Summer Project

School of Mathematics, Statistics & Physics

Summer 2017

Workbook 3

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Introduction

This project is intended to give current A-Level students (or equivalent) an opportunity to demonstrate their mathematical knowledge and experience, and to give such students a taste of some new material which most will not have seen before. A brief recap of some concepts encountered in the A-level syllabus is provided, with a more detailed description of new concepts to provide assistance when preparing solutions to questions posed in the project. This project consists of three components:

- A written assignment sheet
- Some computer based questions using our NUMBAS online testing system
- A more open ended, research-style question

The questions are designed to be of varying difficulty to challenge students at different skill levels and we would expect a good student to take up to five days to complete all components. Any questions marked with a " * " will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the information

in this booklet carefully before attempting the questions. Completed assignments will be graded and successful students will be awarded with a certificate and general feedback. More details about this summer project can be found on our dedicated webpage: `http://www.mas.ncl.ac.uk/~nlf8/summer-project/`.

Written assignment

There are eight questions to work through here. Most of these questions will be based on material covered in this booklet. However, there may be one or two surprises! Solutions to these questions can be hand-written or typed. Hand-written solutions should be scanned and emailed to `mathsandstats@ncl.ac.uk`. Solutions prepared in a word-processing package (eg. Microsoft Word) can be emailed directly, although we recommend exporting to portable document format (PDF) first, to help prevent loss of formatting. Students must submit their solutions by **4pm, Friday 1st September GMT**.

Computer based test

This uses Newcastle University's *NUMBAS online testing system*. The test will be available in two "modes"; *PRACTICE* and *ASSESSED*. Students can access the practice test as many times as they like. Each time they open a test, the format of the questions within that particular test will remain the same, but the numbers might change. The assessed mode will be active for two weeks only, from **Monday 21st August–Friday 1st September**. Students must login and complete the test by **4pm, Friday 1st September GMT**. Only **one** attempt will be allowed, but students can pause the test as often as they wish – there are no time constraints. The practice mode will be available throughout August and September. Students are strongly advised to work through our *Introduction to NUMBAS* first – available at
`https://numbas.mathcentre.ac.uk/test-yourself/getting-started/`. A link to the actual test, and further instructions about submission of the test will, be available on our webpage soon.

Research-style question

This is the final part of the project. Students will be given a choice of three questions, from which they are required to attempt only **one**. We expect solutions to any of these questions to be one or two pages long. Students can pick from a range of topics for this question, including options from statistics or number theory. Most questions do not have strict guidelines and we encourage students to be creative with their answers. As with the written assignment, solutions to the open-ended questions can be either hand-written or electronically produced with a word processing package. Answers should be emailed to `mathsandstats@ncl.ac.uk` no later than **4pm, Friday 1st September GMT**.

1 Some Revision

1.1 Quadratic equations

A root of the function $f(x)$ is any value of x which solves $f(x) = 0$. For a quadratic polynomial where $f(x) = ax^2 + bx + c$, there is a maximum of two roots. These can be found via three different methods:

1. Factorisation:

   The quadratic equation $ax^2 + bx + c = 0$ can sometimes be expressed in the form$(dx + e)(jx + k) = 0$.

   This implies that the roots are $x = -\dfrac{e}{d}$ and $x = -\dfrac{k}{j}$.

2. Quadratic formula:

   Substitute the respective values of a, b and c into the quadratic formula:

   $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Completing the square (only when $a = 1$):

   $x^2 + bx + c = (x + b/2)^2 - (b/2)^2 + c = 0 \Rightarrow (x + b/2)^2 = (b/2)^2 - c.$

1.2 Plane geometry

The equation of a straight line in *gradient form* is

$$y = mx + c,$$

where m is the *gradient* and c is the *y-intercept*. The equation of the straight line between points (x_1, y_1) and (x_2, y_2) in the xy-plane can be found using the formulae:

$$y - y_1 = m(x - x_1), \quad \text{where} \quad m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Points of intersection between two functions can be identified by equating them and finding values of x which satisfy the relationship. Some pairs of functions may have many intersection points and others may not have any.

1.3 The factor theorem

If we have a function that can be written as a product of linear factors then equating each factor to zero will give us a root of the function. Mathematically this can be seen more clearly. Consider

$$f(x) = (x - a)(x - b).$$

Clearly if we let $x = a$ or $x = b$ then $f(x) = 0$ implies that both a and b are roots of $f(x)$. The principle can be applied in reverse: if we know that $x = a$ is a root then we know that $(x - a)$ is a factor. The other factors can be found by dividing $f(x)$ by $(x - a)$.

