Summer Project

School of Mathematics, Statistics & Physics

Summer 2019

Workbook 5

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Contents

1 Estimating square roots **4**
1.1 Introduction 4
1.2 Simplifying square roots 5
1.3 Definition of a surd 6
1.4 Estimating the square roots of large numbers 6

2 Egyptian fractions **9**
2.1 Introduction 9
2.2 Different representations of the same fraction 9
2.3 Practical applications 11

3 Written assignment **12**

4 Research question **15**

Introduction

This project is intended to give current GCSE students (or equivalent) an opportunity to demonstrate their mathematical skills, and to give such students a taste of some new material which most will not have seen before. This document consists of two sections of material not directly covered by the GCSE syllabus. Each contains a detailed description of new concepts to provide assistance when preparing solutions to questions posed in the project. This project consists of three components:

- A written assignment sheet
- Some computer based questions using our NUMBAS online testing system
- A more open ended, research-style question

The questions are designed to be of varying difficulty to challenge students at different skill levels and we would expect a good student to take up to five days to complete all components. Most questions will involve concepts beyond the standard GCSE syllabus and we strongly recommend reading through the information in this booklet carefully before attempting the questions. Completed assignments will be graded and successful students will be awarded with a certificate and general feedback. More details about this summer project can be found on our dedicated webpage:
`http://www.mas.ncl.ac.uk/~nlf8/summer-project/`.

Written assignment

There are eight questions to work through here. Most of these questions will be based on material covered in this booklet. However, there may be one or two surprises! Solutions to these questions can be hand-written or typed. Hand-written solutions should be scanned and emailed to `maths.physics@ncl.ac.uk`. Solutions prepared in a word-processing package (eg. Microsoft Word) can be emailed directly, although we recommend exporting to portable document format (PDF) first, to help prevent loss of formatting. Students must submit their solutions by **4pm, Monday 9th September GMT**.

Computer based test

This uses Newcastle University's *NUMBAS online testing system*. The test will be available in two "modes"; *PRACTICE* and *ASSESSED*. Students can access the practice test as many times as they like. Each time they open a test, the format of the questions within that particular test will remain the same, but the numbers might change. The assessed mode will be active for two weeks only, from **Monday 26th August–Monday 9th September**. Students must login and complete the test by **4pm, Monday 9th September GMT**. Only **one** attempt will be allowed, but students can pause the test as often as they wish – there are no time constraints. The practice mode will be available throughout August and September. Students are strongly advised to work through our *Introduction to NUMBAS* first – available at
`https://numbas.mathcentre.ac.uk/test-yourself/getting-started/`. A link to the actual test, and further instructions about submission of the test will be available on our webpage soon.

Research-style question

This is the final part of the project. Students will be given a choice of three questions, from which they are required to attempt only **one**. We expect solutions to any of these questions to be one or two pages long. Students can pick from a range of topics for this question, including options from statistics or number theory. Most questions do not have strict guidelines and we encourage students to be creative with their answers. As with the written assignment, solutions to the open-ended questions can be either hand-written or electronically produced with a word processing package. Answers should be emailed to `maths.physics@ncl.ac.uk` no later than **4pm, Monday 9th September GMT**.

1 Estimating square roots

1.1 Introduction

The square root of a number n is defined as the number you multiply with itself to get n. Put mathematically, $\sqrt{n} \times \sqrt{n} = n$. Working out the square root of a square number such as $1, 4, 9, 25 \cdots$ is fairly straightforward as the square root is a nice whole number. However, you only have to ask a very simple question before things start getting considerably more complicated. For example, what is $\sqrt{2}$? You may be aware that this is an irrational number with infinity many decimal digits; it is approximately equal to 1.414. How can this be calculated?

Let us start with a slightly different example: what is $\sqrt{12}$? The logical first step is to think about the squares that we know? For example, we know $3^2 = 9$ and $4^2 = 16$, and so $\sqrt{9} = 3$ and $\sqrt{16} = 4$. This means $\sqrt{12}$ will have to be between 3 and 4. 12 is slightly closer to 3^2 than 4^2 so it would be sensible to guess that $\sqrt{12} \approx 3.4$. Lets look at a mathematical technique to calculate the square root to a chosen degree of accuracy.

