Summer Project

School of Mathematics, Statistics & Physics

Summer 2019

Workbook 2

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Introduction

This project is intended to give current A-Level students (or equivalent) an opportunity to demonstrate their mathematical knowledge and experience, and to give such students a taste of some new material which most will not have seen before. A brief recap of some concepts encountered in the A-level syllabus is provided, with a more detailed description of new concepts to provide assistance when preparing solutions to questions posed in the project. This project consists of three components:

- A written assignment sheet
- Some computer based questions using our NUMBAS online testing system
- A more open ended, research-style question.

The questions are designed to be of varying difficulty to challenge students at different skill levels and we would expect a good student to take up to five days to complete all components. Any questions marked with a " * " will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the information in this booklet carefully before attempting the questions. Completed assignments will be graded and successful students will be awarded with a certificate and general feedback. More details about this summer project can be found on our dedicated webpage: `http://www.mas.ncl.ac.uk/~nlf8/summer-project/`.

Written assignment

There are eight questions to work through here. Most of these questions will be based on material covered in this booklet. However, there may be one or two surprises! Solutions to these questions can be hand-written or typed. Hand-written solutions should be scanned and emailed to `maths.physics@ncl.ac.uk`. Solutions prepared in a word-processing package (eg. Microsoft Word) can be emailed directly, although we recommend exporting to portable document format (PDF) first, to help prevent loss of formatting. Students must submit their solutions by **4pm, Monday 9th September GMT**.

Computer based test

This uses Newcastle University's *NUMBAS online testing system*. The test will be available in two "modes"; *PRACTICE* and *ASSESSED*. Students can access the practice test as many times as they like. Each time they open a test, the format of the questions within that particular test will remain the same, but the numbers might change. The assessed mode will be active for two weeks only, from **Monday 26th August–Friday 9th September**. Students must login and complete the test by **4pm, Monday 9th September GMT**. Only **one** attempt will be allowed, but students can pause the test as often as they wish – there are no time constraints. The practice mode will be available throughout August and September. Students are strongly advised to work through our *Introduction to NUMBAS* first – available at
`https://numbas.mathcentre.ac.uk/test-yourself/getting-started/`. A link to the actual test, and further instructions about submission of the test will be available on our webpage soon.

Research-style question

This is the final part of the project. Students will be given a choice of three questions, from which they are required to attempt only **one**. We expect solutions to any of these questions to be one/two pages long. Students can pick from a range of topics for this question, including options from statistics or number theory. Most questions do not have strict guidelines and we encourage students to be creative with their answers. As with the written assignment, solutions to the open-ended questions can be either hand-written or electronically produced with a word processing package. Answers should be emailed to `maths.physics@ncl.ac.uk` no later than **4pm, Monday 9th September GMT**.

1 Some Revision

1.1 Quadratic equations

A root of the function $f(x)$ is any value of x which solves $f(x) = 0$. For a quadratic polynomial where $f(x) = ax^2 + bx + c$, there is a maximum of two roots. These can be found via three different methods:

1. Factorisation:

   The quadratic equation $ax^2 + bx + c = 0$ can sometimes be expressed in the form $(dx + e)(jx + k) = 0$.

   This implies that the roots are $x = -\frac{e}{d}$ and $x = -\frac{k}{j}$.

2. Quadratic formula:

   Substitute the respective values of a, b and c into the quadratic formula:

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Completing the square (only when $a = 1$):

   $x^2 + bx + c = (x + b/2)^2 - (b/2)^2 + c = 0 \Rightarrow (x + b/2)^2 = (b/2)^2 - c.$

1.2 Plane geometry

The equation of a straight line in *gradient form* is

$$y = mx + c,$$

where m is the *gradient* and c is the *y-intercept*. The equation of the straight line between points (x_1, y_1) and (x_2, y_2) in the xy-plane can be found using the formulae:

$$y - y_1 = m(x - x_1), \quad \text{where} \quad m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Points of intersection between two functions can be identified by equating them and finding values of x which satisfy the relationship. Some pairs of functions may have many intersection points and others may not have any.