1.4 Binomial expansion

An expression of the form $(a+b)^n$ can be written out as a complete sum of all the individual terms using the binomial formula given below:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where $\binom{n}{k}$ is the binomial coefficient which can be obtained using a calculator or by writing out Pascal's triangle. The binomial coefficient can also be calculated as a quotient of factorials given in the formula below:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

1.5 Logarithms and exponentials

Index form: $y = a^x$ *Logarithmic form:* $\log_a(y) = x$

Here is a list of useful logarithmic identities:

- $\log_a(x) + \log_a(y) = \log_a(xy)$
- $\log_a(x) - \log_a(y) = \log_a(\frac{x}{y})$
- $\log_a(x^k) = k\log_a(x)$

The exponential function $f(x) = e^x$ is a special function because $\dfrac{df}{dx} = f(x)$. The natural logarithm is $\log_e$ and is often denoted ln. If at any point in the exercises you encounter a log without a subscript then it can be assumed it means $\log_e$, not $\log_{10}$ as is sometimes the convention.

1.6 Trigonometric functions

You should all be familiar with $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ and how they link to the ratio of lengths in a right-angled triangle. Radians are a measure of angle where 2π radians $= 360°$. One can assume that the angle is given in radians unless explicitly expressed otherwise. Some important trigonometric identities are:

- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $sin(\theta) = a \quad \Rightarrow \quad \theta = sin^{-1}(a)$
- $cos(\theta) = b \quad \Rightarrow \quad \theta = cos^{-1}(b)$

1.7 Differentiation and integration

For a polynomial function $f(x) = ax^n$, the *first-order derivative* $\dfrac{df}{dx} = f'(x) = anx^{n-1}$. The inverse of this operation is *integration*, where for $f(x) = ax^n$ then

$$\int f(x)\ dx = \frac{a}{n+1}x^{n+1} + c,$$

where c is the constant of integration.

Differentiation can be used to find the gradient at any point of a particular function. Substituting any given x value in to the derivative function will give the gradient of $f(x)$ at that point. Points where the gradient equals zero are called turning points and can be important when trying to find values of x which maximize or minimize a function.

Integration can be used to calculate the area between a function and the x axis. If we want to calculate the area under the curve between two values of x we select the lowest value of x as our lower limit of integration, l, and the higher value as our upper limit of integration, u. Mathematically this can be written as:

$$\int_l^u f(x)\ dx = \left[\int f(x)dx\right]_l^u = \int f(x)dx\Big|_{x=u} - \int f(x)dx\Big|_{x=l},$$

where the notation $|_{x=u}$, for example, means "evaluated at $x = u$". One important thing to consider is that the area below the x-axis will have a negative contribution to the overall integral so it may be necessary to calculate the area above and below the x-axis separately and then add them together.

2 New topic 1: Modular arithmetic

We expect most students to not have seen this topic before. Modular arithmetic is an extremely broad subject, filled with amazing proofs and mathematical techniques. We aim to simply introduce the basic computations using the modulo operator (as seen later). The purpose of including this new section is to give students a chance to familiarize themselves with certain mathematical computations which will be required of them should they pursue Mathematics beyond A-level.

2.1 Division with remainder

Recall, when we divide a by b, where both a and b are integers and $b \neq 0$, we get the following:

$$\frac{a}{b} = q \text{ remainder } r,$$

where :

- a is the *dividend*,
- b is the *divisor*,
- q is the *quotient*,
- r is the *remainder*, where $0 \leq r < b$.

Under certain circumstances, we are interested only in the *remainder*.

2.2 The modulo operator

We will now introduce the modulo operator *(mod)*.

For any integers a and b, we write $a \pmod{b}$ or $[a]_b$ to denote the remainder when a is divided by b.

Example 1: Find 17 (mod 6).

$$\frac{17}{6} = 2 \text{ remainder } 5$$

Remember, we are only interested in the *remainder* when using the modulo operator (mod). Thus, we have:

$$17 \pmod{6} \equiv 5.$$

Example 2: Find $-12 \pmod{17}$.

We know that :

$-12 = (-1) \cdot (17) + 5.$

Thus,

$-12 \pmod{17} \equiv 5.$

Notice here that the remainder is 5 because the remainder has to be strictly greater than 0. Effectively we have to find the first negative multiple of 17 less than -12 and then add back upwards.

