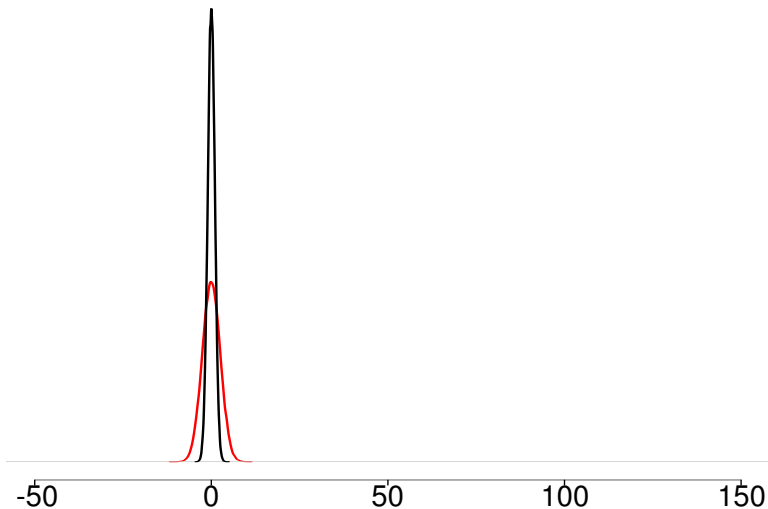
Example 3

Slide-squash



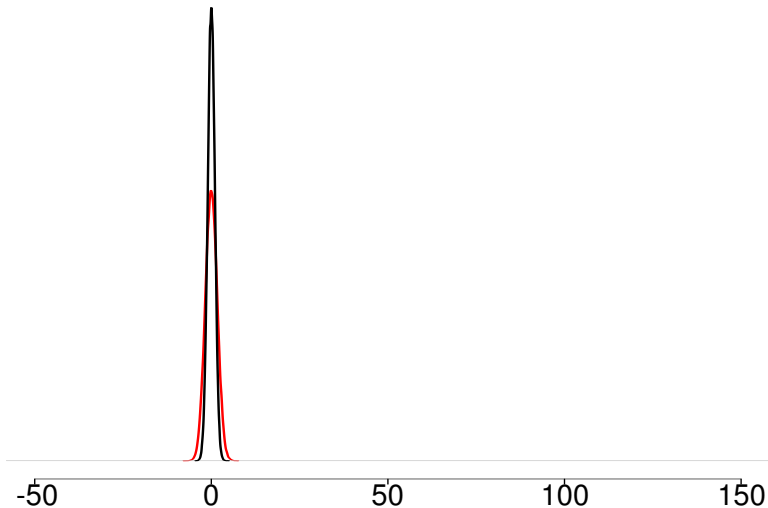
Example 3

Slide-squash



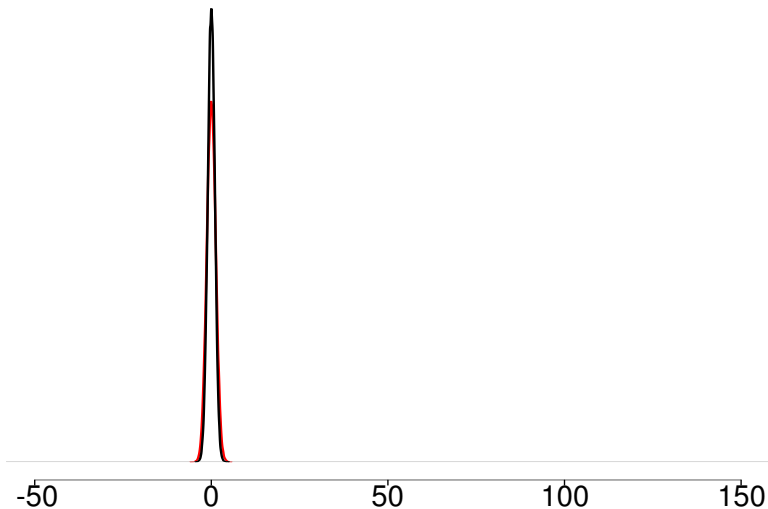
Example 3

Slide-squash



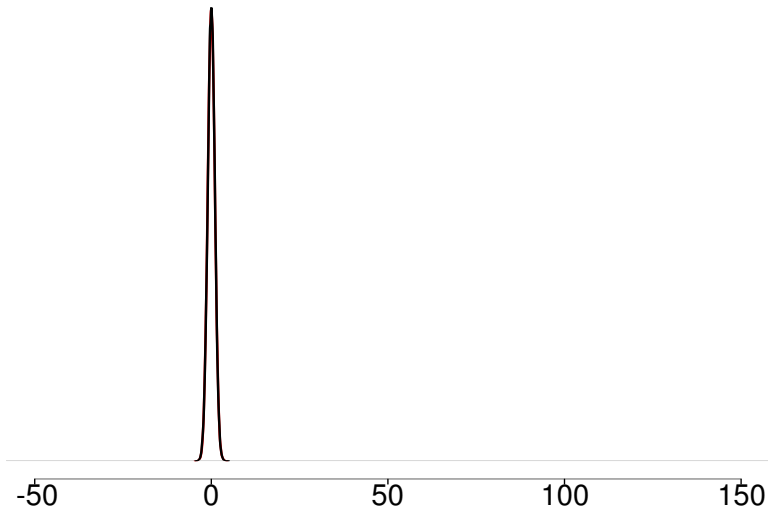
Example 3

Slide-squash



Example 3

Slide-squash



Example 3 (a)

$$\begin{aligned}P(X < 110) &= P\left(Z < \frac{X - \mu}{\sigma}\right) \\&= P\left(Z < \frac{110 - 100}{15}\right) \\&= P(Z < 0.67) \\&= 0.7486.\end{aligned}$$

Example 3 (b)

$$\begin{aligned}P(X > 110) &= 1 - P(X < 110) \\&= 1 - 0.7486 \\&= 0.2514.\end{aligned}$$

Example 3 (c)

$$\begin{aligned}P(X > 125) &= 1 - P(X < 125) \\&= 1 - P\left(Z < \frac{125 - 100}{15}\right) \\&= 1 - P(Z < 1.67) \\&= 1 - 0.9525 \\&= 0.0475.\end{aligned}$$

Example 3 (d)

$$\begin{aligned}P(95 < X < 115) &= P(X < 115) - P(X < 95) \\&= P\left(Z < \frac{115 - 100}{15}\right) - P\left(Z < \frac{95 - 100}{15}\right) \\&= P(Z < 1) - P(Z < -0.33) \\&= 0.8413 - 0.3707 \\&= 0.4706.\end{aligned}$$