



MAS1403

Quantitative Methods for Business Management

Semester 1, 2017–2018

Module leader: Dr. Lee Fawcett

Additional lecturers: Dr. Dave Walshaw and Dr. Ged Cowburn

Announcements

- This week is a **computer practical week**
- That means the standard Thursday tutorials are replaced with computer sessions:

Group	Day	Time
A	Tues	10am
B	Wed	10am
C	Thurs	11am

- All sessions take place in the **Herschel PC cluster**
- You will be introduced to some software for statistical analysis – important for the first written assignment

Announcements

- The **semester 1 written assignment** will be available to view from the course webpage later this week
 - It is worth 10% of this module, so you should treat it like a mini project
 - You will have four full weeks to complete the assignment – the deadline for submission is **4pm, Thursday 14th December 2017**
 - Some questions will require you to use the software you will be introduced to in this week's computer sessions
 - For some questions you will be allocated your own personal dataset
 - Some questions will be open-ended, so you will have to think carefully about how to tackle the problems
 - Time in tutorials will be given for support
- **CBA2** is now live in assessed mode – deadline: **23:59 this coming Friday, 17th November**

Lecture 7

DISCRETE PROBABILITY MODELS

7.1 Probability distributions

The **probability distribution** of a discrete random variable X is the list of all possible values X can take and the probabilities associated with them.

For example, if the random variable X is the outcome of a roll of a die then the probability distribution for X is:

r	1	2	3	4	5	6	Sum
$P(X = r)$	1/6	1/6	1/6	1/6	1/6	1/6	1

7.1 Probability models

In the die-rolling example, we used the interpretation of probability to obtain the probability distribution for X , the outcome of a roll on the die.

7.1 Probability models

In the die-rolling example, we used the **classical** interpretation of probability to obtain the probability distribution for X , the outcome of a roll on the die.

Consider the following **frequentist** example.

Let X be the number of cars observed in half-hour periods passing the junction of two roads. In a five hour period, the following observations on X were made:

2 3 2 5 5 3 4 5 6 7

Obtain the probability distribution of X .

7.1 Probability models

2 3 2 5 5 3 4 5 6 7

We can calculate the following probabilities:

$$P(X = 0) = \frac{0}{10} = 0$$

$$P(X = 1) = \frac{0}{10} = 0$$

$$P(X = 2) = \frac{2}{10} = 0.2$$

$$P(X = 3) = \frac{2}{10} = 0.2$$

7.1 Probability models

2 3 2 5 5 3 4 5 6 7

$$P(X = 4) = \frac{1}{10} = 0.1$$

$$P(X = 5) = \frac{3}{10} = 0.3$$

$$P(X = 6) = \frac{1}{10} = 0.1$$

$$P(X = 7) = \frac{1}{10} = 0.1$$

7.1 Probability models

Thus would give:

x	$P(X = x)$
< 2	0
2	0.2
3	0.2
4	0.1
5	0.3
6	0.1
7	0.1
> 7	0
sum	1

Does this make sense?

7.2 The binomial distribution

In many surveys and experiments data is collected in the form of **counts**. For example,

- the number of people in a survey who bought a CD
- the number of people who said they would vote Labour
- the number of defective items in a sample

All these variables have common features:

- 1 Each person/item has only **two** possible (exclusive) responses (Yes/No, Defective/Not defective etc)
 - this is referred to as a **trial** which results in a **success** or **failure**
- 2 The survey/experiment takes the form of a **random sample**
 - the responses are independent

7.2 The binomial distribution

If:

- There are a fixed number of trials or experiments (n)
- There are only two possible outcomes for each trial ('success' or 'failure')
- There is a constant probability of 'success', p
- The outcome of each trial is independent of any other trial

Then we say that the number of successes, X , follows a **binomial distribution**.

Example 2

Which of the following scenarios could be adequately modelled by a binomial distribution?

- The number of sixes on 3 rolls of a fair six-sided die.
- The number of students who pass MAS1403 this year.

Example 2

Which of the following scenarios could be adequately modelled by a binomial distribution?

- **The number of sixes on 3 rolls of a fair six-sided die.**
- ~~The number of students who pass MAS1403 this year.~~

7.2 The binomial distribution

Suppose we are interested in the **number of sixes** we get from 3 rolls of a die.

Each roll of the die is an experiment or trial which gives a “**six**” (success, or **s**) or “**not a six**” (failure, or **f**).

The probability of a success is $p = P(\text{six}) = 1/6$.

We have $n = 3$ **independent** experiments or trials (rolls of the die).

7.2 The binomial distribution

Let X be the number of sixes obtained.

We can now obtain the full **probability distribution** of X ; a probability distribution is a list of all the possible outcomes for X with along with their associated probabilities.

7.2 The binomial distribution

For example, suppose we want to work out the probability of obtaining **three** sixes: (three “**successes**” — i.e. **sss** — or $P(X = 3)$).

Since the rolls of the die can be considered independent, we get (using the **multiplication law**):

$$P(\text{sss}) = P(s) \times P(s) \times P(s) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \left(\frac{1}{6}\right)^3$$

7.2 The binomial distribution

That one's easy!

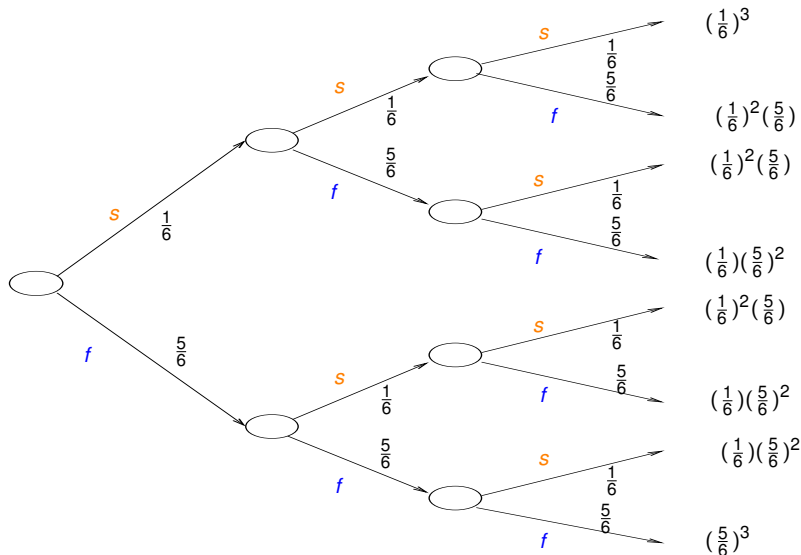
What about the probability that we get **two** sixes — i.e.
 $P(X = 2)$?

This one's a bit more tricky, because that means we need two
s's and one **f**...

...but the **f** (“not six”) could appear on the first roll, or the
second roll, or the third!

Thinking about it, there are actually **eight** possible outcomes
for the three rolls of the die:

7.2 The binomial distribution



7.2 The binomial distribution

So, for $P(X = 2)$, we *could* have:

$$P(\textcolor{blue}{f}\textcolor{brown}{ss}) = \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} = \left(\frac{1}{6}\right)^2 \times \frac{5}{6},$$

or we *could* have:

$$P(\textcolor{brown}{s}\textcolor{blue}{fs}) = \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} = \left(\frac{1}{6}\right)^2 \times \frac{5}{6},$$

or even:

$$P(\textcolor{brown}{ss}\textcolor{blue}{f}) = \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \left(\frac{1}{6}\right)^2 \times \frac{5}{6},$$

7.2 The binomial distribution

Can you see that we therefore get:

$$P(X = 2) = 3 \times \left(\frac{1}{6}\right)^2 \times \frac{5}{6}.$$

Which takes the form:

$$\begin{aligned} P(X = 2) &= \text{Number of ways to get two sixes} \\ &\quad \times P(2 \text{ sixes}) \times P(1 \text{ "not six"}). \end{aligned}$$

7.2 The binomial distribution

Using the same argument as above we can calculate the other probabilities:

$$P(X = 0) = \left(\frac{5}{6}\right)^3 = 0.579$$

$$P(X = 1) = 3 \times \left(\frac{1}{6}\right) \times \left(\frac{5}{6}\right)^2 = 0.347$$

$$P(X = 2) = 3 \times \left(\frac{1}{6}\right)^2 \times \frac{5}{6} = 0.069$$

$$P(X = 3) = \left(\frac{1}{6}\right)^3 = 0.005...$$

7.2 The binomial distribution

... and so the full **probability distribution** for X is:

x	0	1	2	3
$P(X = x)$	0.579	0.347	0.069	0.005

This probability distribution shows that most of the time we would get either 0 or 1 sixes and, for example, 3 sixes would be quite rare.