2.2.1 Addition

When considering a sum within the modulo operator, we can first compute the modulo of each of the components individually and then take the modulo of the whole sum.

Example 3: Compute $[6 + 12 + 8]_5$.

The expression within the modulo operator is a sum of three terms. We can take the modulo of each term individually, and finally take the modulo of the sum of the modulos.

We have: $[6]_5 = 1, [12]_5 = 2$, and $[8]_5 = 3$. Thus:

$$\begin{aligned}
[6 + 12 + 8]_5 &= \big[[6]_5 + [12]_5 + [8]_5\big]_5 \\
&= [1 + 2 + 3]_5 \\
&= [6]_5 \\
&= 1.
\end{aligned}$$

Let us verify our solution. We know the sum $6+12+8 = 26$. We can verify our solution by taking the remainder of 26 divided by 5.

$$\frac{26}{5} = 5 \text{ remainder } 1.$$

Therefore,

$$26 \pmod{5} \equiv 1.$$

2.2.2 Multiplication

Multiplication within the modulo operator works similarly.

Example 4: Compute $[2 \times 4 \times 8]_3$.

The expression within the modulo operator is a product of three terms. We can take the modulo of each term individually, and finally take the modulo of the product of the modulos.

We have: $[2]_3 = 2, [4]_3 = 1$, and $[8]_3 = 2$.

Thus:

$$\begin{aligned}
[2 \times 4 \times 8]_3 &= \big[[2]_3 \times [4]_3 + [8]_3\big]_3 \\
&= [2 \times 1 \times 2]_3 \\
&= [4]_3 \\
&= 1.
\end{aligned}$$

Let us verify our solution. We know the product $2 \times 4 \times 8 = 64$. We can verify our solution by taking the remainder of 64 divided by 3.

$$\frac{64}{3} = 21 \text{ remainder } 1.$$

Therefore,

$$64 \pmod 3 \equiv 1.$$

3 Mathematical Induction

3.1 Introduction

Induction is a powerful method for proving a sequence of statements are all true. We will first take a look at an example of a sequence of statements that is proven with mathematical induction.

Table 1: Sequence of statements.

n	sum of the first n odd numbers	n^2
1	1	$1^2 = 1$
2	$1 + 3 = 4$	$2^2 = 4$
3	$1 + 3 + 5 = 9$	$3^2 = 9$
4	$1 + 3 + 5 + 7 = 16$	$4^2 = 16$
5	$1 + 3 + 5 + 7 + 9 = 25$	$5^2 = 25$

What do you observe? Is there a recurring pattern to the sequence of statements? You are encouraged to spend a few minutes to analyse the statements given in Table 1.

The first thing to notice would be that up until $n = 5$, the sum of the first n odd numbers really does sum to n^2.

The second observation would be that the last number in each sum (the nth odd number in each sum, highlighted red) is $2n - 1$. For example, when $n = 4$, the sum of the first four odd numbers, $1, 3, 5$ and 7 is equal to 16; the *fourth* odd number in the sum, 7, is simply $(2 \times 4) - 1 = 7$.

With our two observations in hand, we can propose that the sum

$$1 + 2 + 3 + 5 + 7 + ... + (2n - 1) = n^2, \quad n \geq 1. \tag{1}$$

This is a general expression of the sums in Table 1. How do we prove if the sum is true for all n greater than or equal to one?

3.2 The principle of induction

Mathematical induction comes in handy when we need to prove that a set of statements are all true. Induction is a two-step process.

Let's say we have the following sequence of statements (from Table 1).

$$\begin{aligned}
&P_1 : 1 = 1^2 \\
&P_2 : 1 + 3 = 2^2 \\
&P_3 : 1 + 3 + 5 = 3^2 \\
&\quad\vdots \\
&P_n : 1 + 3 + 5 + 7 + 9 + \cdots + (2n - 1) = n^2
\end{aligned}$$

Step 1: We have to show that the first statement P_1 is true. P_1 is referred to as the *base case*. This will be fairly easy, because the base case, P_1, is usually the easiest statement in the sequence.

Step 2: We then have to prove that if we know a given statement is true then this implies the next statement must also be true. Given any integer $k \geq 1$, we have to prove that if P_k is true, it implies that P_{k+1} is true, i.e P_k is true $\implies$ P_{k+1} is true. In this step, we start by assuming a statement P_k is true, which in turn, forces P_{k+1} to be true as well. We need not prove that P_k is true, we may simply assume this. Assuming P_k is true is referred to as the *inductive hypothesis*.

