Summer Project

School of Mathematics, Statistics & Physics

Summer 2018

Workbook 4

©Dr. Lee Fawcett, George Loughlin, Inbbavathie Ravi, June 2018.

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Introduction

This project is intended to give current A-Level students (or equivalent) an opportunity to demonstrate their mathematical knowledge and experience, and to give such students a taste of some new material which most will not have seen before. A brief recap of some concepts encountered in the A-level syllabus is provided, with a more detailed description of new concepts to provide assistance when preparing solutions to questions posed in the project. This project consists of three components:

- A written assignment sheet
- Some computer based questions using our NUMBAS online testing system
- A more open ended, research-style question

The questions are designed to be of varying difficulty to challenge students at different skill levels and we would expect a good student to take up to five days to complete all components. Any questions marked with a " * " will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the information in this booklet carefully before attempting the questions. Completed assignments will

be graded and successful students will be awarded with a certificate and general feedback. More details about this summer project can be found on our dedicated webpage: `http://www.mas.ncl.ac.uk/~nlf8/summer-project/`.

Written assignment

There are nine questions to work through here. Most of these questions will be based on material covered in this booklet. However, there may be one or two surprises! Solutions to these questions can be hand-written or typed. Hand-written solutions should be scanned and emailed to `maths.physics@ncl.ac.uk`. Solutions prepared in a word-processing package (eg. Microsoft Word) can be emailed directly, although we recommend exporting to portable document format (PDF) first, to help prevent loss of formatting. Students must submit their solutions by **4pm, Friday 31st August GMT**.

Computer based test

This uses Newcastle University's *NUMBAS online testing system*. The test will be available in two "modes"; *PRACTICE* and *ASSESSED*. Students can access the practice test as many times as they like. Each time they open a test, the format of the questions within that particular test will remain the same, but the numbers might change. The assessed mode will be active for two weeks only, from **Monday 20th August–Friday 31st August**. Students must login and complete the test by **4pm, Friday 31st August GMT**. Only **one** attempt will be allowed, but students can pause the test as often as they wish – there are no time constraints. The practice mode will be available throughout August and September. Students are strongly advised to work through our *Introduction to NUMBAS* first – available at
`https://numbas.mathcentre.ac.uk/test-yourself/getting-started/`. A link to the actual test, and further instructions about submission of the test will be available on our webpage soon.

Research-style question

This is the final part of the project. Students will be given a choice of three questions, from which they are required to attempt only **one**. We expect solutions to any of these questions to be one or two pages long. Students can pick from a range of topics for this question, including options from statistics or number theory. Most questions do not have strict guidelines and we encourage students to be creative with their answers. As with the written assignment, solutions to the open-ended questions can be either hand-written or electronically produced with a word processing package. Answers should be emailed to `maths.physics@ncl.ac.uk` no later than **4pm, Friday 31st August GMT**.

1 Some Revision

1.1 Quadratic equations

A root of the function $f(x)$ is any value of x which solves $f(x) = 0$. For a quadratic polynomial where $f(x) = ax^2 + bx + c$, there is a maximum of two roots. These can be found via three different methods:

1. Factorisation:

   The quadratic equation $ax^2 + bx + c = 0$ can sometimes be expressed in the form $(dx + e)(jx + k) = 0$.

   This implies that the roots are $x = -\dfrac{e}{d}$ and $x = -\dfrac{k}{j}$.

2. Quadratic formula:

   Substitute the respective values of a, b and c into the quadratic formula:

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Completing the square (only when $a = 1$):

   $x^2 + bx + c = (x + b/2)^2 - (b/2)^2 + c = 0 \Rightarrow (x + b/2)^2 = (b/2)^2 - c.$

1.2 Plane geometry

The equation of a straight line in *gradient form* is

$$y = mx + c,$$

where m is the *gradient* and c is the *y-intercept*. The equation of the straight line between points (x_1, y_1) and (x_2, y_2) in the xy-plane can be found using the formulae:

$$y - y_1 = m(x - x_1), \quad \text{where} \quad m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Points of intersection between two functions can be identified by equating them and finding values of x which satisfy the relationship. Some pairs of functions may have many intersection points and others may not have any.