1.3 Binomial expansion

An expression of the form $(a+b)^n$ can be written out as a complete sum of all the individual terms using the binomial formula given below:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where $\binom{n}{k}$ is the binomial coefficient which can be obtained using a calculator or by writing out Pascal's triangle. The binomial coefficient can also be calculated as a quotient of factorials given in the formula below:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

1.4 Logarithms and exponentials

Index form: $y = a^x$ *Logarithmic form:* $\log_a(y) = x$

Here is a list of useful logarithmic identities:

- $\log_a(x) + \log_a(y) = \log_a(xy)$
- $\log_a(x) - \log_a(y) = \log_a(\frac{x}{y})$
- $\log_a(x^k) = k\log_a(x)$

1.5 Trigonometric functions

You should all be familiar with $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ and how they link to the ratio of lengths in a right-angled triangle. Radians are a measure of angle where 2π radians $= 360°$. One can assume that the angle is given in radians unless explicitly expressed otherwise. Some important trigonometric identities are:

- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $sin(\theta) = a \quad \Rightarrow \quad \theta = sin^{-1}(a)$
- $cos(\theta) = b \quad \Rightarrow \quad \theta = cos^{-1}(b)$

1.6 Differentiation and integration

For a polynomial function $f(x) = ax^n$, the *first-order derivative* $\dfrac{df}{dx} = f'(x) = anx^{n-1}$. The inverse of this operation is *integration*, where for $f(x) = ax^n$ then

$$\int f(x)\ dx = \frac{a}{n+1}x^{n+1} + c,$$

where c is the constant of integration.

Differentiation can be used to find the gradient at any point of a particular function. Substituting any given x value in to the derivative function will give the gradient of $f(x)$ at that point. Points where the gradient equals zero are called turning points and can be important when trying to find values of x which maximize or minimize a function.

Integration can be used to calculate the area between a function and the x axis. If we want to calculate the area under the curve between two values of x we select the lowest value of x as our lower limit of integration, l, and the higher value as our upper limit of integration, u. Mathematically this can be written as:

$$\int_l^u f(x)\ dx = \left[\int f(x)\ dx\right]_l^u = \int f(x)\ dx\Big|_{x=u} - \int f(x)dx\Big|_{x=l},$$

where the notation $|_{x=u}$, for example, means "evaluated at $x = u$". One important thing to consider is that the area below the x-axis will have a negative contribution to the overall integral so it may be necessary to calculate the area above and below the x-axis separately and then add them together.

2 New topic 1: Complex numbers

2.1 Introduction

To understand the next few sections, it would be helpful to be familiar with quadratic equations (eg. solving $x^2 - 3x + 2 = 0$; see section 1.1). In order to solve the equation of the form $ax^2 + bx + c = 0$, we can use the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

However, we know for certain that if x is a real number, i.e $x^2 \geq 0$, then there are some quadratic equations which do not have any real solutions.

Example 1: Solve the quadratic equation $x^2 + 6x + 13 = 0$.
Using the formula stated above with $a = 1, b = 6$ and $c = 13$,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{-16}}{2}. \end{aligned}$$

We can see that this equation does not have any real solutions because we cannot evaluate $\sqrt{-16}$ using real numbers.

As $\sqrt{-1}$ cannot be evaluated, mathematicians introduced a new symbol to denote it: i.

Thus, we have:

$$\begin{aligned} \sqrt{-1} &= i \\ -1 &= i^2. \end{aligned}$$

With this new symbol in hand, it follows that:

$$\sqrt{-16} = \sqrt{16 \times (-1)} = \sqrt{16} \times \sqrt{-1} = 4 \times \sqrt{-1} = 4i.$$

Here, $4i$ is an *imaginary number*.

An imaginary number is a number in the form bi, where b is a real number and $i = \sqrt{-1}$.

Now that we have introduced *imaginary numbers*, we can complete the solution for x in Example 1.

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{-16}}{2} \\ &= -3 \pm \frac{4i}{2} \\ &= -3 \pm 2i. \end{aligned}$$

We have just seen that by combining real numbers and imaginary numbers, we can find the solution to any quadratic equation.