Try your own experiment!

Minitab

7.2 The binomial distribution

Now this is a bit long-winded . . . and that was just for **three** rolls of the die!

Imagine what it would be like to calculate for **100** rolls of the die!

We would like a more concise way of working these probabilities out without having to list all the possible outcomes as we did above.

7.2.1 Calculating probabilities

You should see from the tree diagram that we can construct a **general formula**, taking the form:

$$P(X = r) = \# \text{ ways to get } r \text{ successes out of } n \text{ trials} \\ \times P(r \text{ successes}) \times P(n - r \text{ failures})$$

We can write this more succinctly as

$$P(X = r) = {}^nC_r \times p^r \times (1 - p)^{n-r}, \quad r = 0, 1, \dots, n.$$

The **binomial coefficient** ${}^nC_r$ works out how many ways we can choose r objects out of n , and so is commonly read as “ n choose r ”: **button on the calculator!**

Example 3

What is the probability of getting 2 sixes from three rolls of a fair six-sided die?

We can just use our table of derived results from earlier... but let's use the binomial formula directly!

We have X : Number of sixes on three rolls of the die, and $X \sim \text{Bin}(3, 1/6)$. Thus

$$P(X = r) = {}^nC_r \times p^r \times (1 - p)^{n-r}$$

$$\begin{aligned} P(X = 2) &= {}^3C_2 \times 1/6^2 \times (1 - 1/6)^{3-2} \\ &= 3 \times \frac{1}{36} \times \frac{5}{6} \\ &= \frac{5}{72} = 0.069. \end{aligned}$$

Example 4

If $X \sim \text{Bin}(10, 0.2)$ calculate:

(a) $P(X = 2)$

(b) $P(X \leq 2)$

(c) $P(X < 3)$

(d) $P(X > 1)$

Example 4(a): $P(X = 2)$

$$P(X = 2) = {}^{10}C_2 \times 0.2^2 \times 0.8^8 = 0.302.$$

Example 4(b): $P(X \leq 2)$

For $P(X \leq 2)$, we need to add the answers to $P(X = 0)$, $P(X = 1)$ and $P(X = 2)$.

$$P(X = 0) = {}^{10}C_0 \times 0.2^0 \times 0.8^{10} = 0.107$$

$$P(X = 1) = {}^{10}C_1 \times 0.2^1 \times 0.8^9 = 0.268$$

$$P(X = 2) = 0.302 \quad \text{from part (a)}$$

So

$$P(X \leq 2) = 0.107 + 0.268 + 0.302 = 0.677.$$

Example 4(c): $P(X < 3)$

The possible outcomes are:

0 1 2 3 4 5 6 7 8 9 10

Example 4(c): $P(X < 3)$

The possible outcomes are:

0 1 2 3 4 5 6 7 8 9 10

Therefore

$$\begin{aligned}P(X < 3) &= P(X = 0) + P(X = 1) + P(X = 2) \\&= P(X \leq 2) \\&= 0.677.\end{aligned}$$

Example 4(d): $P(X > 1)$

The possible outcomes are:

0 1 2 3 4 5 6 7 8 9 10

Example 4(d): $P(X > 1)$

The possible outcomes are:

0 1 2 3 4 5 6 7 8 9 10

Therefore

$$P(X = 0) = {}^{10}C_0 \times 0.2^0 \times 0.8^{10} = 0.107$$

$$P(X = 1) = {}^{10}C_1 \times 0.2^1 \times 0.8^9 = 0.268$$

Therefore

$$P(X > 1) = 1 - (0.107 + 0.268) = 0.625.$$

7.2.2 Mean and variance

If $X \sim \text{Bin}(n, p)$, then its mean (or “expected value”) and variance are

$$\begin{aligned} E[X] &= n \times p & \text{and} \\ \text{Var}(X) &= n \times p \times (1 - p). \end{aligned}$$

Example 5

If $X \sim \text{Bin}(10, 0.2)$ calculate:

- (a) $E[X]$
- (b) $\text{Var}(X)$
- (c) $\text{SD}(X)$

$$E[X] = 10 \times 0.2 = 2$$

$$\text{Var}(X) = 10 \times 0.2 \times 0.8 = 1.6$$

$$\text{SD}(X) = \sqrt{1.6} = 1.265.$$

Example 6

A salesperson has a 50% chance of making a sale on a customer visit and she arranges 6 visits in a day.

- (a) Assuming sales at each visit are independent, suggest an appropriate distribution for the number of sales she makes in a day.
- (b) Calculate her expected number of sales.

Example 6

(a) X : Number of sales per day; $X \sim \text{Bin}(6, 0.5)$.

(b)

$$E[X] = 6 \times 0.5 = 3 \text{ sales.}$$



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Announcements

- Back to standard classroom-based tutorials this week, with a twist...
- The class on Thursday at 11, with Ged, will be in **KGVI LT4** this week, not the usual LT1! (My classes at 1 and 2 remain unchanged)
- The **semester 1 written assignment** is now available to download from the course webpage – **deadline: 4pm, Thursday 14th December**

Lecture 8

MORE DISCRETE PROBABILITY MODELS

8.1 The Poisson distribution

The Poisson distribution is another very important discrete probability distribution.

- 1 It is often used to model **count data**
- 2 Unlike the binomial distribution, there is **no known fixed upper limit** to the number of events
- 3 The **rate** of occurrence, λ , is the parameter here – we assume events occur independently, with constant rate λ

If these conditions are reasonable, then we say the number of events, X , occurring in a given interval, has a **Poisson distribution** with parameter λ .

Example 1

Which of the following random variables could be modelled by a Poisson distribution? Suggest an alternative if the Poisson distribution is not appropriate, and state the values of any parameters.

- (a) Calls are received at a call centre at a constant rate of 3 per minute on average. Let X be the number of calls received in a 1 minute period.
- (b) An operator at a tele-sales marketing firm has 20 calls to make in an hour. History suggests that calls will be answered 55% of the time. Let Y be the number of answered calls in an hour.
- (c) Newcastle United score goals at a constant rate of 2.4 in 90 minutes, on average. Let Z be the number of goals scored in 45 minutes.

Example 1

X : Number of calls received in a 1 minute period

- Could be Poisson: we have count data, we have a fixed rate of occurrence (3 per minute) and we could assume independent events
- We have $\lambda = 3$

Y : Number of answered calls in an hour

- Cannot be Poisson: We have no rate of occurrence (λ), and there is an upper limit to Y (20)
- The **binomial distribution** could be used: we have a fixed number of independent trials, each with two outcomes (“success” and “failure”), and we have a probability of success
- Specifically, we have $Y \sim \text{Bin}(n = 20, p = 0.55)$.

Example 1

Z: Number of goals scored in 45 minutes

- Could be Poisson: we have count data and we have a fixed rate of occurrence (2.4 per 90 minutes)
- We have $\lambda = 1.2$

8.1.1 Probabilities, means and variances

If X follows a Poisson distribution we write $X \sim \text{Po}(\lambda)$, and

$$P(X = r) = \frac{\lambda^r e^{-\lambda}}{r!}, \quad r = 0, 1, \dots$$

If $X \sim \text{Po}(\lambda)$, then

$$\begin{aligned} E[X] &= \lambda \quad \text{and} \\ \text{Var}(X) &= \lambda. \end{aligned}$$

Example 2

If $X \sim \text{Po}(5)$ calculate:

- (a) $P(X = 4)$
- (b) $P(X \leq 1)$
- (c) $P(X > 0)$
- (d) $E[X]$
- (e) $\text{SD}(X)$

Example 2

We know that

$$P(X = r) = \frac{\lambda^r e^{-\lambda}}{r!},$$

and we have $\lambda = 5$.