The inductive principle can be thought of as being similar to a chain of dominoes. Once the first domino falls it knocks over the next, and then the next, and so on. Similarly when performing proof by induction we begin by proving the first statement (knocking over the first domino) and then showing that each subsequent statement implies the next (each domino knocks down the one following it).

Generally,

Step 1: Prove that the base case is true.
Step 2: Use inductive hypothesis (IH) to show that P_k is true $\implies$ P_{k+1} is true.

Conclusion : All statements $P_1, P_2, P_3, \cdots, P_n$ are true.

We will now apply the *principle of induction* to two different examples. Example 1 uses the set of statements from Table 1. Example 2 will be a completely new set of statements. Students are encouraged to study both these examples and see for themselves that proof by induction is quite a straight-forward mathematical technique.

Example 1: We can rephrase the sequence of statements from Table 1 as follows:

Prove by induction that $1 + 3 + 5 + \cdots + (2n - 1) = n^2, \quad \text{for all} \quad n \geq 1.$

Solution:

Step 1 : The base case.
Our base case is at $n = 1$. Thus, $P_1 : 1 = 1^2$, which is true just by observation.

Step 2: (IH).
We now have to prove P_k is true $\implies$ P_{k+1} is true, for any $k \geq 1$.

We have :

$$P_k : 1 + 3 + 5 + \cdots + (2k - 1) = k^2 \tag{2}$$
$$P_{k+1} : 1 + 3 + 5 + [2(k + 1) - 1] = (k + 1)^2 \tag{3}$$

We use induction hypothesis, (IH) to assume P_k is true.

Suppose $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ is true. Then, for P_{k+1}, we have:

$$\begin{aligned} P_{k+1} : 1 + 3 + 5 + ... + [2(k + 1) - 1] &= 1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= \{1 + 3 + 5 + \cdots + (2k - 1)\} + [2(k + 1) - 1] \\ \text{substituting equation (2)} \quad &= k^2 + [2(k + 1) - 1] \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \quad \text{equal to equation (3)} \end{aligned}$$

by assuming P_k is true, we have just shown that P_{k+1} is also true. This means that for any $k \geq 1$, the subsequent statement after P_k will also be true. For example, if $k = 3$, P_3 would be true, and hence P_4 would also be true. P_4 being true in turn implies that P_5 is true, and so on.

Since our base case, P_1 is true, it means that P_2 is also true, and subsequently P_n is true, for $n \geq 1$.

Conclusion: By induction, $P_n : 1 + 3 + 5 + \cdots + (2n - 1) = n^2$ holds for all $n \geq 1$.

Example 2:

Prove by induction that $1+2+3+4+\cdots+n = \dfrac{n(n+1)}{2}$, for all $n \geq 1$.

Solution:

Step 1 : The base case.
Our base case is at $n = 1$. Thus, $P_1 : 1 = \dfrac{1(1+1)}{2}$, Which is true.

Step 2: (IH).

We have :

$$P_k : 1+2+3+4+\cdots+k = \frac{k(k+1)}{2} \tag{4}$$

$$P_{k+1} : 1+2+3+4+\cdots+(k+1) = \frac{(k+1)[(k+1)+1]}{2} \tag{5}$$

We use induction hypothesis, (IH) to assume P_k is true.

Suppose $1+2+3+4+\cdots+k = \dfrac{k(k+1)}{2}$ is true. Then, for P_{k+1}, we have:

$$\begin{aligned}
P_{k+1} : 1+2+3+4+\cdots+(k+1) &= 1+2+3+4+\cdots+k+(k+1) \\
&= \{1+2+3+4+\cdots+k\}+(k+1) \\
\text{substituting equation (4)} \quad &= \frac{k(k+1)}{2}+(k+1) \\
\text{using simple algebra} \quad &= \frac{k(k+1)+2(k+1)}{2} \\
\text{using simple algebra} \quad &= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)[(k+1)+1]}{2}. \quad \text{same as equation (5)}
\end{aligned}$$

Since our base case, P_1, is true, it means that P_2 is also true, and subsequently P_n are all true, for $n \geq 1$.

Conclusion: By induction, P_n holds for all $n \geq 1$.

4 Written assignment

Questions marked with a " * " will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the attached notes carefully before attempting these questions.