- **Step one:**
  What is the biggest square number that is less than 12? In this case it is $9 = 3^2$.

- **Step two:**
  Divide 12 by the square root of that square number. The square root of 9 is 3 and so we calculate $\frac{12}{3} = 4$.

- **Step three:**
  Find the mean value between the first square root (step 1) and the quotient (step 2). Here, we have $\frac{3+4}{2} = 3.5$.

- **Step four:**
  Square the mean value and see how close it is to 12. Here, $3.5^2 = 12.25$.

- **Step five:**
  Repeat steps 2, 3 and 4 replacing the initial square root with the mean value. Repeat this procedure to get closer and closer to the true square root.

So, to finish the example,

$$\begin{aligned} 12 \div 3.5 &= 3.442857 \\ \frac{3.5 + 3.442857}{2} &= 3.4643 \\ 3.4643^2 &= 12.001 \end{aligned}$$

This is pretty close to 12 so we can stop here but the procedure can be continued to get an even more accurate square root. The conclusion is $\sqrt{12} \approx 3.4643$. This is a process of iteration. This means we keep trying new values continually getting closer to the true result.

Now let's perform the same process to calculate $\sqrt{2}$.

Example 1: Estimate $\sqrt{2}$ to 4 decimal places.

The largest square number less than 2 is $1 = 1^2$.

First iteration:

$$\begin{aligned} 2 \div 1 &= 2 \\ \frac{1+2}{2} &= 1.5 \\ 1.5^2 &= 2.25 \end{aligned}$$

Second iteration:

$$\begin{aligned} 2 \div 1.5 &= 1.3333 \\ \frac{1.5 + 1.3333}{2} &= 1.4167 \\ 1.4167^2 &= 2.0069 \end{aligned}$$

Third iteration:

$$\begin{aligned} 2 \div 1.4167 &= 1.4118 \\ \frac{1.4167 + 1.4118}{2} &= 1.4142 \\ 1.4142^2 &= 1.99996 \end{aligned}$$

So to 4 decimal places $\sqrt{2} = 1.4142$.

1.2 Simplifying square roots

Manipulation can be done within the square root sign. $\sqrt{50}$, for example, can be written as $\sqrt{25 \times 2}$. The square root of a product can be calculated as the product of the square roots, so $\sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$. Similar simplifications can be done for the square root of any number which has a square number as a factor. Interestingly, what this means is that as we know $\sqrt{2}$ then to calculate $\sqrt{50}$ we simply calculate $5 \times \sqrt{2}$. So,

$$\sqrt{50} = 5 \times \sqrt{2} = 5 \times 1.4142 = 7.071.$$

Example 2: Estimate $\sqrt{98}$ to 3 decimal places.

$$\sqrt{98} = \sqrt{49 \times 2} = \sqrt{49} \times \sqrt{2} = 7 \times 1.4142 = 9.899$$

A similar process can be applied in reverse when we already know a larger square root.

Example 3: Estimate $\sqrt{3}$ to 3 decimal places.

$$\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2 \times \sqrt{3}$$

Earlier we calculated $\sqrt{12} = 3.4643$; this implies that $2\sqrt{3} = 3.4643$, so

$$\sqrt{3} = \frac{3.4643}{2} = 1.732.$$

1.3 Definition of a surd

A surd is any square root which does not simplify to a whole number. So for example $\sqrt{25}$ is not a surd because $\sqrt{25} = 5$ and 5 is a nice whole number. $\sqrt{12}$, however is a surd, despite the fact it can be simplified to $2\sqrt{3}$, because $2\sqrt{3}$ is not a whole number.