(a)

$$P(X = 4) = \frac{5^4 e^{-5}}{4!} = 0.175.$$

(b)

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \frac{5^0 e^{-5}}{0!} + \frac{5^1 e^{-5}}{1!} \\ &= 0.0067 + 0.0337 = 0.0404. \end{aligned}$$

Example 2

(c)

$$\begin{aligned}P(X > 0) &= 1 - P(X = 0) \\&= 1 - 0.0067 = 0.9933.\end{aligned}$$

(d)

$$E[X] = \lambda = 5.$$

(e)

$$\text{Var}(X) = \lambda = 5.$$

Therefore

$$\text{SD}(X) = \sqrt{5} = 2.236.$$

Example 3

A new Mercedes-Benz car franchise forecasts that it will sell around three of its most expensive models each day.

- (a) What probability distribution might be reasonable to use to model the number of cars sold each day?
- Cannot be binomial - we don't have the probability of success or the number of trials (i.e. there is no known fixed upper limit to the number of cars sold each day)
 - Could be Poisson - we have a rate of occurrence

So X : Number of expensive models sold each day

$$X \sim \text{Po}(3)$$

Example 3

- (b) What is the expected number and standard deviation of the number of cars sold each day?

$$E[X] = \lambda = 3 \text{ cars per day.}$$

Also

$$\text{Var}(X) = \lambda = 3,$$

and so

$$SD(X) = \sqrt{3} = 1.732 \text{ cars per day}$$

Example 3

- (c) What is the probability that 3 cars are sold on a particular day?

$$P(X = 3) = \frac{3^3 e^{-3}}{3!} = 0.224.$$

- (d) What is the probability that no cars are sold on a particular day?

$$P(X = 0) = \frac{3^0 e^{-3}}{0!} = 0.0498.$$

- (e) What is the probability that at least one car is sold on a particular day?

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - 0.0498 = 0.9502. \end{aligned}$$

Example 3

- (f) Sales will be monitored over the next seven days and the sales team at the franchise will receive a warning if they make no sales on at least 1 of the 7 days. What is the probability that they receive a warning?

Let Y : Number of days on which zero sales are made

$$Y \sim \text{Bin}(7, 0.0498).$$

Then

$$\begin{aligned} P(Y \geq 1) &= 1 - P(Y = 0) \\ &= 1 - {}^7C_0 \times 0.0498^0 \times 0.9502^7 \\ &= 1 - 0.699 = 0.301. \end{aligned}$$

Extra example

Recall the example at the start of the lecture last week, used to help motivate the study of probability models.

Let X : Number of cars observed every half an hour over a five hour period. We have

2 3 2 5 5 3 4 5 6 7

This gives

x	$P(X = x)$	Poisson
< 2	$\frac{0}{10} = 0$	0.078
2	$\frac{2}{10} = 0.2$	0.132
3	$\frac{2}{10} = 0.2$	0.185
4	$\frac{1}{10} = 0.1$	0.194



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Announcements: Written assignment (mini project)

You should be working on this now; worth 10% of the module;
deadline for submission: **4pm, Thursday 14th December**

Hints:

- Graphs: when comparing two or more groups, use the same scales (e.g. percentage rel. freq. histograms or polygons, x-axes etc.)...
- ...and where appropriate overlay graphs on the same panel
- 'Produce *appropriate* graphical / numerical summaries...':
Numerical – one measure of average + one measure of spread per dataset; Graphical – one or two at most per dataset
- Comments: Average? Where does the graph 'peak'?
Spread / dispersion? Outliers? Symmetric / asymmetric distribution? Normal distribution?

Announcements: Written assignment (mini project)

Submission:

- Hard-copy, posted through the **Stage 1 homework submission letterbox** on the 3rd floor of the **Herschel Building**
- Must have a personalised NESS cover sheet attached
- Personalised datasets for question 2!
- Marks for presentation: Type up solutions in WORD?

Announcements

- CBA3: Will go live at 00:01 this coming Saturday, 2nd December in both practice and assessed modes
- Deadline: **23:59 Friday 15th December**

Lecture 9

CONTINUOUS PROBABILITY MODELS

9. Continuous probability models

We have seen how **discrete** random variables can be modelled by discrete probability distributions such as the **binomial** and **Poisson** distributions.

We now consider how to model **continuous** random variables.

9. Continuous probability models

A variable is **discrete** if it takes a **countable** number of values.

For example,

- the number of **blue** cars that I count in a 5 minute period
- the number of **heads** observed when I flip a coin ten times
- Shoe sizes: $1, \dots, 12, 13, 1, 2, \dots$
- $r = 0, 0.1, 0.2, \dots, 0.9, 1.0$

In contrast, the values which a **continuous variable** can take form a **continuous scale**, with no “jumps”.

For example,

- Height
- Weight
- Temperature

An example

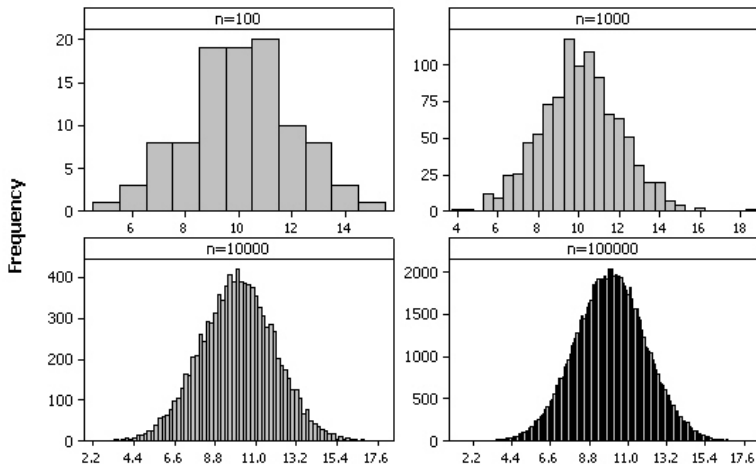
Think about **height**.

- In practice, we might only record height to the **nearest cm**
- If we could measure height *exactly* we'd find that everyone had a **different height**
- This is the essential difference between discrete and continuous variables
- If there are n people on the planet, the probability that someone's height is x would be $\frac{1}{n}$
- As n gets bigger and bigger, this probability tends to zero!!

An example

Consider taking a sample of values from the continuous random variable X .

Histograms of continuous r.v. for increasing sample sizes



An example

- As the sample size gets bigger, the **interval widths get smaller**
- the jagged profile of the histogram **smooths out** to become a curve
- When the sample size is infinitely large, this curve is known as the **probability density function** (pdf)

Features of the probability density function

The key features of pdfs are:

- 1 the area under a pdf is one: $P(-\infty < X < \infty) = 1$
- 2 areas under the curve correspond to probabilities
- 3 $P(X \leq x) = P(X < x)$ since $P(X = x) = 0$.

9. Continuous probability models

Over the next two weeks we will consider some particular probability distributions that are often used to describe continuous random variables.

We start with the most **important**, most **widely-used** statistical distribution of all time...

...wait for it...