1.3 Binomial expansion

An expression of the form $(a+b)^n$ can be written out as a complete sum of all the individual terms using the binomial formula given below:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where $\binom{n}{k}$ is the binomial coefficient which can be obtained using a calculator or by writing out Pascal's triangle. The binomial coefficient can also be calculated as a quotient of factorials given in the formula below:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

1.4 Geometric progression

A sum of the form $a + ar + ar^2 + \cdots$ is called a geometric progression or series where a is the first term and r is the common ratio. Sums to an nth term can be found using the following formula:

$$\sum_{k=0}^{n-1} (ar^k) = a\frac{1-r^n}{1-r}.$$

If $|r| < 0$ then each term in the series gets smaller with higher powers of r. In this situation it is possible to calculate the limit as the sum tends to infinity using the following formula:

$$\sum_{k=0}^{\infty} (ar^k) = a\frac{1}{1-r}, \qquad \text{for } |r| < 0.$$

1.5 Logarithms and exponentials

Index form: $y = a^x$ *Logarithmic form:* $\log_a(y) = x$

Here is a list of useful logarithmic identities:

- $\log_a(x) + \log_a(y) = \log_a(xy)$
- $\log_a(x) - \log_a(y) = \log_a(\frac{x}{y})$
- $\log_a(x^k) = k\log_a(x)$

1.6 Trigonometric functions

You should all be familiar with $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ and how they link to the ratio of lengths in a right-angled triangle. Radians are a measure of angle where 2π radians $= 360°$ One can assume that the angle is given in radians unless explicitly expressed otherwise. Some important trigonometric identities are:

- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $sin(\theta) = a \quad \Rightarrow \quad \theta = sin^{-1}(a)$
- $cos(\theta) = b \quad \Rightarrow \quad \theta = cos^{-1}(b)$

1.7 Differentiation and integration

For a polynomial function $f(x) = ax^n$, the *first-order derivative* $\dfrac{df}{dx} = f'(x) = anx^{n-1}$. The inverse of this operation is *integration*, where for $f(x) = ax^n$ then

$$\int f(x)\ dx = \frac{a}{n+1}x^{n+1} + c,$$

where c is the constant of integration.

Differentiation can be used to find the gradient at any point of a particular function. Substituting any given x value in to the derivative function will give the gradient of $f(x)$ at that point. Points where the gradient equals zero are called turning points and can be important when trying to find values of x which maximize or minimize a function.

Integration can be used to calculate the area between a function and the x axis. If we want to calculate the area under the curve between two values of x we select the lowest value of x as our lower limit of integration, l, and the higher value as our upper limit of integration, u. Mathematically this can be written as:

$$\int_l^u f(x)\ dx = \left[\int f(x)\ dx\right]_l^u = \int f(x)\ dx\Bigg|_{x=u} - \int f(x)\ dx\Bigg|_{x=l},$$

where the notation $|_{x=u}$, for example, means "evaluated at $x = u$". One important thing to consider is that the area below the x-axis with have a negative contribution to the overall integral so it may be necessary to calculate the area above and below the x-axis separately and then add them together.

2 Linear programming

2.1 Introduction

A BRIEF HISTORY

Operational research (OR) is a method of advanced mathematical analysis of large-scale operations. It is used to find optimal solutions to complex decision-making problems. OR first originated as a discipline of applied mathematics in World War 1. the modern-field of OR was then developed during World War 2 to analyse the large-scale operations of war and the economic support of it. Decades later, OR techniques are applied to smaller-scale operations in business, companies and governments.

We will only consider the optimisation method of *linear programming (LP)*, which is the most important and developed tool of *operational research*.

2.2 A motivating example

We will now look at the following question in order to visualize the type of decision-making problems that can be solved using linear programming.

A brewer makes two varieties of beer, namely Pale Ale (P) and Bitter (B). Both types are made from two ingredients, Malt(M) and Yeast(Y) (all other ingredients are ignored in this question). The brewer has the following information available to him:

- 1 unit of P requires 2 units of M and 3 units of Y;
- 1 unit of B requires 4 units of M and 2 units of Y;
- Profit from selling one unit of P is £2;
- Profit from selling one unit of B is £3;
- There are currently 12 units of M and 10 units of Y in stock.

How many of each type of beer should the brewer make to maximize his profit?