1.4 Estimating the square roots of large numbers

We have seen how we can accurately estimate the square root of small numbers, but here is a trick to loosely estimate the square root of very large numbers which can be done faster than someone on a calculator. Before we see the method let's look at an interesting pattern.

$$\begin{aligned}
\sqrt{4} &= 2 \\
\sqrt{40} &= 6.324555 \\
\sqrt{400} &= 20 \\
\sqrt{4000} &= 63.24555 \\
\sqrt{40000} &= 200 \\
\sqrt{400000} &= 632.4555
\end{aligned}$$

Here we can see that there are two different square roots repeating in a cycle. Let n denote the number of digits in the number within the square root. When n is odd the square root is $2 \times 10^{\frac{n-1}{2}}$ and when n is even the square root is $6.324555 \times 10^{\frac{n-2}{2}}$. We can see that these formulae hold by substituting in values of n. The reader is encouraged to do so.

This is the principle the following method to estimate the square root of an arbitrary large number is based on.

- **Step one:**
  Suppose we are trying to find the square root of 9,412,381, a number with seven digits. Separate the number into blocks of two digits starting with the right hand side. This gives:
  $$| 9 | 41 | 23 | 81 |$$
- **Step two:**
  Estimate the square root of the number in the left most section, in this case 9. Luckily $\sqrt{9} = 3$ so this is fairly easy.
- **Step three:**
  The number of digits in the final estimate is equal to the number of two digit sections, so in this case 4. (Although 9 is only one digit it counts as a new "two" digit section... it could alternatively be written as 09).
- **Step four:**
  In order to get the right number of digits in our final estimated square root we can multiply our initial estimate by 10^{k-1}, where k is the number of two digit sections. In this example, $k = 4$, so the estimated square root of 9,412,381 is $3 \times 10^3 = 3000$.

The true value is $\sqrt{9412381} = 3067.96$ so we were not too far away at all. Let us consider another example.

Example 1: Approximate the square root of 56,723,412 to the nearest hundred.

$$| 56 | 72 | 34 | 12 |$$

In this case we will estimate $\sqrt{57}$ as apposed to $\sqrt{56}$ because the number rounds up to 57,000,000.

$$\sqrt{57} \approx 7.5$$
$$7.5 \times 10^3 = 7500$$

So this implies $\sqrt{56723412} \approx 7500$. The true square root is 7531.5.

In order to apply this method you only have to able to reasonably accurately estimate the square root of any two digit number. There are even tricks for this too.

$10^2 = 100$ so this means that the square root of any two digit number will be less than 10. We should all be familiar with the squared numbers less than 100, eg. 1, 4, 9, 25, etc. To estimate the square root of a two digit number such as 57 first identify the two square numbers it is between. So for 57, it is between $49 = 7^2$ and $64 = 8^2$, so we know that $\sqrt{57}$

will be 7 point something. The first decimal place can be guessed simply by considering how close the number is to the two square numbers. In this case 57 is pretty much in the middle between 49 and 64 so 7.5 seems like a sensible choice.

A slightly more mathematical approach to estimating the decimal place is to consider the ratio of the number between the square numbers. This can be done by calculating the difference between the lower square and the number, divided by the difference between the two square numbers. So we get

$$\frac{49 - 57}{49 - 64} = \frac{8}{15} = 0.53333$$

which verifies that 7.5 was a good choice. Because we are dealing with squared quantities the ratio is not exact but gives a good estimate.

If nothing else this final section will provide you with a fantastic party trick to demonstrate your mathematical prowess!

2 Egyptian fractions

2.1 Introduction

Ancient Egyptians had a unique way of expressing fractions. They would express each fraction as a finite sum of *distinct, unit fractions*. This means that all fractions are expressed as a sum of unit fractions, and none of the denominators are repeated.

KEY POINTS

A *unit fraction* has 1 as its numerator and a *positive integer* as its denominator. For example: $\frac{1}{3}, \frac{1}{5}$ and $\frac{1}{15}$ are all unit fractions.

This means that instead of writing $\frac{2}{7}$ as it is, they would express it as a sum of *different* unit fractions.

Let us see how $\frac{2}{7}$ can be expressed with distinct, unit fractions.