The Normal Distribution

9.1 The Normal distribution

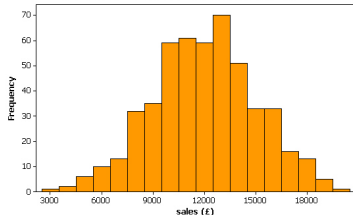
The **Normal distribution** is without doubt the most widely-used statistical distribution in many practical applications:

- Normality arises **naturally** in many physical, biological and social measurement situations
- Normality is important in **Statistical inference** (see *Semester 2 material*)
- The normal distribution has many guises:
 - Gaussian distribution
 - Laplacean distribution
 - “bell-shaped curve”

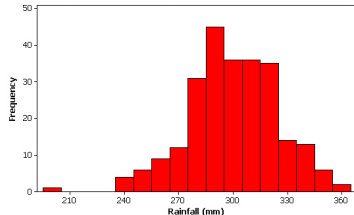


Some real-life examples

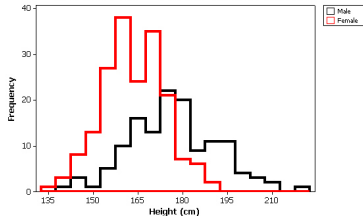
Histogram of weekly sales at 500 branches of HMV



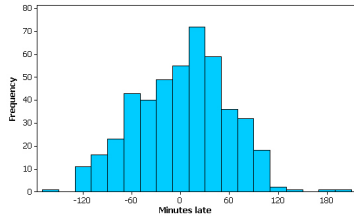
Histogram of monthly rainfall totals at 250 locations in the USA



Histograms to show the heights of MAS1403 students



Histogram to show arrival times of flights at Heathrow



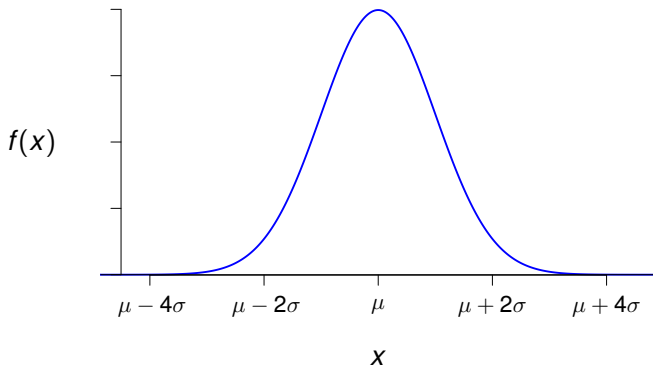
9.1 The Normal distribution

Recall the “parameters” of the binomial and Poisson distributions:

- The **binomial distribution** has two parameters, n and p
- the **Poisson distribution** has one parameter λ
- The **Normal distribution** has two parameters: the mean, μ , and the standard deviation, σ

9.1 The Normal distribution

The probability density function (pdf) of the Normal distribution has a “**bell-shaped**” profile:



9.1 The Normal distribution

We can think of the pdf as a smoothed percentage relative frequency histogram: **the area under the curve is 1**.

The (rather nasty!) formula for this pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}.$$

Unlike the binomial and Poisson distributions, there is **no simple formula** for calculating probabilities.

Don't worry though, probabilities from the Normal distribution can be determined using **statistical tables** (see page 51) or statistical packages such as `Minitab`.

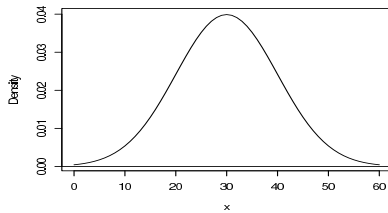
Characteristics of the Normal distribution

There are four important characteristics of the Normal distribution:

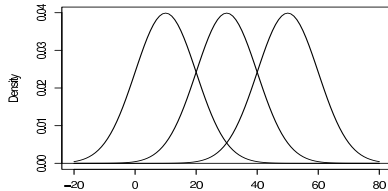
- 1 It is **symmetrical** about its mean, μ .
- 2 The mean, median and mode all **coincide**.
- 3 The area under the curve is equal to 1.
- 4 The curve extends in both directions to infinity (∞).

On the next slide are plots of the pdf for Normal distributions with different values of μ and σ .

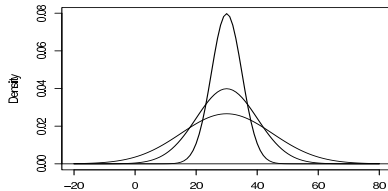
Normal pdf with mean 30 and sd 10



Normal pdfs with mean 10, 30, 50 and sd 10



Normal pdfs with mean 30 and sds 5, 10, 15



Notation

If a random variable X has a Normal distribution with mean μ and variance σ^2 , then we write

$$X \sim N(\mu, \sigma^2).$$

For example, a random variable X which follows a Normal distribution with mean **10** and variance **25** is written as

$$\begin{aligned} X &\sim N(10, 25) && \text{or} \\ X &\sim N(10, 5^2). \end{aligned}$$

It is important to note that the second parameter in this notation is the **variance** and not the **standard deviation**.

9.1.1 The *standard* Normal distribution

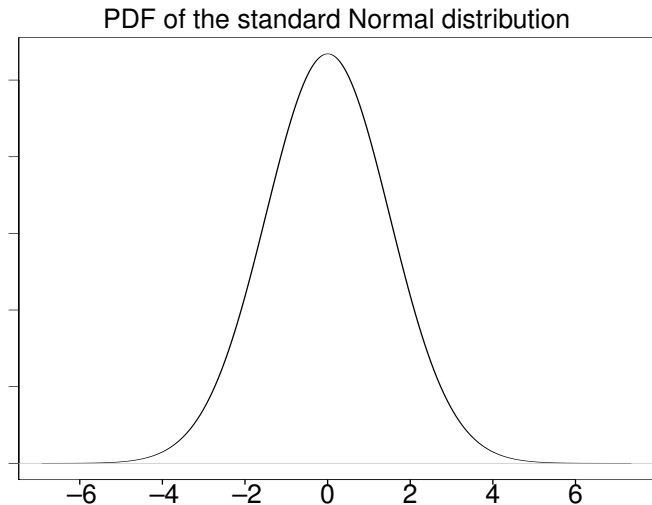
The **Standard** Normal distribution has a mean of **0** and a variance of **1**.

A random variable with this *standard* Normal distribution is usually given the letter Z , and so we say

$$Z \sim N(\mathbf{0}, \mathbf{1}).$$

If our random variable follows a **standard** Normal distribution, then we can obtain *cumulative probabilities* from statistical tables (see page 51 of the notes), which give “**less than or equal to**” probabilities.

Probability density function for Z



Example 1

For example, if $Z \sim N(0, 1)$:

- (a)** The probability that Z is less than or equal to -1.46 is $P(Z \leq -1.46)$. Therefore we look for the probability in tables corresponding to $z = -1.46$: row labelled -1.4 , column headed -0.06 .

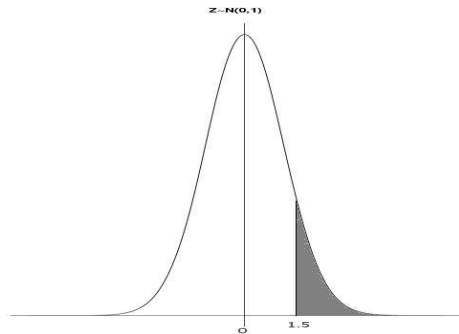
This gives $P(Z \leq -1.46) = \mathbf{0.0721}$.

- (b)** The probability that Z is less than or equal to 0.01 is $P(Z \leq 0.01)$. Therefore we look for the probability in tables corresponding to $z = 0.01$: row labelled 0.0 , column headed 0.01 .

This gives $P(Z \leq 0.01) = \mathbf{0.5040}$.

Example 1

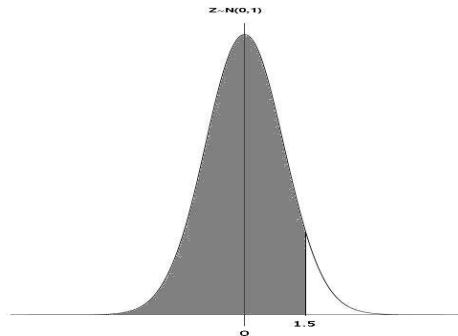
- (c) The probability that Z is greater than 1.5 is $P(Z > 1.5)$.
Now our tables give “less than” probabilities, and here we want a “greater than” probability.



So we find $P(Z < 1.5) = 0.9332$ and subtract this from 1 to give **0.0668**.

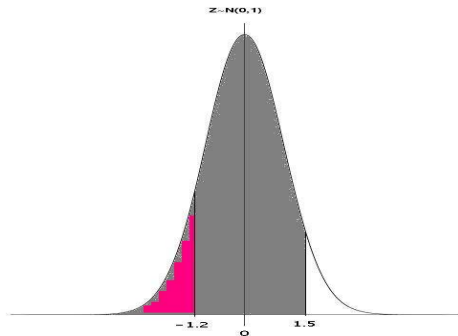
Example 1

- (d) What about the probability that Z lies between -1.2 and 1.5 ? It helps to think about this graphically.