Solution:

Step 1: Defining the decision variables.
We first have to define our *decision variables* accurately. To find the decision variables, read through the question and identify what you need to know to solve the problem. For the question above, we need to know how many of each type of beer the brewer has to produce to maximize profits. Thus, our decision variables can be defined as follows:

- Let x be the number of units of Pale Ale (P) to make.
- Let y be the number of units of Bitter (B) to make.

Step 2: Translating the constraints.
It is important to read the question carefully and translate the constraints posed in our problem. You'll notice that there's a lot of information given in the question – this is typical of a linear programming problem. Sometimes it's easier to summarise the information given in a table like the one below:

Type of beer	Units of Malt(M) required	Units of Yeast(Y) required	Profit(£)
Pale Ale(P)	2	3	2
Bitter(B)	4	2	3
Total units available	12	10	

Identifying the constraints is probably the hardest bit. To better understand how the formulation of constraints work, try analysing the information regarding each ingredient individually. Let us look at the information regarding the units of Malt(M) required for both types of beer:

Since:	The production of 1 unit of P requires 2 units of M,
Then:	The production of x units of P requires $(2 \times x) = 2x$ units of M.
Similarly:	The production of 1 unit of B requires 4 units of M,
So:	The production of y units of B requires $(4 \times y) = 4y$ units of M.

The total units of M required is therefore:

$$(2x + 4y) \quad \text{units.}$$

Since only 12 units of M are available, one of our constraints is:

$$2x + 4y \leq 12 \text{ units}, \qquad \text{or simply}$$
$$2x + 4y \leq 12.$$

We repeat the same process with the information we have in hand regarding the units of Yeast(Y) required to make both types of beer.

Since:	The production of 1 unit of P requires 3 units of Y,
Then:	The production of x units of P requires $(3 \times x) = 3x$ units of Y.
Similarly:	The production of 1 unit of B requires 2 units of Y,
So:	The production of y units of B requires $(2 \times y) = 2y$ units of Y.

The total units of Y required is therefore:

$$(3x + 2y) \quad \text{units.}$$

Since only 10 units of Y are available, our second constraint is:

$$\begin{aligned} &3x + 2y \leq 10 \text{ units}, \qquad \text{or simply} \\ &3x + 2y \leq 10. \end{aligned}$$

Additional constraints that we must be aware of are the *non-negativity constraints*. Keep in mind that x and y represent the number of units of P and B respectively. This means that they *cannot* be negative (we cannot make a negative number of beers!).

The *non-negativity constraints* are:

$$x \geq 0 \quad , \quad y \geq 0.$$

Step 3: Identifying the objective function.
The objective function is the quantity that needs to be *maximized* or *minimized*. For this question, we want to *maximize* our profit.

Since:	The profit from selling 1 unit of P is £2,
Then:	selling x number of P gives us a profit of £$(2 \times x) =$ £$2x$.
Similarly:	The profit from selling 1 unit of B is £3.
So,	selling y number of B gives us a profit of £$(3 \times y) =$ £$3y$.

The total profit that we can make, depending on the values of x and y is:

$$£z = 2x + 3y.$$

Thus, we want to maximize

$$z = 2x + 3y,$$

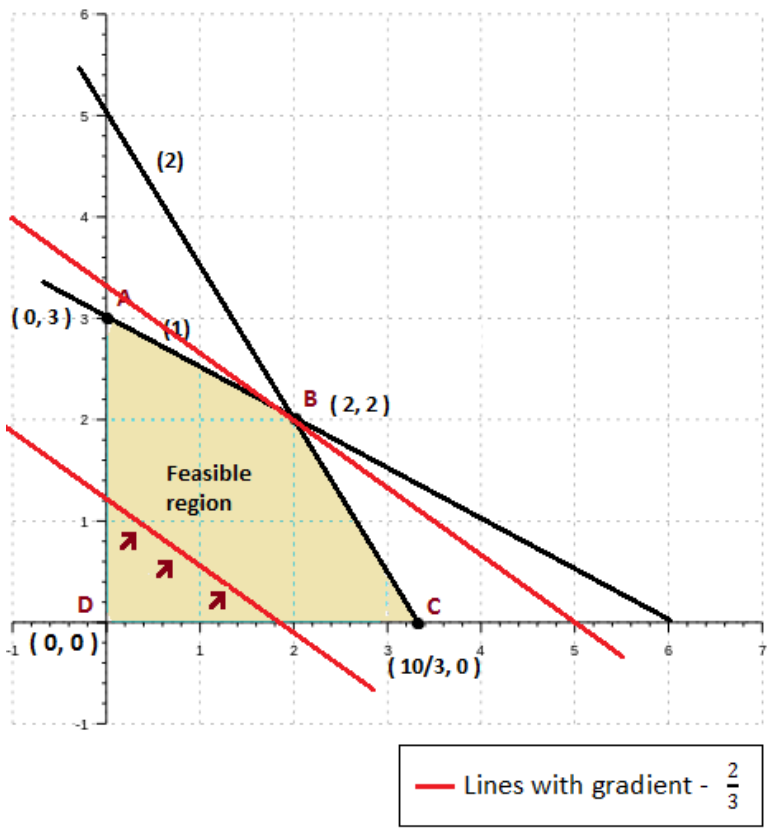
where z is the *objective function*.

Now that we have translated all the information from the question into constraints and identified the objective function, we can summarize the linear-programming problem as follows:

$$\begin{aligned} \text{maximize} \quad z = 2x + 3y \quad & \text{subject to the constraints:} \\ 2x + 4y &\leq 12; \qquad (1) \\ 3x + 2y &\leq 10; \qquad (2) \\ x \geq 0, \quad y &\geq 0. \end{aligned}$$

Note: In this project, you will not be required to sketch any linear-programming graphs. You will only be required to identify the objective function of an LP problem, and express the problem in terms of constraints. However, the following section will be useful when attempting the computer-based assignment.

Step 4: Sketching the graph.
In order to solve an LP problem graphically, you should be able to sketch the graph accurately and read off it.



The breakdown of all the information to be included in the LP graph is as follows:

i Draw the lines representing each constraint and label them. The constraint lines in the graph above are labeled (1) and (2) respectively, according to the constraints listed in *step 3*.

ii Shade in the regions represented by all the constraints. The overlap of all the wanted regions is called the *feasible region*.

iii Find the coordinates of the vertices of the feasible region. For this question, the vertices are points A, B, C and D respectively. If the graph is drawn to scale, you would be able to identify the coordinates of the vertices simply by inspection. Alternatively, the more fail-proof method would be to use *simultaneous equations* to solve for the point of intersection of the two lines.

iv Find the optimal solution. There are two methods that we can use.

(a) The objective line method.
Draw any line representing the objective function (has the same gradient as the objective function). If maximising, move the objective function line upwards (to the right), and if minimizing, move the objective function line downwards (to the left). The last vertex that the objective function line touches is the optimal solution to our LP problem.

(b) The vertex method.
Using this method, simply substitute each of the vertices of the feasible region into the objective function and evaluate which vertex gives us the maximum or minimum solution.

In this example, we used the objective line method. The red line in the graph above represents the objective line. By moving the objective function upwards and outwards, We see that the outermost vertex that the objective line touches is (B). This means that the optimal solution is at $x = 2$ and $y = 2$. By substituting these values into the objective function, we see that the brewer should make 2 Pale Ales and 2 Bitters to make a profit of:

$$z = 2(2) + 3(2) = £10.$$

Minimization LP problems are dealt with in a similar fashion. The constraints, however, tend to involve $\geq$ rather than $\leq$.

3 Mathematical Proof

3.1 Introduction

KEY POINTS

- A *conditional statement* is of the form *if* P *is* $\cdots$, *then* Q *is* $\cdots$.
- the word "integers" refers to *whole numbers.*
- The following definitions are used for even and odd numbers:
  1. An *even number* is an integer which is evenly divisible by 2. Thus, $2n$ is an even number, for any integer n.
  2. An *odd number* is an integer which cannot be divided by 2. Thus, $2n+1$ is an odd number, for any integer n.

3.2 Direct proof

Direct proof is the most basic method of proof that can be used in any mathematical context. The general structure of a direct proof is as follows.

Conditional statement: If P is $\cdots$, then Q is $\cdots$.

Proof:

Suppose P is $\cdots$.
$\vdots$
Therefore Q is $\cdots$.