$\frac{2}{7} = \frac{1}{7} + \frac{1}{7}$. Since $\frac{1}{7}$ is repeated, this is not allowed! WRONG

Or:

$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$. Since $\frac{1}{4}$ and $\frac{1}{28}$ are distinct, this is allowed! CORRECT

Or:

$\frac{2}{7} = \frac{1}{7} + \frac{1}{8} + \frac{1}{56}$. Since $\frac{1}{7}, \frac{1}{8}$ and $\frac{1}{56}$ are distinct, this is allowed! CORRECT

It is important to note here that we have seen the fraction $\frac{2}{7}$ expressed as *distinct, unit* fractions in two different ways. Are there any other combinations of distinct unit fractions which add up to $\frac{2}{7}$?

2.2 Different representations of the same fraction

We will use the fraction $\frac{2}{3}$ as an example here.

Take:

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}.$$

This is NOT the only representation of $\frac{2}{3}$ in terms of distinct unit fractions.

Consider this :

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}. \tag{1}$$

Using equation (1), we will expand the expression for $\frac{2}{3}$ as follows:

$$\begin{aligned}
\frac{2}{3} &= \frac{1}{2} + \frac{1}{6} \\
\frac{2}{3} &= \frac{1}{2} + \left(\frac{1}{6} \times 1\right) \\
\frac{2}{3} &= \frac{1}{2} + \left[\frac{1}{6} \times \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)\right] \quad \text{Substituting from equation (1)} \\
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}. \qquad (2)
\end{aligned}$$

Subsequently, the last fraction of the Egyptian fraction representation of $\frac{2}{3}$ in equation (2) can be expanded using equation (1) again. We will then have:

$$\begin{aligned}
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \\
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \left(\frac{1}{36} \times 1\right) \\
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \left[\frac{1}{36} \times \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)\right] \quad \text{Substituting from equation (1)} \\
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \frac{1}{72} + \frac{1}{108} + \frac{1}{216}. \qquad (3)
\end{aligned}$$

We have just seen the fraction $\frac{2}{3}$ expressed as 3 different sums of distinct, unit fractions. Using equation (1), the smallest unit fraction in the sum of unit fractions can always be expanded further into smaller unit fractions.

We can simply repeat the process to expand $\frac{1}{216}$, only to repeat it again on the next smallest unit fraction.

This shows us that each time we repeat the process of expanding the Egyptian fraction, we get a new, different set of unit fractions which add up to $\frac{2}{3}$.

We can safely conclude that once we are able to express any given fraction as a sum

of distinct, unit fractions, we can derive infinitely many other representations of the same fraction simply by multiplying the smallest fraction with equation (1).

You may have noticed there has been no mention of a method for explicitly calculating the Egyptian fraction form of a common fraction. Can you work this part out for yourself? One small hint is to begin by thinking: "What is the largest unit fraction that is less than the common fraction you are considering?"

2.3 Practical applications

The exact reason for which the ancient Egyptians used such a system to express fractions remains a mystery, but there are many practical uses for being able to express a fraction as a sum of distinct, unit fractions.

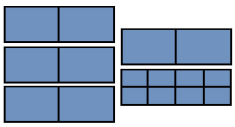
We will take a look at an everyday problem that may be solved using Egyptian fractions.

Suppose 8 workers in Egypt are to divide 5 loaves of bread equally amongst themselves. How could they do this?

Firstly, expressing $\frac{5}{8}$ as decimals is not going to be of much help. If we use Egyptian fractions, however, $\frac{5}{8}$ could be expressed as:

$$\frac{1}{2} + \frac{1}{8}.$$

This means that each worker would receive half a loaf, and an additional $\frac{1}{8}$ of a loaf.



The advantage of using the Egyptian fraction in this scenario is that only one loaf of bread has to be cut into eight equal slices. The remaining four loaves only have to be cut in half, and this minimises the number of slices that have to be made!

3 Written assignment

Questions marked with a " * " will involve concepts beyond the standard GCSE syllabus and we strongly recommend reading through the attached notes carefully before attempting these questions.

Question 1

Solve $\dfrac{8}{3x+1} = 2$.

[3 marks]

Question 2

Find the equation for a line perpendicular to $y = -2x + 3$ which passes through the point $(4, 1)$.