2.2 The general complex number

A complex number is simply a number in the form $z = a + bi$, where a and b are real numbers, and $i = \sqrt{-1}$. The *real* part of z is called a, and the *imaginary* part of z is called b. We often use the following notations to denote the real and imaginary parts of z respectively :

$$a = \mathbf{Re}(z) \qquad\qquad b = \mathbf{Im}(z)$$

- If $\mathbf{Re}(z) = 0$, then $z = bi$, an imaginary number.
- If $\mathbf{Im}(z) = 0$, then $z = a$, a real number.

2.3 Complex arithmetic

Complex numbers can be added, subtracted and multiplied, just like real numbers.

2.3.1 Addition and subtraction

When adding or subtracting complex numbers, we add their real and imaginary components separately.

Example 2: Let $z_1 = 3 + 4i$ and $z_2 = 2 + 3i$. Find $z_1 + z_2$ and $z_1 - z_2$.

$$\begin{aligned} z_1 + z_2 &= (3 + 4i) + (2 + 3i) \\ &= 3 + 4i + 2 + 3i \qquad \text{(Rearranging this)} \\ &= (3 + 2) + (4 + 3)i \\ &= 5 + 7i. \end{aligned}$$

Similarly,

$$\begin{aligned} z_1 - z_2 &= (3+4i) - (2+3i) \\ &= 3 + 4i - 2 - 3i \qquad \text{(Rearranging this)} \\ &= (3-2) + (4-3)i \\ &= 1 + i \end{aligned}$$

2.3.2 Multiplication

In order to multiply complex numbers, we have to keep in mind that $i^2 = -1$. Once we have expanded the brackets, we only have to "collect" the real and imaginary terms before presenting our solution.

Example 3: Using the values of z_1 and z_2 fom the previous example, calculate the value of $z_1 \cdot z_2$.

$$\begin{aligned} z_1 \cdot z_2 &= (3+4i) \cdot (2+3i) \\ &= 6 + 9i + 8i + 12i^2 \qquad \text{(Since } i^2 = -1\text{)} \\ &= 6 + 9i + 8i + 12(-1) \qquad \text{(Rearranging this)} \\ &= (6-12) + (9+8)i \\ &= -6 + 17i. \end{aligned}$$

Generally:
If $z_1 = a + bi$ and $z_2 = c + di$, then:

- $z_1 + z_2 = (a+c) + (b+d)i$
- $z_1 - z_2 = (a-c) + (b-d)i$
- $z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$

2.4 The conjugate of a complex number

The *complex conjugate* of $z = a + bi$ is the number $a - bi$. We use a bar to denote the conjugate of a complex number.

Thus, the conjugate of $z = 3 + 7i$ is $\bar{z} = \overline{3 + 7i} = 3 - 7i$.

The one property of the complex conjugate that we will be looking at is if $z = a + bi$ and $\bar{z} = a - bi$, then the product of $z \cdot \bar{z}$ is a *real number.*

Proof:

$$\begin{aligned} z \cdot \bar{z} &= (a + bi) \cdot (a - bi) \\ &= a^2 - abi + abi - b^2 i \\ &= a^2 + b^2 \end{aligned}$$

Therefore, when a complex number is multiplied by its conjugate, the imaginary parts are cancelled, and the product is a real number.

We will see in the following section that the division of complex numbers is slightly more complicated than multiplication, and usually we will have to make use of complex conjugates.

2.4.1 Division

Example 4: Let $z_1 = 3 + 4i$ and $z_2 = 2 + 3i$. Find $\dfrac{z_1}{z_2}$.

$$\frac{z_1}{z_2} = \frac{3 + 4i}{2 + 3i}$$

In order to make the denominator a real number, we have to multiply both the numerator and the denominator by $\bar{z_2}$.

Hence, we have:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{(3+4i)(2-3i)}{(2+3i)(2-3i)} \\ &= \frac{18-i}{2^2+3^2} \\ &= \frac{18-i}{13} \\ &= \frac{18}{13} - \frac{1}{13}i.\end{aligned}$$

3 New topic 2: Egyptian fractions

3.1 Introduction

Ancient Egyptians had a unique way of expressing fractions. They would express each fraction as a finite sum of *distinct, unit fractions*. This means that all fractions are expressed as a sum of unit fractions, and none of the denominators are repeated.

KEY POINTS

A *unit fraction* has 1 as its numerator and a *positive integer* as its denominator. For example: $\frac{1}{3}, \frac{1}{5}$ and $\frac{1}{15}$ are all unit fractions.