Example 1: Prove that if n is even, then n^2 is also even.

With reference to the general structure of direct proofs (green box above),

P here is n is even, Q here is n^2 is even.

Proof:

Suppose n is even. Then $n = 2m$, where m is an integer. This is stated with reference to the definition of an even integer.

Squaring both sides, we have

$$\begin{aligned} n^2 &= (2m)^2 \\ &= 4m^2 \\ \text{Here, } n^2 \text{ is a multiple of 2} \quad &= 2(2m^2). \end{aligned}$$

Since m is an integer, $2m^2$ is also an integer. Thus,

$$n^2 = 2k, \quad \text{where} \quad k = 2m^2.$$

Therefore n^2 is even, by the definition of an even integer.

3.3 Proof by contraposition

Proof by contraposition is an alternative method to the direct proof. It has a fairly similar set up as well. In order to understand how *proof by contraposition* works, we must first understand what a contrapositive statement is.

Let us take the conditional statement:

if Steve is a man, then he is human.

The *contraposition* of this statement is:

if Steve is NOT human, then he is NOT a man.

The law of contraposition states that a conditional statement is identical to its contraposition. They are just different ways of conveying the same information. Thus, if the conditional statement is true, its contraposition is also true. Alternatively, if the conditional statement is false, its contraposition is also false. We will use this property to our advantage when applying the method of proof by contraposition.

The general structure of a proof by contraposition is as follows.

Conditional statement: If P is $\cdots$, then Q is $\cdots$. Recall, this statement is logically equivalent to its contraposition: If Q is not $\cdots$, then P is not $\cdots$.

Proof:

Suppose Q is not $\cdots$.
$\vdots$
Therefore P is not $\cdots$.

Since the contrapositive statement: if Q is not $\cdots$, then P is not $\cdots$ is equivalent to

the actual statement we were required to proof, by proving the contrapositive statement to be true, we also prove that the actual statement is true as well.

Example 2: Prove by contraposition that if n^2 is odd, then n is odd.

With reference to the general structure of proof by contraposition (green box above),

The contraposition of this conditional statement is:

If n is not odd, then n^2 is not odd.

This can then be rephrased as:

$$\text{If } n \text{ is even, then } n^2 \text{ is even.} \tag{3}$$

We can then proceed to proof the contrapositive statement (3) derived from the actual statement given in the example using *direct proof. Note that the contrapositive statement (3) is exactly the same as the conditional statement in* ***Example 1***.

The remainder of this proof is identical to **Example 1**.

Proof:

Suppose n is even. Then $n = 2m$, where m is an integer. This is stated with reference to the definition of an even integer.

Squaring both sides, we have

$$\begin{aligned} n^2 &= (2m)^2 \\ &= 4m^2 \\ \text{Here, } n^2 \text{ is a multiple of 2} \quad &= 2(2m^2). \end{aligned}$$

Since m is an integer, $2m^2$ is also an integer. Thus,

$$n^2 = 2k, \quad \text{where} \quad k = 2m^2.$$

Therefore n^2 is even, by the definition of an even integer.

Since the contrapositive statement (3) is equivalent to the actual statement we were required to proof, we have just proven by contraposition that if n^2 is odd, then n is odd.

4 Written assignment

Questions marked with a " * " will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the attached notes carefully before attempting these questions.

Question 1

Find the values of x which satisfy the following relationship:

$$\frac{x+5}{x-1} = \frac{2x-5}{x+1}.$$

[3 marks]

Question 2

Evaluate the following expression for $a = 5$ and $b = 8$:

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

[4 marks]

Question 3

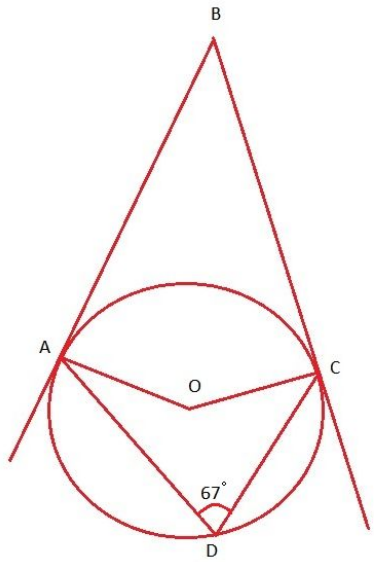


Figure 1

Find the internal angle at the point B of Figure 1.