[4 marks]

Question 3

A goat is tied to the middle of one side of a 10m square barn, using a 10m long rope. The barn is in the middle of a field. What is the area of grass the goat can graze?

[4 marks]

Question 4

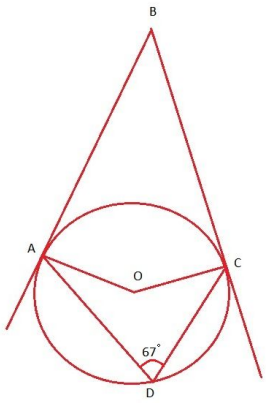


Figure 1

Find the internal angle at the point B of Figure 2.

[5 marks]

Question 5*

(a) Show how $\sqrt{28}$ can be estimated to be 5.2915.

(b) Use the answer from part (a) to estimate $\sqrt{7}$ to 4 decimal places.

(c) Use your answer from part (b) estimate $\sqrt{175}$ to 4 decimal places.

[6 marks]

Question 6*

Complete this question without using a calculator, Showing all your working.

(a) Estimate the square root of 713,261 to the nearest ten.

(b) Estimate the square root of 891,677,128 to the nearest thousand.

(c) Estimate the square root of 23,416,948 to the nearest hundred.

[6 marks]

Question 7*

Express the fraction $\frac{3}{5}$ as an Egyptian fraction (in terms of distinct unit fractions). Once you have the first Egyptian fraction expression of $\frac{3}{5}$, use equation (1) from the notes in Section 3 of this booklet to find 3 other expressions of the same fraction.

[5 marks]

Question 8*

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{12} \quad \frac{1}{20}$$

The fractions listed above are all unit fractions. Observe the following sums:

$$\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$$
$$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Does the sum of any two different unit fractions always add up to a fraction with a numerator greater than 1? Using the unit fractions given above, can you find the three equations of two different unit fractions that add up to another unit fraction?

Bonus Question: Is there any pattern in the pairs of unit fractions which add up to another unit fraction? If there is a pattern, write down a general formula for a pair of distinct unit fractions that add up to form another unit fraction.

[8 marks]

4 Research question

The final part of the project is a more open ended research based question. We expect a one/two page report presenting all the information related to the research question.

Choose **one** of the following research topics,

[20 marks]

Fibonacci sequence and the golden ratio

What is the Fibonacci sequence? Investigate its history and the man "Fibonacci". What is the golden ratio? How are the Fibonacci sequence and the golden ratio related? Give examples of both the Fibonacci sequence and the golden ratio in the real world.

Prime numbers and the fundamental theorem of arithmetic

What is a prime number? Investigate prime numbers and how they are involved in the fundamental theorem of arithmetic. With reference to the fundamental theorem of arithmetic what makes primes so important? Is there a biggest prime number and if so what is it? How can prime numbers be used to find the greatest common divisor between two numbers and the lowest common multiple? Include any other interesting facts about prime numbers.

Decrypting the code

The following code is encrypted using the Caesar Cipher. Research the Caesar Cipher as an encryption technique. Use the information you gathered on the Caesar Cipher to decode the message given below:

Znk rout, znk cozin gtj znk cgxjxuhk gtj Znk muuj, znk hgj gtj znk amre gxk zcu ul se lgbuxozk lorsy.

A good way to start to attempt to decrypt the code is to think of the most commonly used words in the English language. Two very frequently used words are "and" and "the". Can you see any patterns in the encrypted text that could correspond to one of these words?

The following code is encrypted using the Vigenere Cipher. Research how the Vigenere code is used, and describe step by step how the Vigenere table (Figure 2) is used to decipher messages. The keyword to decipher the message below is "NEWCASTLE". Refer to the Vigenere table included below.

fijf uk t zrr waptwgni ziouayx fwvrc dola elr gwgssk nmclat afw elr zeiefxci pmljej.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
B	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
C	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
D	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
F	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
G	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
H	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
I	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
J	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
K	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
L	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
M	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
N	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
O	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
P	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Q	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
R	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
U	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
V	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
W	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
X	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
Y	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
Z	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

Figure 2