This means that instead of writing $\frac{2}{7}$ as it is, they would express it as a sum of *different* unit fractions.

Let us see how $\frac{2}{7}$ can be expressed with distinct, unit fractions.

$\frac{2}{7} = \frac{1}{7} + \frac{1}{7}$. Since $\frac{1}{7}$ is repeated, this is not allowed! WRONG

Or:

$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$. Since $\frac{1}{4}$ and $\frac{1}{28}$ are distinct, this is allowed! CORRECT

Or:

$\frac{2}{7} = \frac{1}{7} + \frac{1}{8} + \frac{1}{56}$. Since $\frac{1}{7}, \frac{1}{8}$ and $\frac{1}{56}$ are distinct, this is allowed! CORRECT

It is important to note here that we have seen the fraction $\frac{2}{7}$ expressed as distinct, unit fractions in two different ways. Are there any other combinations of distinct unit fractions which add up to $\frac{2}{7}$?

3.2 Different representations

We will use the fraction $\frac{2}{3}$ as an example here.

Take:

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}.$$

This is NOT the only representation of $\frac{2}{3}$ in terms of distinct unit fractions.

Consider this :

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}. \tag{1}$$

Using equation (1), we will expand the expression for $\frac{2}{3}$ as follows:

$$\begin{aligned}
\frac{2}{3} &= \frac{1}{2} + \frac{1}{6} \\
\frac{2}{3} &= \frac{1}{2} + \left(\frac{1}{6} \times 1\right) \\
\frac{2}{3} &= \frac{1}{2} + \left[\frac{1}{6} \times \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)\right] \quad \text{Substituting from equation (1)} \\
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}. \qquad (2)
\end{aligned}$$

Subsequently, the last fraction of the Egyptian fraction representation of $\frac{2}{3}$ in equation (2) can be expanded using equation (1) again. We will then have:

$$\begin{aligned}
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36} \\
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \left(\frac{1}{36} \times 1\right) \\
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \left[\frac{1}{36} \times \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)\right] \quad \text{Substituting from equation (1)} \\
\frac{2}{3} &= \frac{1}{2} + \frac{1}{12} + \frac{1}{18} + \frac{1}{72} + \frac{1}{108} + \frac{1}{216}. \qquad (3)
\end{aligned}$$

We have just seen the fraction $\frac{2}{3}$ expressed as 3 different sums of distinct, unit fractions. Using equation (1), the smallest unit fraction in the sum of unit fractions can always be expanded further into smaller unit fractions.

We can simply repeat the process to expand $\frac{1}{216}$, only to repeat it again on the next smallest unit fraction.

This shows us that each time we repeat the process of expanding the Egyptian fraction, we get a new, different set of unit fractions which add up to $\frac{2}{3}$.

We can safely conclude that once we are able to express any given fraction as a sum

of distinct, unit fractions, we can derive infinitely many other representations of the same fraction simply by multiplying the smallest fraction with equation (1).

You may have noticed there has been no mention of a method for explicitly calculating the Egyptian fraction form of a common fraction. Can you work this out for yourself? One small hint is to begin by thinking: "What is the largest unit fraction that is less than the common fraction you are considering?"

3.3 Practical applications

The exact reason for which the ancient Egyptians used such a system to express fractions remains a mystery, but there are many practical uses for being able to express a fraction as a sum of distinct, unit fractions.

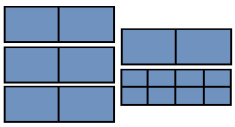
We will take a look at an everyday problem that may be solved using Egyptian fractions.

Suppose 8 workers in Egypt are to divide 5 loaves of bread equally amongst themselves. How could they do this?

Firstly, expressing $\frac{5}{8}$ as decimals is not going to be of much help. If we use Egyptian fractions, however, $\frac{5}{8}$ could be expressed as:

$$\frac{1}{2} + \frac{1}{8}.$$

This means that each worker would receive half a loaf, and an additional $\frac{1}{8}$ of a loaf.



The advantage of using the Egyptian fraction in this scenario is that only one loaf of bread has to be cut into eight equal slices. The remaining four loaves only have to be cut in half, and this minimises the number of slices that have to be made!