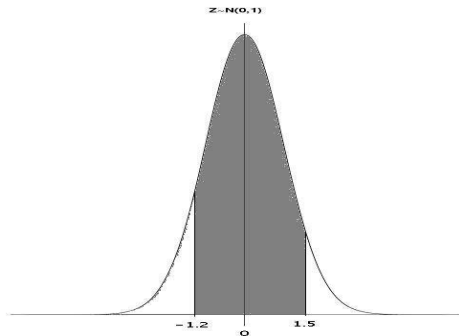
Example 1

- (d) What about the probability that Z lies between -1.2 and 1.5 ? It often helps to think about this graphically.



Example 1

- (d) What about the probability that Z lies between -1.2 and 1.5 ? It often helps to think about this graphically.



Doing so, gives

$$\begin{aligned} P(-1.2 < Z < 1.5) &= P(Z < 1.5) - P(Z \leq -1.2) \\ &= 0.9332 - 0.1151 \end{aligned}$$

Example 1

(e)

$$\begin{aligned}P(Z < 1.5) &= 1 - P(Z > 1.5) \\&= 1 - 0.0668 \quad \text{From part (c)} \\&= 0.9332.\end{aligned}$$

9.1.2 Probabilities from *any* Normal distribution

So how do we calculate probabilities for **any** Normal distribution, not just the **standard** Normal distribution (for which we have tables)?

Idea: “make” the Normal distribution that we have “look like” the standard Normal distribution, and then we can just use the tables as before!

But how? Use the **slide-squash** technique!

9.1.2 Probabilities from *any* Normal distribution

The formula which changes **any** Normal random variable X into the **standard** Normal random variable Z is given by

$$Z = \frac{X - \mu}{\sigma},$$

where

- μ is the mean
- σ is the standard deviation

This can be translated into probability statements:

$$P(X \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right),$$

which can be looked up in tables.

Example 2

If $X \sim N(10, 2^2)$, calculate $P(X \leq 8)$.

Translate X into Z using the slide-squash rule:

$$\begin{aligned} Z &= \frac{X - \mu}{\sigma} \\ &= \frac{8 - 10}{2} \\ &= -1. \end{aligned}$$

Then, from the table on page 51,

$$P(Z \leq -1) = 0.1587.$$

Example 3

Suppose X is the IQ of a randomly selected 18–19 year old and that X follows a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$. Thus, we have:

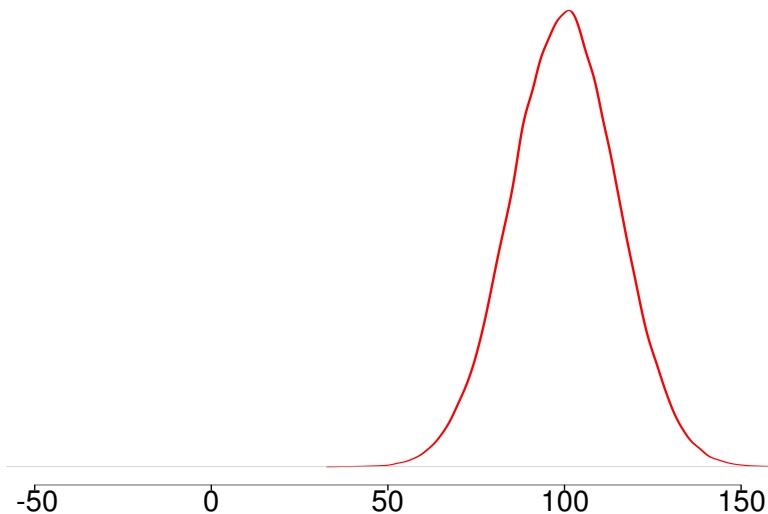
$$X \sim N(100, 15^2).$$

Find the following probabilities.

- (a) The probability that an 18–19 year old has an IQ less than 110.
- (b) The probability that an 18–19 year old has an IQ greater than 110.
- (c) The probability that an 18–19 year old has an IQ greater than 125.
- (d) The probability that an 18–19 year old has an IQ between 95 and 115.