[5 marks]

Question 4

Find a value of θ which satisfies the following expression:

$$sin^2(\theta) + cos(\theta) + 1 = 0, \qquad 0 < \theta < 2\pi.$$

[5 marks]

Question 5

Find a value of x which satisfies $3^{2^x} = 81$.

[3 marks]

Question 6

Long ago a "wizard" beats a King at a game of chess. The King offers the wizard a reward of his choosing. The Wizard shows the King the chess board. He asks for one Square meter of land for the first square, two for the second, four for the third and so on, doubling with each square. The King agrees. Explain numerically why it was not a good idea for the King to agree to this reward. (The kingdom is the same size as the U.K.)

[5 marks]

Question 7*

Express the following word-problem in terms of equations involving constraints. Identify the objective function. You are advised to summarise the information from the question in a table before formulating the constraints. You are *NOT* required to sketch the LP graph or find the optimal solution.

A carpenter makes wooden shelves and tables. Each shelf requires 3 hours to assemble, 2 hours to sand, and another 1.5 hours to varnish. The tables require 2.5 hours to assemble, 2 hours to sand, and only 1 hour to varnish. The carpenter has allocated 18 hours to assemble, 15 hours to sand, and 10 hours to varnish all furniture each week. The profit made from the sales are £4 for each wooden shelf sold, and £3 for each table sold. How many wooden shelves and tables does the carpenter have to make to maximize his profit his week?

[6 marks]

Question 8*

(a) Using the direct proof method, show that if n is odd, then n^2 is also odd.

(b) Using the proof by contraposition method, show that if n^2 is even, then n is also even.

[6 marks]

Question 9*

Using contrapositive proof, show that if $3x^2 + 5$ is odd, then x is even.

[6 marks]

5 Research question

The final part of the project is a more open ended research based question. We expect a one to two page report presenting all the information related to the research question.

Choose **one** of the following research topics.

[20 marks]

Divergent infinite series

Some infinite series have caused heated debate between mathematicians for centuries. One famous example is Grandi's series $1-1+1-1+1-1+1-\cdots$ it can be shown mathematically that this sums to 3 different values: $0, 1$ and most importantly for this research question $1/2$. Show how Grandi's series can sum to $1/2$ and then use this result to prove a frankly unbelievable result, that $1 + 2 + 3 + 4 + \cdots = -1/12$. Comment on the validity of the proof.

Probability of winning the lottery

We are all well aware that matching all six balls in the U.K national lottery is no easy feat. With this in mind millions of people still buy a lottery ticket every week. Calculate the probability of matching all six numbers in the lottery of 59 balls with one ticket. Will it increase your chances if you use the same numbers each week as apposed to changing them? If there is no winning ticket matching all six balls then the jackpot rolls over to the next week adding to the jackpot. It is estimated 32 million people play the lottery on a regular basis. If the jackpot for a usual week is £14 million then what is the probability of a jackpot of £56 million being won after 4 weeks?

Decrypting the code

The following code is encrypted using the Caesar Cipher. Research the Caesar Cipher as an encryption technique. Use the information you gathered on the Caesar Cipher to decode the message given below:

Znk rout, znk cozin gtj znk cgxjxuhk gtj Znk muuj, znk hgj gtj znk amre gxk zcu ul se lgbuxozk lorsy.

A good way to start to attempt to decrypt the code is think of the most commonly used words in the English language. Two very frequently used words are "and" and "the". Can you see any patterns in the encrypted text that could correspond to one of these words.

The following code is encrypted using the Vigenere Cipher. Research how the Vigenere code is used, and describe step by step how the Vigenere table is used to decipher messages.

The keyword to decipher the message below is “NEWCASTLE”. Refer to the Vigenere table included below.

fijf uk t zrr waptwgni ziouayx fwvrc dola elr gwgssk nmclat afw elr zeiefxci pmljej.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
B	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
C	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
D	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
F	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
G	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
H	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
I	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
J	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
K	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
L	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
M	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
N	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
O	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
P	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Q	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
R	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
U	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
V	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
W	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
X	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
Y	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
Z	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

Figure 2