4 Written assignment

Questions marked with a " * " will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the attached notes carefully before attempting these questions.

Question 1

Tony started a savings account in 2005. He deposited a certain amount of money into his account each week. When he plotted his total savings each week, his graph looked like the graph in Figure 1 below.

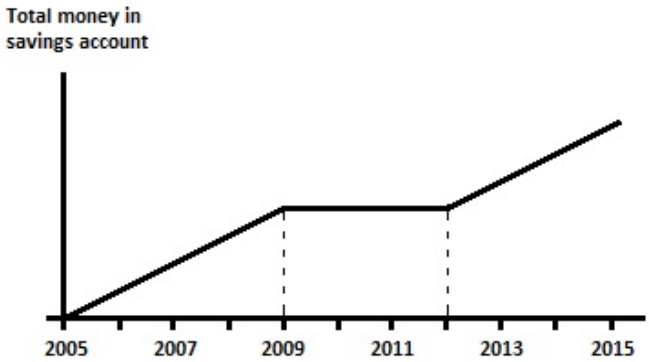


Figure 1

Sketch a graph to represent his weekly additions to his savings account over the same period (the graph does not have to be drawn to scale).

[3 marks]

Question 2

Find a value of x which satisfies:

$$log_6(x-12) + log_6(x-3) = 2$$

[4 marks]

Question 3

Show that $\sin^2(\theta)+\cos^2(\theta) = 1$.

(Hint: draw and label a right-angled triangle)

[5 marks]

Question 4

A palindrome is a number that reads the same forwards and backwards, for example 4334. Are all 4 digit palindromes exactly divisible by 11? (Hint: write out a generic palindrome as $1000A + 100B + 10B + A$)

[4 marks]

Question 5*

For a given function $f(x) = x^2 + 6x + 18$;

(a) Sketch the function $f(x)$. Label any turning points and points of intersection with the x-axis and the y-axis.

(b) Find the roots of the function $f(x)$.

[5 marks]

Question 6*

Let $z_1 = 16 + 12i$ and $\frac{z_1}{z_2} = 2 + 2i$. Find the value of z_2 in the form $a + bi$. Show your working clearly.

[4 marks]

Question 7*

Express the fraction $\frac{3}{5}$ as an Egyptian fraction (in terms of distinct unit fractions). Once you have the first Egyptian fraction expression of $\frac{3}{5}$, use equation (1) from the notes in Section 3 of this booklet to find 2 other expressions of the same fraction.

[5 marks]

Question 8*

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{12} \quad \frac{1}{20}$$

The fractions listed above are all unit fractions. Observe the following sums:

$$\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$$
$$\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Does the sum of any two different unit fractions always add up to a fraction with a numerator greater than 1? Using the unit fractions given above, can you find the three equations of two different unit fractions that add up to another unit fraction.

[5 marks]

Bonus Question: Is there any pattern in the pairs of unit fractions which add up to another unit fraction? If there is a pattern, write down a general formula for a pair of distinct unit fractions that add up to form another unit fraction.

[3 marks]

5 Research question

The final part of the project is a more open-ended research based question. We expect a one/two page report presenting all the information related to the research question.

Choose **one** of the following research topics.

[20 marks]

Geometry of complex numbers

We have shown you a brief introduction to the world of complex numbers. Investigate the geometry of complex numbers. How can they be drawn? What is their argument and modulus? Include any other information you think could be relevant or interesting.

Fibonacci sequence and the golden ratio

What is the Fibonacci sequence? Why is it called the Fibonacci sequence? What is the golden ratio? How are the Fibonacci sequence and the golden ratio relate? Give examples of both the Fibonacci sequence and the golden ratio in the real world. Include anything else you consider to be of relevance.

Premier League spending

The finances available for Premier League teams is commonly considered to be an important influence on final league position. By looking at the 2015/16 English Premier League season data, investigate the correlation between the overall turnover and final league position. Find an equation for the line of regression between the two variables and superimpose this on your scatterplot. Provide an interpretation of the regression coefficient.

Newcastle United are a newly promoted team going into the Premier League with an expected turnover of around £130 million. Predict their final league position and use this to suggest whether the fans should fear relegation.