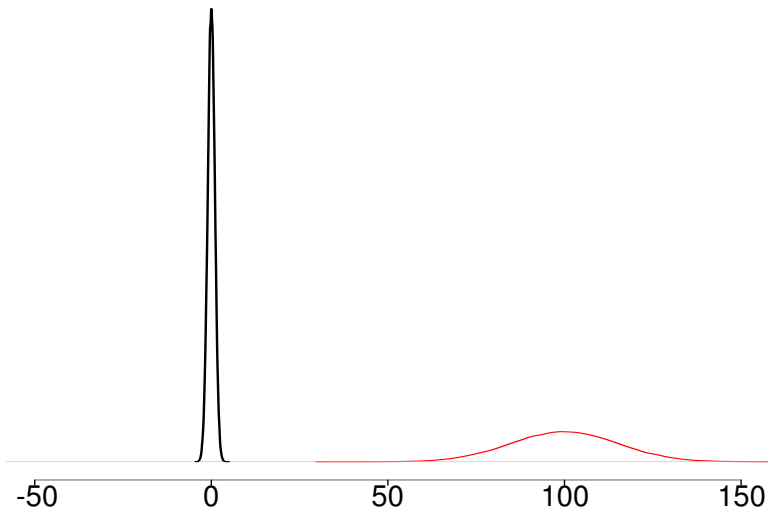
Example 3

Distribution of IQs



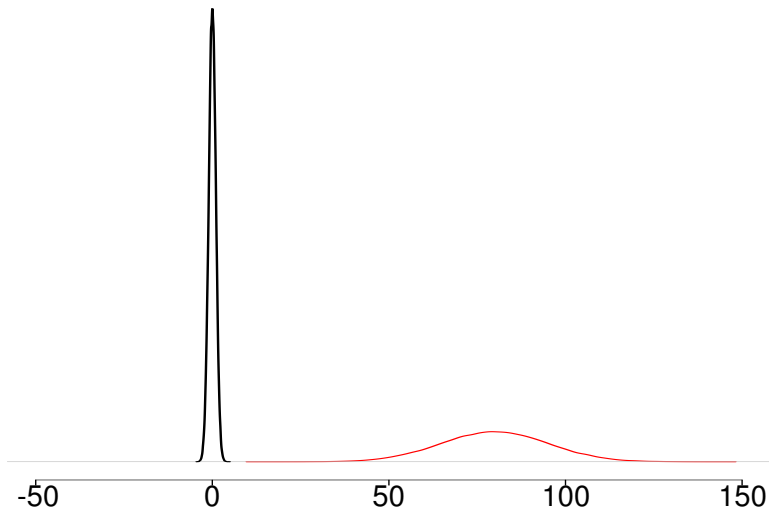
Example 3

Slide-squash



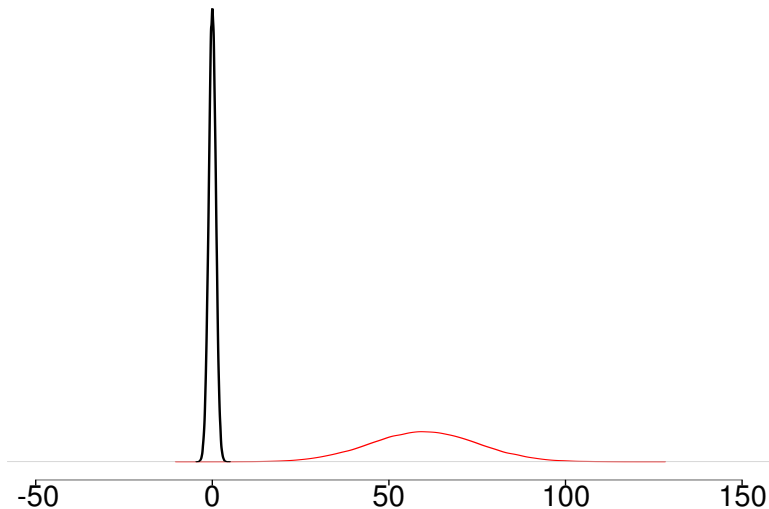
Example 3

Slide-squash



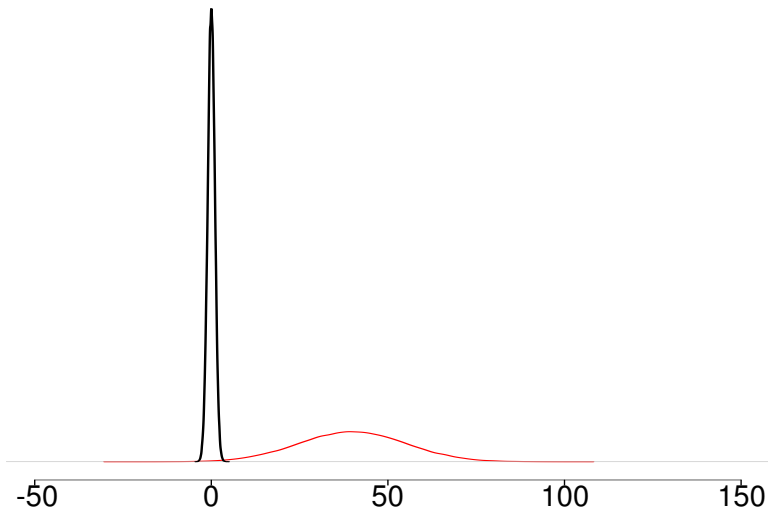
Example 3

Slide-squash



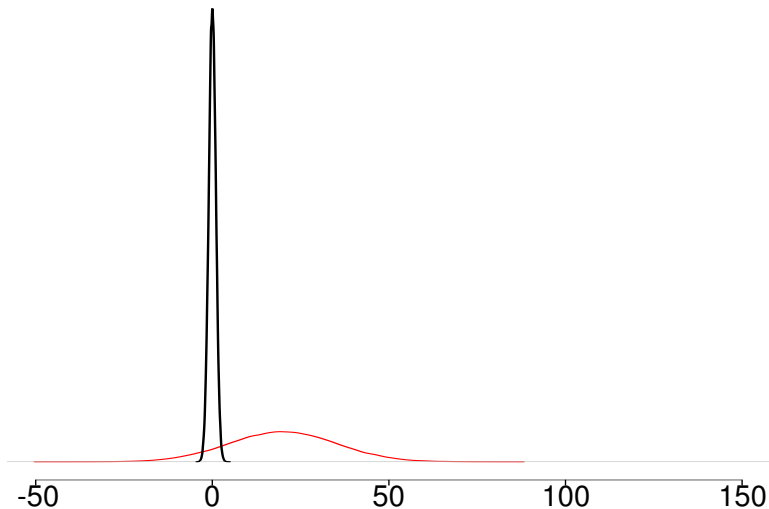
Example 3

Slide-squash



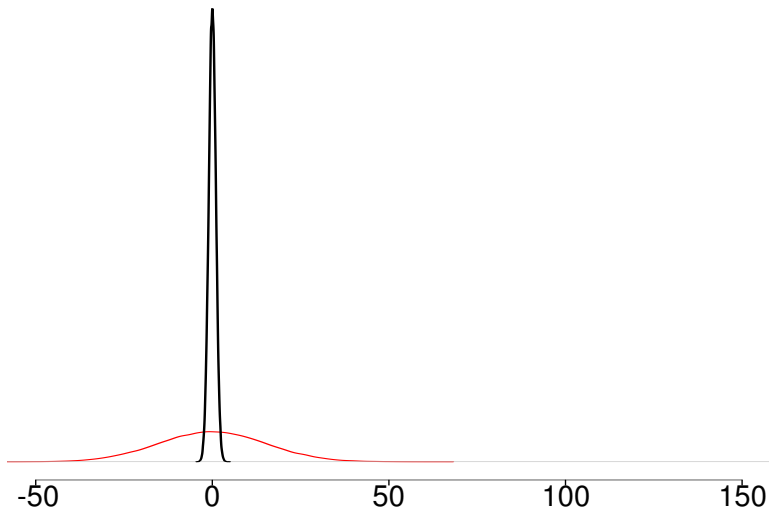
Example 3

Slide-squash



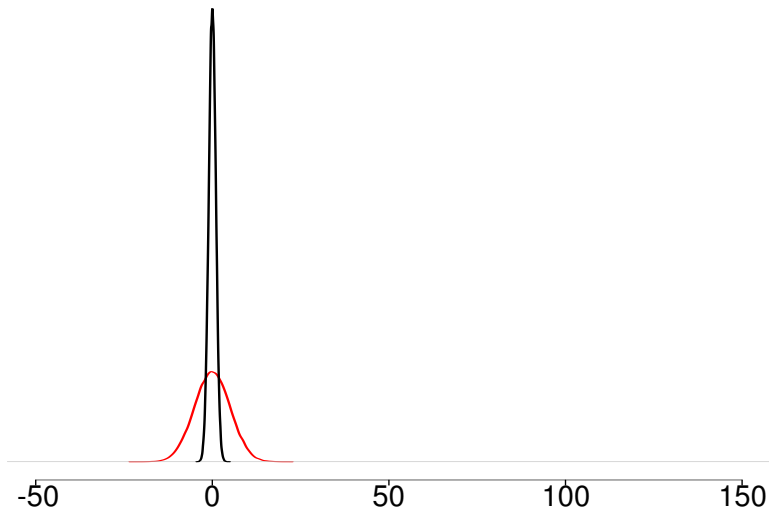
Example 3

Slide-squash



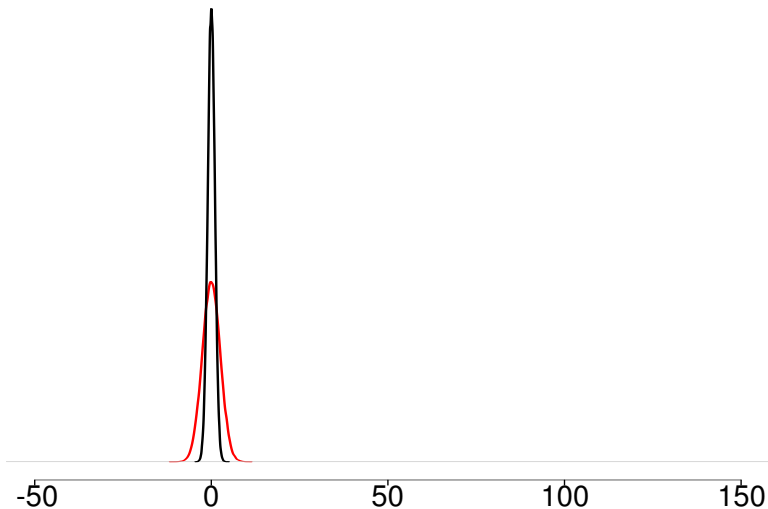
Example 3

Slide-squash



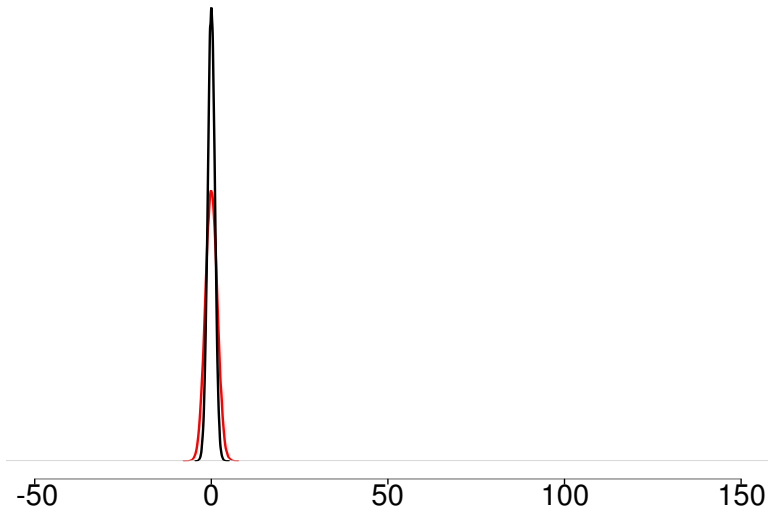
Example 3

Slide-squash



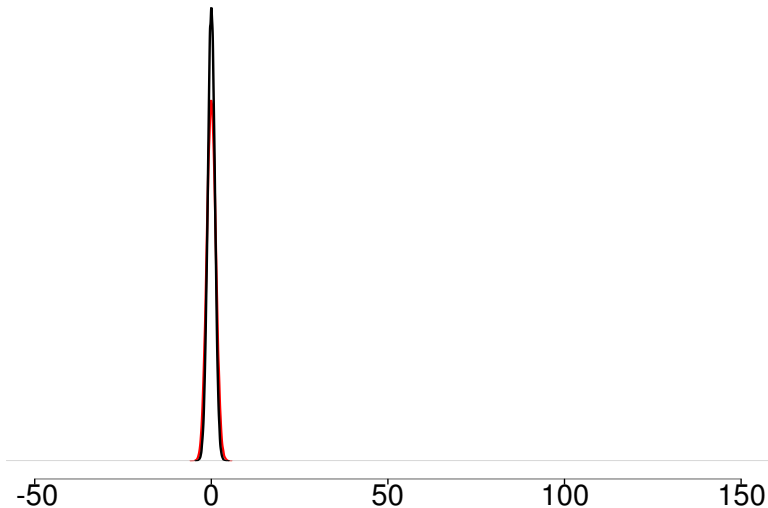
Example 3

Slide-squash



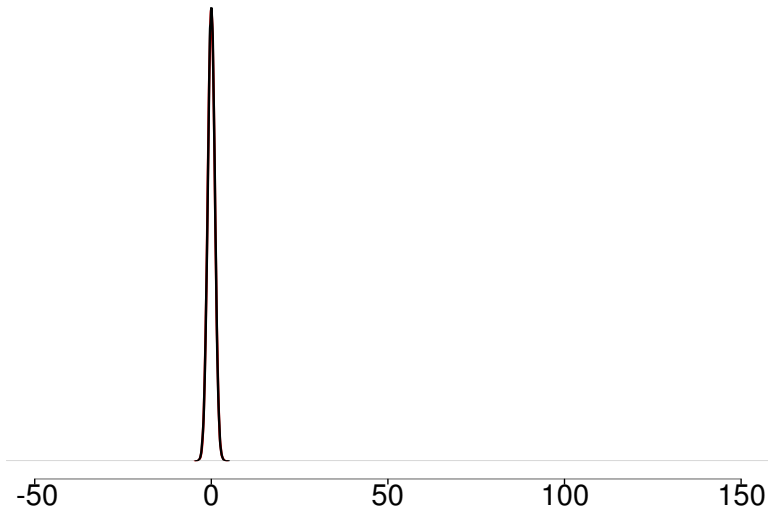
Example 3

Slide-squash



Example 3

Slide-squash



Example 3 (a)

$$\begin{aligned}P(X < 110) &= P\left(Z < \frac{X - \mu}{\sigma}\right) \\&= P\left(Z < \frac{110 - 100}{15}\right) \\&= P(Z < 0.67) \\&= 0.7486.\end{aligned}$$

Example 3 (b)

$$\begin{aligned}P(X > 110) &= 1 - P(X < 110) \\&= 1 - 0.7486 \\&= 0.2514.\end{aligned}$$

Example 3 (c)

$$\begin{aligned}P(X > 125) &= 1 - P(X < 125) \\&= 1 - P\left(Z < \frac{125 - 100}{15}\right) \\&= 1 - P(Z < 1.67) \\&= 1 - 0.9525 \\&= 0.0475.\end{aligned}$$

Example 3 (d)

$$\begin{aligned}P(95 < X < 115) &= P(X < 115) - P(X < 95) \\&= P\left(Z < \frac{115 - 100}{15}\right) - P\left(Z < \frac{95 - 100}{15}\right) \\&= P(Z < 1) - P(Z < -0.33) \\&= 0.8413 - 0.3707 \\&= 0.4706.\end{aligned}$$



MAS1403

Quantitative Methods for Business Management

Semester 1, 2017–2018

Module leader: Dr. Lee Fawcett

Additional lecturers: Dr. Dave Walshaw and Dr. Ged Cowburn

Semester 1 nearly over!

This week

- CBA3 in practice mode **and** assessed mode
- Should now be working through written assignment

Next week

- Written assignment due in Thursday
- CBA3 due in Friday
- Lecture running as normal: Drop-in for last-minute help with assignment

Monday 8th January 2018

- Last week of semester 1