Question 1

Find the values of x which satisfy the following relationship:

$$\frac{x+5}{x-1} = \frac{2x-5}{x+1}.$$

[3 marks]

Question 2

Sketch the function $f(x) = 2x^2 - x - 15$. Label all turning points and points of intersection.

[3 marks]

Question 3

Given that $x = a^n$ and $y = a^m$ show that $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$.

[4 marks]

Question 4

What is the area of intersection of the two squares shown in the diagram? The point P is the centre of the small square which has sides of length 10 cm. The top left corner of the larger square is also at point P.

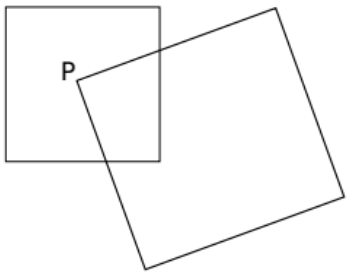

Figure 1

[4 marks]

Question 5*

The following question involves modular arithmetic where $[X]_n$ denotes $X \pmod{n}$. Determine answers to the following expressions:

(a) $[47]_3$

(b) $[-23]_5$

(c) $[15 + 30 + 45 + ... + 300]_5$

(d) $[2 \times 4 \times 6 \times ... \times 16]_3$

[4 marks]

Question 6*

Steve turns 20 on a Friday in 2017. Using your knowledge of modular arithmetic, find out on what day of the week Steve was born. Show all the steps in your working clearly.

[5 marks]

Question 7*

Prove by induction that, for all $n \geq 1$,

$$2 + 4 + 6 + \cdots + 2n = n(n + 1)$$

.

[5 marks]

Question 8*

Prove by induction that, for all $n \geq 1$,

$$2 + 6 + 12 + \cdots + n(n + 1) = \frac{n(n + 1)(n + 2)}{3}$$

.

[7 marks]

5 Research question

The final part of the project is a more open-ended research based question. We expect a one/two page report presenting all the information related to the research question.

Choose **one** of the following research topics.

[20 marks]

Prime numbers and the fundamental theorem of arithmetic

What is a prime number? Investigate prime numbers and how they are involved in the fundamental theorem of arithmetic. With reference to the fundamental theorem of arithmetic what makes primes so important? Is there a biggest prime number and if so what is it? How can prime numbers be used to find the greatest common divisor between two numbers and the lowest common multiple? Include any other interesting facts about prime numbers.

Analysing summer project scores

The summer project marks from 10 students at two different schools are given in the tables below:

Table 2

St John's High										
Student number	1	2	3	4	5	6	7	8	9	10
Summer project percentage	74	89	67	51	61	84	76	42	93	59

Table 3

St Mary's College										
Student number	1	2	3	4	5	6	7	8	9	10
Summer project percentage	59	63	96	73	41	71	63	53	59	86

First, analyse both datasets individually. What could an appropriate probability distribution be to fit to each set? Justify your answer. Estimate the parameter(s) in your chosen distribution and sketch their *probability density functions*. Find the 5th and the 95th percentiles for each distribution. Compare the two groups. Is there any evidence to suggest that the marks from the two groups are statistically different? Justify all of your conclusions.

Decrypting the code

The following code is encrypted using the Caesar Cipher. Research the Caesar Cipher and how modular arithmetic plays a role in this encryption technique. Use the information you gathered on the Caesar Cipher to decode the message given below:

Znk rout, znk cozin gtj znk cgxjxuhk gtj Znk muuj, znk hgj gtj znk amre gxk zcu ul se lgbuxozk lorsy.

A good way to start to attempt to decrypt the code is to think of the most commonly used words in the English language. Two very frequently used words are “and” and “the”. Can you see any patterns in the encrypted text that could correspond to one of these words.

The following code is encrypted using the Vigenere Cipher. Research how the Vigenere code is used, and describe step-by-step how the Vigenere table (Figure 2) is used to decipher messages. The keyword to decipher the message below is “NEWCASTLE”. Refer to the Vigenere table included below.

fijf uk t zrr waptwgni ziouayx fwvrc dola elr gwgssk nmclat afw elr zeiefxci pmljej.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
B	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
C	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
D	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
F	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
G	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
H	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
I	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
J	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
K	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
L	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
M	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
N	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
O	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
P	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Q	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
R	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
U	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
V	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
W	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
X	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
Y	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
Z	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

Figure 2