- Revision week... but no January exam, so lectures cancelled for this module!

Semester 2 starts Monday 29th January 2018

Lecture 10

MORE CONTINUOUS PROBABILITY MODELS

10.1 The normal distribution: using tables in reverse

Last week we looked at the Normal distribution as a probability model for **continuous random variables**.

As a refresher, suppose X : IQ of a randomly selected 18-19 year old and that $X \sim N(100, 15^2)$.

1. What is the mean IQ, μ ?
2. What is the standard deviation, σ ?
3. What is the probability that an 18-19 year old has an IQ greater than 100?
4. What is the probability that an 18-19 year old has an IQ less than 120?
5. Below what IQ are 95% of the population?

10.1 The normal distribution: using tables in reverse

1. $\mu = 100$
2. $\sigma = 15$
3. $P(X > 100) = 0.5$
- 4.

$$\begin{aligned}P(X < 120) &= P\left(Z < \frac{120 - 100}{15}\right) \\&= P(Z < 1.33) \\&= 0.9082 \quad (\text{tables page 51})\end{aligned}$$

10.1 The normal distribution: using tables in reverse

5. Below what IQ are 95% of the population?

From tables on page 51, we find that

$$P(Z \leq 1.64) = 0.9495 \quad \text{and}$$

$$P(Z \leq 1.65) = 0.0505.$$

Therefore,

$$P(Z \leq \mathbf{1.645}) = 0.95 = 95\%.$$

Now that's on the Z -scale, and we know that:

$$Z = \frac{x - \mu}{\sigma}$$

$$1.645 = \frac{x - 100}{15}$$

$$1.645 \times 15 = x - 100$$

$$1.645 \times 15 + 100 = x$$

and so $x = 124.7 \approx 125$.

Over the past few weeks we have talked about some “**standard**” probability distributions which can be used to model data. So far, we have looked at:

1. Discrete distributions

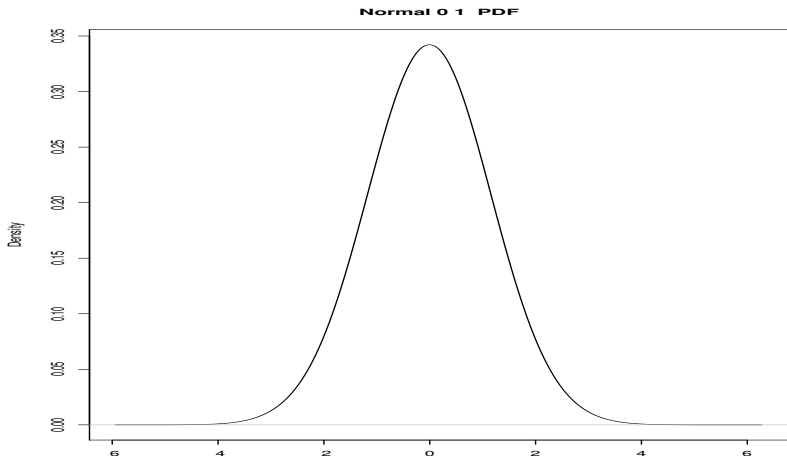
- The **Binomial** distribution
- The **Poisson** distribution

2. Continuous distributions

- The **Normal** distribution

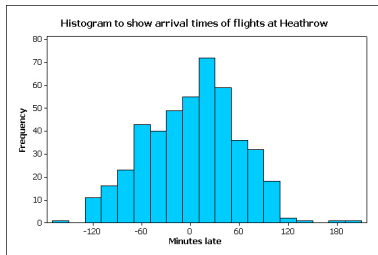
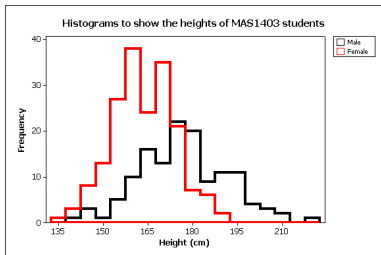
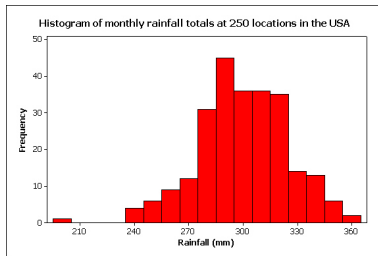
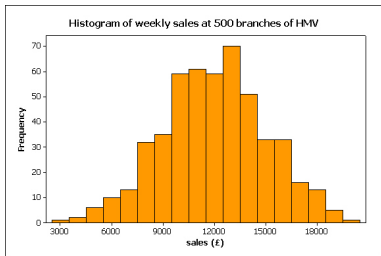
Other probability models

Recall the **probability density function** of the Normal distribution, which is often referred to as a “**bell-shaped curve**”:



Other probability models

Recall also from last week that many naturally occurring measurements seem to follow this distribution:



Example of “non–Normality”

- You manage a group of Environmental Health Officers and need to decide at what time they should inspect a local hotel
- You decide that any time during the working day (9.00 to 18.00) is okay
- You want to decide the time “randomly”

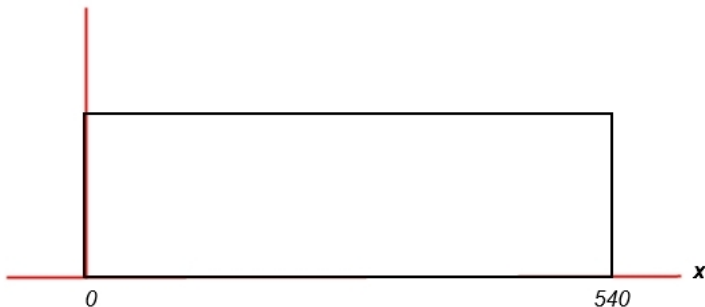
Here, “**randomly**” is a short–hand for

“a random time, where all times in the working day are equally likely to be chosen”

10.2 The Uniform distribution

Let X be the time to their arrival at the hotel, measured in terms of minutes from the start of the day.

Then X is a **Uniform** random variable between 0 and 540:



10.2 The Uniform distribution

As with the Normal distribution, the total area (base $\times$ height) under the pdf must equal one.

Therefore, as the base is 540, the height must be $1/540$.

Hence the **probability density function** (pdf) for the continuous random variable X is

$$f(x) = \begin{cases} \frac{1}{540} & \text{for } 0 \leq x \leq 540 \\ 0 & \text{otherwise.} \end{cases}$$

10.2 The Uniform distribution

In general, we say that a random variable X which is equally likely to take any value between a and b has a **uniform distribution** on the interval a to b , i.e.

$$X \sim U(a, b).$$

The random variable has **probability density function** (pdf)

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and probabilities can be calculated using the formula

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b. \end{cases}$$

10.2 The Uniform distribution

Therefore, for example, the **probability that the inspectors visit the hotel in the morning** (within 180 minutes after 9am) is

$$P(X \leq 180) = \frac{180 - 0}{540 - 0} = \frac{1}{3}.$$

The **probability of a visit during the lunch hour** (12.30 to 13.30) is

$$\begin{aligned} P(210 \leq X \leq 270) &= P(X \leq 270) - P(X < 210) \\ &= \frac{270 - 0}{540 - 0} - \frac{210 - 0}{540 - 0} \\ &= \frac{270 - 210}{540} \\ &= \frac{60}{540} = \frac{1}{9}. \end{aligned}$$

10.2 The Uniform distribution

Recall that:

- If $X \sim \text{bin}(n, p)$, then
 - $E(X) = n \times p$ and
 - $\text{Var}(X) = n \times p \times (1 - p)$
- If $X \sim \text{Po}(\lambda)$, then
 - $E(X) = \lambda$ and
 - $\text{Var}(X) = \lambda$

We have equivalent formulae for $X \sim U(a, b)$:

$$\begin{aligned} E(X) &= \frac{a + b}{2} \\ \text{Var}(X) &= \frac{(b - a)^2}{12}. \end{aligned}$$

10.2 The Uniform distribution

In the above example, we have

$$E(X) = \frac{a+b}{2} = \frac{0+540}{2} = 270,$$

so that the mean arrival of the inspectors is 9am+270 minutes = 13.30.

Also

$$\text{Var}(X) = \frac{(540-0)^2}{12} = 24300,$$

and therefore $SD(X) = \sqrt{\text{Var}(X)} = \sqrt{24300} = 155.9$ minutes.

10.3 The Exponential Distribution

The **exponential distribution** is another common distribution that is used to describe continuous random variables.

It is often used to model lifetimes of products and times between “random” events, for example:

- Arrival of customers in a queueing system
- Arrival of orders

10.3 The Exponential Distribution

The distribution has one parameter, λ . If our random variable X follows an **exponential distribution**, then we say

$$X \sim \exp(\lambda).$$

Its **probability density function** is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

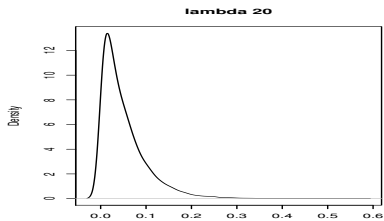
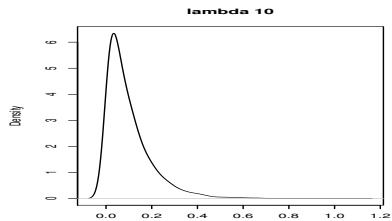
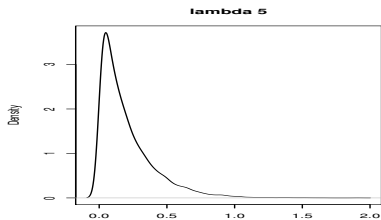
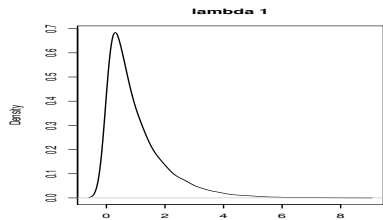
and probabilities can be calculated using

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x > 0. \end{cases}$$

10.3 The Exponential Distribution

The main **features** of this distribution are:

- 1 an exponentially distributed random variable can only take **positive** values
- 2 larger values are **increasingly unlikely** – “**exponential decay**”
- 3 the value of λ fixes the **rate of decay** – *larger values correspond to more rapid decay.*



10.3 The Exponential Distribution

Consider an example in which the time (in minutes) between successive users of a pay phone can be modelled by an exponential distribution with $\lambda = 0.3$.

The probability of the gap between phone users being less than 5 minutes is

$$P(X < 5) = 1 - e^{-0.3 \times 5} = 1 - 0.223 = 0.777.$$

Also the probability that the gap is more than 10 minutes is

$$P(X > 10) = 1 - P(X \leq 10) = 1 - (1 - e^{-0.3 \times 10}) = e^{-0.3 \times 10} = 0.050$$

and the probability that the gap is between 5 and 10 minutes is

$$P(5 < X < 10) = P(X < 10) - P(X \leq 5) = 0.950 - 0.777 = 0.173.$$

Mean and Variance

The mean and variance of the exponential distribution can be shown to be

$$E(X) = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}.$$

10.3.1 Poisson process

One of the main uses of the exponential distribution is as a model for the **times between events occurring randomly in time**.

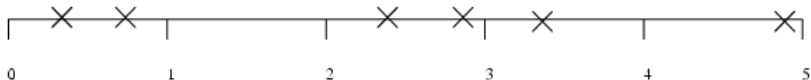
We have previously considered events which occur at random points in time in connection with the **Poisson distribution**.

The Poisson distribution describes probabilities for the number of events taking place in a given time period.

The exponential distribution describes probabilities for the times between events. Both of these concern events occurring randomly in time (at a constant average rate, say λ). This is known as a **Poisson process**.

10.3.1 Poisson process

Consider a series of randomly occurring events such as calls at a credit card call centre. The times of calls might look like



We can view these data in **two ways**:

- The number of calls in each minute (here 2, 0, 2, 1 and 1)
- the times between successive calls

10.3.1 Poisson process

For the **Poisson process**,

- the number of calls has a **Poisson** distribution with parameter λ , and
- the time between successive calls has an **exponential** distribution with parameter λ .